HEDGING OF SWING GAME OPTIONS IN CONTINUOUS TIME.

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Abstract. The paper introduces and studies hedging for game (Israeli) style extension of swing options in continuous time considered as multiple exercise derivatives of a base security evolving according to the geometric Brownian motion. Assuming that the underlying security can be traded without restrictions we derive a formula for valuation of multiple exercise options via classical hedging arguments. The paper extends to the continuous time case the discrete time valuation results from [3] which requires substantial additional machinery such as, for instance, reqularity results for value processes of Dynkin's games and the study of multiple stopping Dynkin's games. Earlier papers on valuation of multiple exercise American options such as [1] viewed it only as a multiple stopping problem.

1. Introduction

Swing contracts emerging in energy and commodity markets are often modeled by multiple exercising of American style options which leads to multiple stopping problems (see [1], [2] and references there). Most closely such models describe options consisting of a package of claims or rights which can be exercised in a prescribed (or in any) order with some restrictions such as a delay time between successive exercises. Observe that peculiarities of multiple exercise options are due only to restrictions such as an order of exercises and a delay time between them since without restrictions the above claims or rights could be considered as separate options which should be dealt with independently.

Attempts to valuate swing options in multiple exercises models are usually reduced to maximizing the total expected gain of the buyer which is the expected payoff in the corresponding multiple stopping problem deviating from what now became classical and generally accepted methodology of pricing derivatives via hedging and replicating arguments. This digression is sometimes explained by difficulties in using an underlying commodity in a hedging portfolio in view of the high cost of storage, for instance, in the case of electricity. We will not discuss here in depth practical possibilities of hedging in energy markets but only observe that the seller of a swing option could, for instance, use for hedging certain index linked securities or futures for a corresponding commodity (electricity, gas, oil etc.). Still, from the beginning we prefer to avoid an extensive economics discussion and to stay on a precise mathematical ground of multiple exercise options based on an underlying security evolving according to the geometric Brownian motion which can be used for hedging without restrictions. Thus, in what follows we use the names swing options and multiple exercise options for the same object.

Observe that multiple exercise options may appear in their own rights when an investor wants to buy or sell an underlying security in several instalments at times of his choosing and, actually, any usual American or game option can be naturally extended to the multi exercise setup so that they may emerge both in commodities, energy and in different financial markets. Suppose, for instance, that a European

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car producer (having most expenses in euros or pounds) plans to supply autos to US during a year in several shipments and it buys a multiple exercise option which guaranties a favorable dollar–euro or pund exchange rate at time of shipments (of its choosing). The seller of such option can use currencies as underlying for his hedging portfolio. A multiple exercise option could be cheaper then a basket of usual one exercise options if the former stipulates certain delay time between exercises which is quite natural in the above example. Furthermore, the acting sides above may prefer to deal with game rather than American multiple exercise options since the former is cheaper for the buyer and safer (because of cancellation clause) for the seller.

Clearly, the study of hedging for multiple exercise options is sufficiently motivated from the financial point of view and it leads to interesting mathematical problems. In this paper we assume that the underlying security evolves according to the geometric Brownian motion and it can be used for construction of a hedging portfolio without restrictions as in the usual theory of derivatives. Moreover, we will deal here with the game (Israeli) option (contingent claim) setup when both the buyer (holder) and the seller (writer) of the option can exercise or cancell, respectively, the claims (or rights) in a given order but as in [6] each cancellation entails a penalty payment by the seller. This is a more general and more complicated than American options setup and it required us, in particular, to extend Dynkin's games machinery to the multiple stopping setup.

In this paper a continuous time swing (multiple exercise) game option is a contract between its seller and the buyer which allows to the seller to cancel (or terminate) and to the buyer to exercise L specific claims or rights in a particular order. Such contract is determined given $2l$ payoff processes $X_i(t) \geq Y_i(t) \geq 0$, $t \geq 0$, $i = 1, 2, ..., l$ adapted to a filtration $\mathcal{F}_t$, $t \geq 0$ generated by the stock (underlying risky security) S_t, $t \geq 0$ evolution. If the buyer exercises the k-th claim $k \leq l$ at the time t then the seller pays to him the amount $Y_k(t)$ but if the latter cancels the claim k at the time t before the buyer he has to pay to the buyer the amount $X_k(t)$ and the difference $\delta_k(t) = X_k(t) - Y_k(t)$ is viewed as the cancelation penalty. In addition, we require a time delay $\delta > 0$ between successive exercises and cancellations. Observe that unlike some other papers (cf. [1]) we allow payoffs depending on the exercise number so, for instance, our options may change from call to put and vice versa after different exercises.

The goal of this paper is to develop a mathematical theory for pricing of continuous time swing game options. In [3] we did this for the discrete time situation but, as usual, the continuous time case requires substantial additional work. The standard definition of the fair price of a derivative security in a complete market is the minimal initial capital needed to create a (perfect) hedging portfolio, and so we have to start with a precise definition of a perfect hedge. Observe that a natural definition of a perfect hedge in a multiple exercise framework is not a straightforward extension of a standard one and it has certain peculiarities. Namely, the seller of the option does not know in advance when the buyer will exercise the $(j-1)$-th claim but his hedging strategy of the j-th claim should depend on this (random) time and on the capital he is left with in the portfolio after the $(j-1)$-payoff. Thus, in addition to the usual dependence on the stock evolution a perfect hedge of the j-th claim should depend on the past behavior of both seller and the buyer of the option. Actually, an optimal portfolio allocation depends also on the payoff processes of the future claims.

Several papers dealt with mathematical analysis of swing American options (see, for instance, [1], [2] and references there) but none of these papers defined explicitly what is a perfect hedge and what is the option price. For instance, in [1] the authors studied an optimal multiple stopping problem for continuous time models but they did not explained why the value of the above problem under the martingale measure in a complete market is the option price. In this paper we define the notion of a perfect hedge for continuous time swing game options which generalize swing American options and prove that in the standard Black–Scholes market the option price V^* is equal to the value of the multiple stopping Dynkin game with discounted payoffs under the unique martingale measure. Observe also that discrete time multiple stopping Dynkin's games were considered first in [3] and their continuous time version which is much more involved were not studied before the present paper. The proofs in the continuous time case require substantial technical work in order to derive regularity properties of value processes of usual and multi stopping Dynkin games which enables us to employ discrete time approximations.

Our exposition proceeds as follows. In Section 2 we define explicitly the notion of a (perfect) hedge and relying on this we define the option price. Then we state Theorem 2.5 which yields the option price together with the corresponding perfect hedge. Next, we formulate Theorem 2.6 which specify the results for Markovian payoffs. In Section 3 we derive auxiliary lemmas on regularity of Dynkin's games values which we need in the proof and which seem to be new. In Section 4 we introduce the concept of a continuous time multiple stopping Dynkin game and prove existence of a saddle point for this game. Section 5–7 are devoted to the proofs of Theorem 2.5 and Theorem 2.6.

2. Preliminaries and main results

We start with a financial market which consists of $m+1$ assets. One asset is a risk free bond with time evolution

$$B_t = B_0 e^{rt} \quad B_0 > 0 \ \ r \geq 0$$

and the other m assets are stocks whose prices $S_{1,t}, .., S_{m,t}$ satisfy the stochastic differential equations

$$dS_{i,t} = S_{i,t}(\mu_i dt + \sum_{j=1}^{m} \kappa_{i,j} dW_{j,t}). \tag{2.1}$$

Here $W_t = (W_{1,t}, ..., W_{m,t})$, $t \geq 0$ is a standard m-dimensional Wiener process with continuous paths starting at 0 with a nonsingular covariance matrix $\kappa = (\kappa_{i,j})$. Let $(\Omega, \mathcal{F}, P)$ be the corresponding probability space and $\{\mathcal{F}_t\}$, $t \geq 0$ be the continuous filtration generated by this Wiener process completed by all P-zero sets. We will use also that every martingale with respect to this filtration has a continuous modification since it can be represented as a stochastic integral (see [7]). All processes will be considered up to some finite time $T > 0$ which is called a horizon.

The solution of (2.1) is given explicitly by

$$S_{i,t} = S_{i,0} \exp\big((\mu_i - \frac{1}{2}\sum_{j=1}^{m} \kappa_{i,j}^2)t + \sum_{j=1}^{m} \kappa_{i,j} W_{j,t}\big) \tag{2.2}$$

where $(S_{1,0}, .., S_{m,0})$ are initial values of the stocks. We denote by v^t the transpose of any vector or matrix v and by $\bar{r}$ the row vector $(r, .., r) \in \mathbb{R}^m$. Set $\theta = \kappa^{-1}(\mu - \bar{r})^t$ and

$$G_t = \exp\big(-\theta \cdot (W_t - \frac{1}{2}\theta)\big)$$

then by Girsanov's theorem (see [7]) the probability measure $\tilde{P}$ on $\mathcal{F}_T$ given by

$$\tilde{P}(A) = E\big(G_T \mathbf{1}_A\big), \quad A \in \mathcal{F}_T$$

is equivalent to P and the process $\tilde{W}_t = W_t + \theta t$ is an m-dimensional Wiener process on $(\Omega, \mathcal{F}_T, \tilde{P}_T)$.

Denote by $\tilde{E}$ the expectation with respect to $\tilde{P}$ and let $\tilde{S}_t = e^{-tr}S_t$ be the discounted process. Then by Itô formula

$$d\tilde{S}_{i,t} = \tilde{S}_{i,t} \sum_{j=1}^{m} \kappa_{i,j} d\tilde{W}_{j,t}, \tag{2.3}$$

and so the discounted process $\tilde{S}_t$ is a martingale with respect to $\tilde{P}$. Using again Itô formula we obtain

$$dS_t = b(S_t)dt + \varsigma(S_t)d\tilde{W}_t \tag{2.4}$$

where $b:\mathbb{R}^m\to\mathbb{R}^m$ and $\varsigma:\mathbb{R}^m\to\mathbb{R}^{m\times m}$ are given by $b_i(x)=rx_i$ and $\varsigma_{i,j}=x_i\kappa_{i,j}$, and so the generator of the Itô diffusion is given by

$$A=\sum_{i=1}^m rx_i\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j}(\sum_{k=1}^m\kappa_{i,k}\kappa_{k,j})x_ix_j\frac{\partial^2}{\partial x_i\partial x_j}. \tag{2.5}$$

We consider a swing or multiple exercise option of the game type which has the i-th payoff, $i=1,...,l$ having the form

$$R_i(s,t)=X_i(s)\mathbf{1}_{\{s<t\}}+Y_i(t)\mathbf{1}_{\{s\geq t\}} \tag{2.6}$$

where $X_i(t)\geq Y_i(t)\geq 0$, $i=1,...,l$, $t\in[0,T]$ are $\mathcal{F}_t$, $t\in[0,T]$-adapted stochastic processes having a.s. (almost surely) continuous paths (though in Section 3 some results are proved assuming that these processes are only right continuous with left limits (RCLL)) and such that for $i=1,2,...,l$,

$$E\big(sup_{0\leq t\leq T}X_i(t)\big)<\infty. \tag{2.7}$$

This means that if the option seller cancells and the option buyer exercises the i-claim (right) at the times s and t, respectively, then the former pays to the latter the amount $R^{(i)}(s,t)$. The setup includes also a necessary delay time $\delta>0$ between cancellations and exercises, and so the game swing option is determined by the triple (X_i,Y_i,δ), $i=1,...,l$.

Let $\mathcal{T}$ be the set of all stopping times σ with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\mathcal{T}_{s,t}$ be the set of stopping times $\sigma\in\mathcal{T}$ such that $s\leq\sigma\leq t$.

2.1**. Definition.** For every $t\geq 0$ a *stopping strategy* with a horizon t is a function $\mathbf{t}:\{1,...,l\}\times[0,T)\to\mathcal{T}_{0,t}$ such that $\mathbf{t}(1,0)\in\mathcal{T}_{0,t}$ for $i=1$ and $\mathbf{t}(i,s)\in\mathcal{T}_{s+\delta,t\wedge(s+\delta)}$ for $1<i\leq l$ where $\delta>0$ is the delay between successive stoppings (exercises). Furthermore, the function $\mathbf{t}(i,\rho)$ is supposed to be $\mathcal{F}$-measurable.

For every $t\geq 0$ denote by $\mathcal{S}_t$ the set of all stopping strategies with a horizon t. Let $F:\mathcal{S}_T\times\mathcal{S}_T\to\mathcal{T}^l_{0,T}\times\mathcal{T}^l_{0,T}$ be a function defined by

$$F(\mathbf{s},\mathbf{t})=((\sigma_1,...,\sigma_l),(\tau_1,...,\tau_l))$$

where

$$\sigma_1=\sigma_1(\mathbf{s},\mathbf{t})=\mathbf{s}(1,0),\tau_1=\tau(\mathbf{s},\mathbf{t})=\mathbf{t}(1,0)$$

and for $1<i\leq l$,

$$\sigma_i=\sigma_i(\mathbf{s},\mathbf{t})=\mathbf{s}(i,\sigma_{i-1}\wedge\tau_{i-1}),\ \tau_i=\tau_i(\mathbf{s},\mathbf{t})=\mathbf{t}(i,\sigma_{i-1}\wedge\tau_{i-1}).$$

Let $\mathcal{L}$ be the set of all sequences $(t_1,..,t_i)$, $1\leq i\leq l-1$ such that $(t_i+\delta)\wedge T\leq t_{i+1}\leq T$. We assume that $\mathcal{L}$ contains also the empty sequence ϕ.

2.2**. Definition.** A *portfolio strategy* is a function π on the set $\mathcal{L}$ such that

$$\pi(t_1,...,t_i)=\{\beta^\pi_s(t_1,...,t_i),\gamma^\pi_{1,s}(t_1,...,t_i),...,\gamma^\pi_{m,s}(t_1,...,t_i)\}.$$

Here $\beta^\pi_s(t_1,...,t_i)$ and $\gamma^\pi_{j,s}(t_1,...,t_i)$, $1\leq j\leq m$ are progressively measurable processes with respect to $\{\mathcal{F}_s\}_{t_i\leq s}$ which satisfy

$$\int_{t_i}^T\beta_s(t_1,..,t_i)ds<\infty,\quad \int_{t_i}^T(\gamma_s(t_1,..,t_i)\cdot S_s)^2ds<\infty \tag{2.8}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^m$ and for every $t_i\leq s\leq T$,

$$\begin{aligned}Z^\pi_s(t_1,...,t_i)&=Z^\pi_{t_i}(t_1,...,t_i)+\textstyle\int_{t_i}^s\beta^\pi_u(t_1,...,t_i)dB_u+\int_{t_i}^s\gamma^\pi_u(t_1,...,t_i)\cdot dS_u\\&=\beta^\pi_s(t_1,...,t_i)B_s+\gamma^\pi_s(t_1,...,t_i)\cdot S_s.\end{aligned} \tag{2.9}$$

Hence, $\pi(t_1,...,t_i)$ is a self financing portfolio strategy starting at time t_i with a value process $Z^\pi_s(t_1,...,t_i)$. For any portfolio strategy π we also require that

$$G^\pi(t_1,...,t_i)=Z^\pi_{t_i}(t_1,...,t_{i-1})-Z^\pi_{t_i}(t_1,...,t_i)\geq 0$$

for every $(t_1,...,t_i)\in\mathcal{L}$ which means that there is no infusion of capital but some money can be withdrawn at exercise or cancellation times for a payment to the option buyer. Note that in the case of the empty sequence we have

$$G^\pi(\phi)=\pi_0-Z_0(\phi)\geq 0.$$

In the case $i=l$ we define

$$G^\pi(t_1,...,t_l)=Z^\pi_{t_l}(t_1,...,t_{l-1}).$$

2.3**. Definition.** A *hedge* is a pair $(\pi,\mathbf{s})$ of a portfolio strategy and a stopping strategy such that if for $\mathbf{t}\in\mathcal{S}_T$,

$$F(\mathbf{s},\mathbf{t})=((\sigma_1,...,\sigma_l),(\tau_1,...,\tau_l))$$

then

(i) $Z^\pi_\rho(\phi)$ is integrable for each $0\leq\rho\leq\sigma_1$ and for every ρ',ρ such that $0\leq\rho'\leq\rho\leq\sigma_1$,

$$E\big(e^{-\rho t}Z^\pi_\rho(\phi)|\mathcal{F}_{\rho'}\big)=e^{-\rho' t}Z^\pi_\rho(\phi);$$

(ii) $Z^\pi_\rho(\sigma_1\wedge\tau_1,...,\sigma_i\wedge\tau_i)$ is integrable for any $1<i\leq l-1$ and ρ satisfying $\sigma_i\wedge\tau_i+\delta\leq\rho'\leq\rho\leq\sigma_{i+1}$ and

$$E\big(e^{-r\rho}Z^\pi_\rho(\sigma_1\wedge\tau_1,...,\sigma_i\wedge\tau_i)|\mathcal{F}_{\rho'}\big)=e^{-r\rho'}Z^\pi_{\rho'}(\sigma_1\wedge\tau_1,...,\sigma_i\wedge\tau_i);$$

(iii) For each $1<i\leq l-1$,

$$E\big(e^{-r(\sigma_i\wedge\tau_i+\delta)}Z^\pi_{\sigma_i\wedge\tau_i+\delta}(\sigma_1\wedge\tau_1,...,\sigma_i\wedge\tau_i)|\mathcal{F}_{\sigma_i\wedge\tau_i}\big)=e^{-r(\sigma_i\wedge\tau_i)}Z^\pi_{\sigma_i\wedge\tau_i}(\sigma_1\wedge\tau_1,...,\sigma_i\wedge\tau_i).$$

If, in addition,

(iv) $G^\pi(\sigma_1\wedge\tau_1,...,\sigma_i\wedge\tau_i)\geq R_i(\sigma_i,\tau_i)$ for every $0\leq i\leq l$ the hedge is called a *perfect hedge* for the swing game option (X_i,Y_i,δ), $1\leq i\leq l$.

2.4**. Definition.** We define the *fair price* of the swing game option (X_i,Y_i,δ), $1\leq i\leq l$ to be the infimum of all x such that there exists a perfect hedge $(x,\pi,\mathbf{s})$ with the initial capital x.

Let $\mathbf{s},\mathbf{t}\in\mathcal{S}_T$ and $F(\mathbf{s},\mathbf{t})=((\sigma_1,...,\sigma_l),(\tau_1,...,\tau_l))$. Set

$$H(\mathbf{s},\mathbf{t})=\tilde{E}\big(\sum_{i=1}^{l}e^{-r\sigma_i\wedge\tau_i}R_i(\sigma_i,\tau_i)\big)$$

We now can state our main result concerning the price of a swing game option (cf. Theorem 2.4 in [3]).

2.5**. Theorem.** *For every* $0\leq t\leq T$ *set*

$$X^{(1)}_t=e^{-rt}X_l(t),\ Y^{(1)}_t=e^{-rt}Y_l(t), \tag{2.10}$$

$$V^{(1)}_t=esssup_{\tau\in\mathcal{T}_{t,T}}essinf_{\sigma\in\mathcal{T}_{t,T}}\tilde{E}\big(R^{(1)}(\sigma,\tau)|\mathcal{F}_t\big)$$

and for $1<i\leq l$,

$$X^{(i)}_t=e^{-rt}X_{l-i+1}(t)+\tilde{E}\big(V^{(i-1)}_{(t+\delta)\wedge T}|\mathcal{F}_t\big),\ Y^{(i)}_t=e^{-rt}Y_{l-i+1}(t)+\tilde{E}\big(V^{(i-1)}_{(t+\delta)\wedge T}|\mathcal{F}_t\big), \tag{2.11}$$

$$and\quad V^{(i)}_t=esssup_{\tau\in\mathcal{T}_{t,T}}essinf_{\sigma\in\mathcal{T}_{t,T}}\tilde{E}\big(R^{(i)}(\sigma,\tau)|\mathcal{F}_t\big)$$

where by the definition

$$R^{(i)}(\sigma,\tau)=X^{(i)}_\sigma\mathbb{1}_{\{\sigma<\tau\}}+Y^{(i)}_\tau\mathbb{1}_{\{\sigma\geq\tau\}}. \tag{2.12}$$

Then the fair price V^* *for the swing game option is given by*

$$V^*=V^{(l)}_0=inf_{\mathbf{s}\in\mathcal{S}_T}sup_{\mathbf{t}\in\mathcal{S}_T}H(\mathbf{s},\mathbf{t}). \tag{2.13}$$

Furthermore, let $\mathbf{s}^*, \mathbf{t}^* \in \mathcal{S}_T$ *be stopping strategies given by*

$$\mathbf{s}^*(1,0) = \inf\{0 \le t : V_t^{(l)} = X_t^{(l)}\} \wedge T,\ \ \mathbf{t}^*(1,0) = \inf\{0 \le t : V_t^{(l)} = Y_t^{(l)}\} \quad \text{and} \tag{2.14}$$

$$\mathbf{s}^*(i,\rho) = \inf\{t \ge \rho+\delta : V_t^{(l-i+1)} = X_t^{(l-i+1)}\} \wedge T,\ \mathbf{t}^*(i,\rho) = \inf\{t \ge \rho+\delta : V_t^{(l-i+1)} = Y_t^{(l-i+1)}\} \tag{2.15}$$

for $1 < i \le l$ *and* $\rho \in \mathcal{T}$*. Then for all* $\mathbf{s}, \mathbf{t} \in \mathcal{S}_T$*,*

$$H(\mathbf{s}^*, \mathbf{t}) \le H(\mathbf{s}^*, \mathbf{t}^*) \le H(\mathbf{s}, \mathbf{t}^*) \tag{2.16}$$

and there exists a portfolio strategy π^* *such that* $(V_0^{(l)}, \pi^*, \mathbf{t}^*)$ *is a perfect hedge.*

Next, we consider the Markovian case. In this situation we analyze the swing option price as a function of the initial stock prices assuming that the option value at each moment depends only on the stock prices at this particular moment. For every $x \in \mathbb{R}_+^m$ denote by S_t^x the vector of stock values at time t having initial prices x and evolving according to (2.2). In the case of finite horizon the time left till option expiration plays here an important role, and so it will be useful to consider the process $\bar{S}_t^{[x,u]} = (S_t^x, u+t)$ for $[x,u] \in \mathbb{R}_+^m \times [0,T]$. Then $\{\bar{S}_t^{[x,u]}\}_{[x,u] \in \mathbb{R}_+^m \times [0,T]}$ is a Markov process (see [9]) with the Markovian family $\{\tilde{P}_{[x,u]}\}_{[x,u] \in \mathbb{R}_+^m \times [0,T]}$ and corresponding expectations $\{\tilde{E}_{[x,u]}\}_{[x,u] \in \mathbb{R}_+^m \times [0,T]}$ (see [7]). In the Markovian case a swing game option is given by l pairs of functions $\{g_i, f_i\}_{i=1}^l$ such that for each $1 \le i \le l$ we have $g_i(x,u) \ge f_i(x,u) \ge 0$ for every $(x,u) \in \mathbb{R}_+^m \times [0,T]$. Here we assume that for each $1 \le i \le l$ these functions satisfy the following conditions

(i) For every $x \in \mathbb{R}_+^m$ the functions $g_i(x,u), f_i(x,u)$ are continuous in $u \in [0,T]$;
(ii) For every $u \in [0,T]$ the functions $g_i(x,u), f_i(x,u)$ are Lipschitz continuous in $x \in \mathbb{R}_+^m$ with a Lipschitz constant not depending on u;
(iii) There is some $p > 1$ such that for every $(x,u) \in \mathbb{R}_+^m$,

$$E_{[x,u]}[\big(\sup_{0 \le s \le t} g_i(\bar{S}_s^{[x,u]})\big)^p] < \infty. \tag{2.17}$$

For every $i = 1, ..., l$ set

$$X_i^{[x,u]}(t) = g_i(\bar{S}_t^{[x,u]}) \quad \text{and} \quad Y_i^{[x,u]}(t) = f_i(\bar{S}_t^{[x,u]}).$$

For $i = 1$ and $0 \le s \le t \le T$ we define

$$V_{s,t}^{(1)}(x,u) = esssup_{\tau \in \mathcal{T}_{s,t}} essinf_{\sigma \in \mathcal{T}_{s,t}} E_{[x,u]}[R^{(1)}(\sigma,\tau)|\mathcal{F}_s].$$

and for $1 < i \le l$ and $0 \le s \le t \le T$,

$$X_t^{(i),[x,u]} = e^{-rt} X_{l-i+1}^{[x,u]}(t) + E_{[x,u]}[V_{(t+\delta)\wedge(T-u),T-u}^{(i-1)}(x,u)|\mathcal{F}_t] \tag{2.18}$$

$$Y_t^{(i),[x,u]} = e^{-rt} Y_{l-i+1}^{[x,u]}(t) + E_{[x,u]}[V_{(t+\delta)\wedge(T-u),T-u}^{(i-1)}(x,u)|\mathcal{F}_t]$$

$$V_{s,t}^{(i)}(x,u) = esssup_{\tau \in \mathcal{T}_{s,t}} essinf_{\sigma \in \mathcal{T}_{s,t}} E_{[x,u]}[R^{(i)}(\sigma,\tau)|\mathcal{F}_s]. \tag{2.19}$$

Here for every $(x,u) \in \mathbb{R}_+^m \times [0,T]$ the function $R^{(i)}(\sigma,\tau)$ is the same as defined in (2.12) but with respect to the probability measure $\tilde{P}_{[x,u]}$. For each $(x,u) \in \mathbb{R}^m \times [0,T]$ we define $V^*(x,u)$ to be the fair price of the swing game option $\{g_i, f_i\}_{i=1}^l$ at a time $u \in [0,T]$ assuming that the stock price at this time is x.

2.6**. Theorem.** *For every* $1 \le i < l$ *set*

$$\hat{V}^{(i)}(x,u) = V_{0,T-u}^{(i)}(x,u). \tag{2.20}$$

Define

$$g^{(1)}(x,u) = g_l(x,u), \quad f^{(1)}(x,u) = f_l(x,u) \tag{2.21}$$

and for $1 < i \leq l$,

$$g^{(i)}(x,u) = g_{l-i+1}(x,u) + e^{-r(\delta\wedge(T-u))}E_{[x,u]}\big(\hat{V}^{(i-1)}(\bar{S}^{[x,u]}_{\delta\wedge(T-u)})\big) \quad \textit{and} \tag{2.22}$$

$$f^{(i)}(x,u) = f_{l-i+1}(x,u) + e^{-r(\delta\wedge(T-u))}E_{[x,u]}\big(\hat{V}^{(i-1)}(\bar{S}^{[x,u]}_{\delta\wedge(T-u)})\big).$$

Then for every $1 \leq i \leq l$ *the functions* $g^{(i)}(x,u)$ *and* $f^{(i)}(x,u)$ *satisfy the conditions (i) and (ii) above and, furthermore,*

$$X^{(i),[x,u]}_t = e^{-rt}g^{(i)}(\bar{S}^{[x,u]}_t) \quad \textit{and} \quad Y^{(i),[x,u]}_t = e^{-rt}f^{(i)}(\bar{S}^{[x,u]}_t). \tag{2.23}$$

Next, for every $1 \leq i \leq l$ *let*

$$\mathcal{C}^{(i)}_s = \{(x,u) : \hat{V}^{(i)}(x,u) = g^{(i)}(x,u)\} \quad \textit{and} \quad \mathcal{C}^{(i)}_b = \{(x,u) : \hat{V}^{(i)}(x,u) = f^{(i)}(x,u)\}.$$

Then for each $i = 1, ..., l$,

$$\hat{V}^{(i)}(x,u) = E_{[x,u]}\big(R^{(i)}(\rho(\mathcal{C}^{(i)}_s), \rho(\mathcal{C}^{(i)}_b))\big) \tag{2.24}$$

where $\rho(\mathcal{C}) = \inf\{0 \leq t : \bar{S}_t \in \mathcal{C}\}$ *for each subset* $\mathcal{C} \subset \mathbb{R}^m_+ \times [0,T]$. *Furthermore, for every* $i = 1, .., l$ *the function* $v(x,u) = \hat{V}^{(i)}(x,u)$ *solves the parabolic free boundary problem*

$$Av(x,u) + \frac{\partial}{\partial u}v(x,u) = 0 \quad \textit{with} \quad v(x,u)\mathbf{1}_{\mathcal{C}_s\cup\mathcal{C}_b} = g^{(i)}(x,u)\mathbf{1}_{\mathcal{C}_s\setminus\mathcal{C}_b} + f^{(i)}(x,u)\mathbf{1}_{\mathcal{C}_b} \tag{2.25}$$

where $v(x,u) \in C^{2,1}(\mathbb{R}^m_+ \times [0,T])$ *and* A *is given in (2.5).*

3. Properties of the game value process

In this section we provide general results concerning Dynkin games on an arbitrary probability space $(\Omega, \mathcal{F}, P)$ evolving on a time interval $[0,T]$ with a left continuous filtration $\{\mathcal{F}_t\}_{0\leq t}$, $0 \leq t \leq T$ satisfying the usual conditions (see [7]). Let $0 \leq Y_t \leq X_t$ be two processes adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and suppose that they both have a.s. RCLL paths and that $-X_t, Y_t$ can have only positive jumps at point of discontinuity. Assume also that

$$E\big(sup_{0\leq t\leq T}X_t\big) < \infty. \tag{3.1}$$

Let again $\mathcal{T}_{t,T}$ be the collection of all stopping times with values in $[t,T]$ and set

$$V_t = esssup_{\tau\in\mathcal{T}_{t,T}}essinf_{\sigma\in\mathcal{T}_{t,T}}E\big(R(\sigma,\tau)|\mathcal{F}_t\big). \tag{3.2}$$

Then (see [8]),

$$V_t = E\big(R(\sigma_t,\tau_t)|\mathcal{F}_t\big).$$

where

$$\sigma_t = \inf\{s \geq t : X_s = V_s\} \wedge T \quad \text{and} \quad \tau_t = \inf\{s \geq t : Y_s = V_s\}.$$

Moreover, for every $0 \leq t \leq T$ the pair (σ_t, τ_t) is a saddle point, i.e. for every $\sigma, \tau \in \mathcal{T}_{t,T}$,

$$E\big(R(\sigma_t,\tau)|\mathcal{F}_t\big) \leq E\big(R(\sigma_t,\tau_t)|\mathcal{F}_t\big) \leq E\big(R(\sigma,\tau_t)|\mathcal{F}_t\big) \quad \text{a.s.} \tag{3.3}$$

Furthermore, $Y_t \leq V_t \leq X_t$ a.s. and the process V_t has a right continuous modification, so from now on we assume that a.s V_t has right continuous paths. We also assume that $X_s = X_T, Y_s = Y_T$ and $V_s = V_T$ for $s > T$ so, in particular, $V_\rho = V_{\rho\wedge T}$ for every stopping time ρ. Set also $\mathcal{F}_s = \mathcal{F}_T$ if $s > T$.

Let $\{Z_t\}_{0\leq t\leq T}$ be any process adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and let ρ be a stopping time. Define $\rho_n = \sum_{k=0}^n n_k\mathbf{1}_{\{n_{k-1}<\rho\leq n_k\}}$ where $n_k = \frac{kT}{n}$. It is obvious that if Z has a.s. right continuous paths then

$$\lim_{n\to\infty} X_{\rho_n} = X_\rho \quad \text{a.s.} \tag{3.4}$$

In particular, since for every $\mathcal{F}$ integrable random variable X the martingale $M^X_t = E\big(X|\mathcal{F}_t\big)$ can be taken to be a.s RCLL (cádlág) (see [7]) then for every stopping time ρ we obtain

$$\lim_{n\to\infty} M^X_{\rho_n} = M^X_\rho = E\big(X|\mathcal{F}_\rho\big) \quad \text{a.s.} \tag{3.5}$$

From the definition of σ_t and τ_t it easy to see that

$$\sigma_t \mathbf{1}_{\{\sigma_s \geq t\}} = \sigma_s \mathbf{1}_{\{\sigma_s \geq t\}} \quad \text{for} \quad s \leq t, \tag{3.6}$$

and so on the set $\{\sigma_\rho \wedge \tau_\rho > \rho_n\}$ we have $\sigma_{\rho_n} = \sigma_\rho$ and $\tau_{\rho_n} = \tau_\rho$. Since $E\big(X|\mathcal{F}_\sigma\big)\mathbf{1}_{\{\tau\leq\sigma\}} = E\big(X|\mathcal{F}_{\sigma\wedge\tau}\big)\mathbf{1}_{\{\tau\leq\sigma\}}$ for any two stopping times σ, τ it follows that $V_{\rho_n} = E\big(R(\sigma_{\rho_n}, \tau_{\rho_n})|\mathcal{F}_{\rho_n}\big)$ for any stopping time ρ_n which takes on only a finite number of values. We conclude that on the set $\{\sigma_\rho \wedge \tau_\rho > \rho_n\}$,

$$V_{\rho_n} = E\big(R(\sigma_{\rho_n}, \tau_{\rho_n})|\mathcal{F}_{\rho_n}\big) = E\big(R(\sigma_\rho, \tau_\rho)|\mathcal{F}_{\rho_n}\big) = M^{R_\rho}_{\rho_n}$$

where $R^\rho = R(\sigma_\rho, \tau_\rho)$. Since $\{\sigma_\rho \wedge \tau_\rho > \rho\} = \cup_{0<n}\{\sigma_\rho \wedge \tau_\rho > \rho_n\}$ and V_t is right continuous it follows that

$$\mathbf{1}_{\{\sigma_\rho\wedge\tau_\rho>\rho\}}V_\rho = \lim_{n\to\infty}\mathbf{1}_{\{\sigma_\rho\wedge\tau_\rho>\rho_n\}}V_{\rho_n} = \lim_{n\to\infty}\mathbf{1}_{\{\sigma_\rho\wedge\tau_\rho>\rho_n\}}M^{R_\rho}_{\rho_n} = \mathbf{1}_{\{\sigma_\rho\wedge\tau_\rho>\rho\}}E\big(R(\sigma_\rho,\tau_\rho)|\mathcal{F}_\rho\big).$$

On the other hand,

$$\mathbf{1}_{\{\sigma_\rho\wedge\tau_\rho=\rho\}}V_\rho = \mathbf{1}_{\{\sigma_\rho\wedge\tau_\rho=\rho\}}R(\sigma_\rho,\tau_\rho) = \mathbf{1}_{\{\sigma_\rho\wedge\tau_\rho=\rho\}}E\big(R(\sigma_\rho,\tau_\rho)|\mathcal{F}_\rho\big).$$

This gives

3.1. **Lemma.** *For every stopping time ρ,*

$$V_\rho = E\big(R(\sigma_\rho,\tau_\rho)|\mathcal{F}_\rho\big) \quad a.s. \tag{3.7}$$

Next, let $\rho \leq \tau$ be any stopping times. Then

$$\mathbf{1}_{\{\sigma_\rho\wedge\tau>\rho\}}V_\rho = \lim_{n\to\infty}\mathbf{1}_{\{\sigma_\rho\wedge\tau>\rho_n\}}V_{\rho_n} = \lim_{n\to\infty}\mathbf{1}_{\{\sigma_\rho\wedge\tau>\rho_n\}}E\big(R(\sigma_{\rho_n},\tau_{\rho_n})|\mathcal{F}_{\rho_n}\big)$$

$$\geq \lim_{n\to\infty}\mathbf{1}_{\{\sigma_\rho\wedge\tau>\rho_n\}}E\big(R(\sigma_{\rho_n},\tau)|\mathcal{F}_{\rho_n}\big) = \lim_{n\to\infty}\mathbf{1}_{\{\sigma_\rho\wedge\tau>\rho_n\}}E\big(R(\sigma_\rho,\tau)|\mathcal{F}_{\rho_n}\big) = \mathbf{1}_{\{\sigma_\rho\wedge\tau>\rho\}}E\big(R(\sigma_\rho,\tau)|\mathcal{F}_\rho\big).$$

The inequality here follows from the saddle point property of the pair $<\sigma_{\rho_n}, \tau_{\rho_n}>$ which can be used since $\tau > \rho_n$. Next two equalities follow from (3.6) and (3.5), respectively.

On the other hand, on the set $\{\sigma_\rho = \rho\}$ we have

$$V_\rho = X_\rho \geq E\big(R(\sigma_\rho,\tau)|\mathcal{F}_\rho\big).$$

and on the set $\{\tau = \rho\}$,

$$V_\rho \geq Y_\rho = E\big(R(\sigma_\rho,\tau)|\mathcal{F}_\rho\big)$$

By a similar argument for $\rho \leq \sigma$ we obtain the following.

3.2. **Lemma.** *Let $\rho \leq \sigma, \tau \leq T$ be any stopping times. Then*

$$E\big(R(\sigma_\rho,\tau)|\mathcal{F}_\rho\big) \leq V_\rho = E\big(R(\sigma_\rho,\tau_\rho)|\mathcal{F}_\rho] \leq E\big(R(\sigma,\tau_\rho)|\mathcal{F}_\rho\big) \quad a.s., \tag{3.8}$$

and so

$$V_\rho = esssup_{\tau\in\mathcal{T}_{\rho,T}}essinf_{\sigma\in\mathcal{T}_{\rho,T}}E\big(R(\sigma,\tau)|\mathcal{F}_\rho\big) = E\big(R(\sigma_\rho,\tau_\rho)|\mathcal{F}_\rho\big) \quad a.s..$$

Using the saddle point property we can deduce

3.3. **Lemma.** *For any $\rho \in \mathcal{T}_{0,T}$ the process $V_{\sigma_\rho\wedge(\rho+t)}, V_{\tau_\rho\wedge(\rho+t)}$ and $V_{\sigma_\rho\wedge(\rho+t)}$ is a super martingale, sub martingale and martingale, respectively, with respect to the filtration $\{\mathcal{F}_{\rho+t}\}_{0\leq t\leq T}$.*

Proof. For $s \leq t$,

$$E\big(V_{\sigma_\rho\wedge(\rho+t)}|\mathcal{F}_{\rho+s}\big) = E\big(V_{\sigma_\rho}\mathbf{1}_{\{\sigma_\rho<\rho+t\}}\big) + E\big(R(\sigma_{\rho+t},\tau_{\rho+t})|\mathcal{F}_{\rho+t}]\mathbf{1}_{\{\sigma_\rho\geq\rho+t\}}|\mathcal{F}_{\rho+s}\big)$$

$$= E\big(E\big(X_{\sigma_\rho}\mathbf{1}_{\{\sigma_\rho<\rho+t\}} + X_{\sigma_{\rho+t}}\mathbf{1}_{\{\sigma_{\rho+t}<\tau_{\rho+t}\}\cap\{\sigma_\rho\geq\rho+t\}} + Y_{\tau_{\rho+t}}\mathbf{1}_{\{\sigma_{\rho+t}\geq\tau_{\rho+t}\}\cap\{\sigma_\rho\geq\rho+t\}}|\mathcal{F}_{\rho+t}\big)|\mathcal{F}_{\rho+s}\big)$$

$$= E\big(R(\sigma_\rho,\tau_{\rho+t})|\mathcal{F}_{\rho+s}\big)$$

where the first equality follows by Lemma 3.1, the second holds true since all functions are measurable with respect to $\mathcal{F}_{\rho+t}$ and the last one follows from

$$\{\sigma_{\rho+t} < \tau_{\rho+t}\} \cap \{\sigma_\rho \geq \rho+t\} \cup \{\sigma_\rho < \rho+t\} = \{\sigma_\rho < \tau_{\rho+t}\}$$

and

$$\{\sigma_{\rho+t} \geq \tau_{\rho+t}\} \cap \{\sigma_\rho \geq \rho + t\} = \{\sigma_\rho \geq \tau_{\rho+t}\}.$$

Next,

$$E\big(R(\sigma_\rho, \tau_{\rho+t})|\mathcal{F}_{\rho+s}\big) = E\big(R(\sigma_\rho, \tau_{\rho+t})|\mathcal{F}_{\rho+s}\big)\mathbf{1}_{\{\sigma_\rho < \rho+s\}} + E\big(R(\sigma_\rho, \tau_{\rho+t})|\mathcal{F}_{\rho+s}\big)\mathbf{1}_{\{\sigma_\rho \geq \rho+s\}}$$
$$= V_{\sigma_\rho}\mathbf{1}_{\{\sigma_\rho < \rho+s\}} + E\big(R(\sigma_{\rho+s}, \tau_{\rho+t})|\mathcal{F}_{\rho+s}\big)\mathbf{1}_{\{\sigma_\rho \geq \rho+s\}} \leq V_{\sigma_\rho}\mathbf{1}_{\{\sigma_\rho < \rho+s\}}$$
$$+E\big(R(\sigma_{\rho+s}, \tau_{\rho+s})|\mathcal{F}_{\rho+s}\big)\mathbf{1}_{\{\sigma_\rho \geq \rho+s\}} = V_{\sigma_\rho}\mathbf{1}_{\{\sigma_\rho < \rho+s\}} + V_{\rho+s}\mathbf{1}_{\{\sigma_\rho \geq \rho+s\}} = V_{\sigma_\rho \wedge \rho+s}.$$

Here the second equality follows from the fact that $\tau_{\rho+t} \geq (\rho+s)$ and the inequality holds true by Lemma 3.2. This gives us the first statement of the lemma while the second statement can be proved in a similar way and the third is a consequence of the first two. □

Next, we show that the process V_t has a.s. RCLL paths. Let Θ be the set of all the rational vectors $\theta = (\alpha, \beta, \gamma, \delta)$ such that $0 \leq \alpha < \beta < T$ and $0 \leq \gamma < \delta$. For each $\theta \in \Theta$ let $\{\nu^n, \nu_n\}_{0=n}^{\infty}$ be the following stopping times

$$\nu_0 = \nu^1 = \alpha,\ \nu_n = \inf\{t \geq \nu^{n-1} : V_t \leq \gamma\} \wedge \beta \quad \text{and} \quad \nu^n = \inf\{t \geq \nu_n : V_t \geq \delta\} \wedge \beta$$

where by definition $\inf \phi = \infty$. Since $\nu_n \leq \nu^n \leq \nu_{n+1} \leq \beta$ these sequences have a limit $\nu = \nu(\theta) = \nu(\theta, \omega) \leq \beta$ and we define $A(\theta) = \{\omega : \nu(\theta, \omega) < \beta\}$.

3.4. **Lemma.** *For every $\theta \in \Theta$, $P(A(\theta)) = 0$.*

Proof. Let $\theta = (\alpha, \beta, \gamma, \delta)$ and assume that $P(A(\theta)) > 0$. Since X_t is a.s RCLL and $X_t \geq V_t$ for every $\omega \in A(\theta)$ we have $\lim_{n\to\infty} X_{\nu_n}(\omega) \geq \limsup_{n\to\infty} V_{\nu^n}(\omega) \geq \delta$, and so there is a subset $B \subset A(\theta)$ which belong to $\mathcal{F}_\nu$ and a positive integer n_0 such that $P(B) > 0$ and for every $\omega \in B$,

$$X_t(\omega) > \frac{\delta + \gamma}{2} \quad \text{whenever} \quad \nu_{n_0}(\omega) \leq t < \nu(\omega).$$

Define the following sequence of stopping times

$$\mu^n = \inf\{t \geq \nu_n : V_t \geq \frac{\delta + \gamma}{2}\}$$

so that $\sigma_{\nu_n}(\omega) \geq \mu^n(\omega)$ for every $n \geq n_0$ and $\omega \in B$. Let $\bar{X} = \sup_{0\leq t\leq T} X_t$. Since $\bar{X}$ is integrable by assumption it follows that for every $\xi > 0$ there is $x(\xi) > 0$ such that $E\big(\bar{X}\mathbf{1}_C\big) \leq \xi$ for every $C \in \mathcal{F}$ satisfying $P(C) \leq x(\xi)$. Let $x_0 = x(\xi_0)$ where $\xi_0 = \frac{\delta-\gamma}{16}P(B)$. Since the filtration is left continuous and $\lim \nu_n = \nu$ it follows that $\mathcal{F}_\nu = \sigma(\cup_{n_0\leq n}\mathcal{F}_{\nu_n})$. Since $\cup_{n_0\leq n}\mathcal{F}_{\nu_n}$ is an algebra and $B \in \mathcal{F}_\nu$ there exist $n_1 > 0$ and a subset $C \in \mathcal{F}_{\nu_n}$, $\forall n \geq n_1$ satisfying $P(B \triangle C) < x_0$. If $n \geq n_0 \vee n_1$ then

$$\tfrac{\delta-\gamma}{2}\mathbf{1}_B \leq (V_{\mu^n} - V_{\nu_n})\mathbf{1}_B = (V_{\mu^n} - V_{\nu_n})\mathbf{1}_C + (V_{\mu^n} - V_{\nu_n})\mathbf{1}_{B\triangle C} \leq (V_{\mu^n} - V_{\nu_n})\mathbf{1}_C + 2\bar{X}\mathbf{1}_{B\triangle C}$$
$$= \big(E\big(R(\sigma_{\mu^n}, \tau_{\mu^n})|\mathcal{F}_{\mu^n}\big) - E\big(R(\sigma_{\nu_n}, \tau_{\nu_n})|\mathcal{F}_{\nu_n}\big)\big)\mathbf{1}_C + 2\bar{X}\mathbf{1}_{B\triangle C} = E\big(R(\sigma_{\mu^n}, \tau_{\mu^n})\mathbf{1}_C|\mathcal{F}_{\mu^n}\big)$$
$$-E\big(R(\sigma_{\nu_n}, \tau_{\mu^n})\mathbf{1}_C|\mathcal{F}_{\nu_n}\big) + 2\bar{X}\mathbf{1}_{B\triangle C} \leq E\big(R(\sigma_{\nu_n}, \tau_{\mu^n})\mathbf{1}_B|\mathcal{F}_{\mu^n}\big)$$
$$-E\big(R(\sigma_{\nu_n}, \tau_{\mu^n})\mathbf{1}_B|\mathcal{F}_{\nu_n}\big)\big) + 2\bar{X}\mathbf{1}_{B\triangle C} + E\big(\bar{X}\mathbf{1}_{B\triangle C}|\mathcal{F}_{\mu^n}\big) + E\big(\bar{X}\mathbf{1}_{B\triangle C}|\mathcal{F}_{\nu_n}\big).$$

Here the first inequality follows from the definition of the stopping times and the first equality follows from Lemma 3.3. The next two inequalities are obtained from Lemma 3.7 and from the fact that $X_t \geq \frac{\delta+\gamma}{2}$ for any $n \geq n_0$ and $\nu_n \leq t < \nu$ on the set B, and so $\sigma_{\nu_n} \geq \mu^n$, which leads to $\sigma_{\nu_n} = \sigma_{\mu^n}$ (cf. 3.6). Now, taking the expectation on both sides above we see that the terms $E\big(R(\sigma_{\nu_n}, \tau_{\mu^n}\mathbf{1}_B|\mathcal{F}_{\mu^n}\big)$ and $E\big(R(\sigma_{\nu_n}, \tau_{\mu^n}\mathbf{1}_B|\mathcal{F}_{\nu_n}\big)$ cancel each other which leaves us with

$$\frac{\delta - \gamma}{2}P(B) \leq E\bigg(2\bar{X}\mathbf{1}_{B\triangle C} + E\big(\bar{X}\mathbf{1}_{B\triangle C}|\mathcal{F}_{\mu^n}\big) + E\big(\bar{X}\mathbf{1}_{B\triangle C}|\mathcal{F}_{\nu_n}\big)\bigg).$$

Since $P(B\triangle C)\leq x_0$ by the choice of C then by the choice of x_0 and ξ_0,

$$\frac{\delta-\gamma}{2}P(B)\leq 4E\big(\bar X\mathbf{1}_{B\triangle C}\big)\leq\frac{\delta-\gamma}{4}P(B)$$

which is a contradiction, and so $P(A(\theta))=0$. □

3.5. **Corollary.** *The process V_t has a.s. RCLL paths.*

Proof. If for some ω the path $V_t(\omega)$ has no left limit at some point t_0 then we can find $\theta\in\Theta$ such that $\omega\in A(\theta)$ and so the set of the paths that V_t is not RCLL contained in $\cup_{\theta\in\Theta}A(\theta)$ but since Θ is countable we obtain from Lemma 3.4 that this set has zero probability. □

Recall that a process Z_t is called regular if $\lim_{n\to\infty}EZ_{\rho_n}=EZ_\rho$ for every non decreasing sequence of stopping times $\{\rho_n\}$ with a limit ρ (see [7]). In order to show that for a.s. continuous X_t,Y_t the process V_t is regular we need following.

3.6. **Lemma.** *Assume that the process X_t,Y_t has a.s. continuous path. Then*

$$\lim_{n\to\infty}\sigma_{\rho_n}=\sigma_\rho\quad\textit{and}\quad\lim_{n\to\infty}\tau_{\rho_n}=\tau_\rho\quad\textit{a.s.}$$

Proof. Ignoring a set of zero probability we can assume that for every $\omega\in\Omega$ the paths $X_t(\omega)$ and $Y_t(\omega)$ are continuous, that $V_t(\omega)$ is RCLL and that $Y_t(\omega)\leq V_t(\omega)\leq X_t(\omega)$. Let $A=\{\lim_{n\to\infty}\sigma_{\rho_n}<\sigma_\rho\}$. If $\sigma_{\rho_{n_0}}\geq\rho$ for some n_0 then $\sigma_{\rho_n}=\sigma_\rho$ for every $n\geq n_0$, and so $\rho_n\leq\sigma_{\rho_n}<\rho$ for any $\omega\in A$ and each n and we have also that $\sigma_\rho>\rho$. From these facts it follows that $V_\rho<X_\rho=\lim_{n\to\infty}X_{\sigma_{\rho_n}}=\lim_{n\to\infty}V_{\sigma_{\rho_n}}$ on A. Since $V_\rho<X_\rho$ for every $\omega\in A$ we deduce that there are $\delta>0$, a positive integer n_0 and a subset $B\in\mathcal{F}_\rho$ such that $B\subset A$, $P(B)>0$ and on B and that $V_{\sigma_{\rho_n}}-V_\rho\geq\delta$ for every $n\geq n_0$. We also claim that for every $\omega\in B$ there is $n(\omega)$ such that $\tau_{\rho_n}(\omega)\geq\rho(\omega)$ for every $n\geq n(\omega)$. Indeed, if $\tau_{\rho_n}(\omega)<\rho(\omega)$ for every n then $\lim_n\tau_{\rho_n}=\rho$ and since V_t has RCLL paths and $\sigma_{\rho_n}<\rho$ we obtain that

$$V_\rho\geq Y_\rho=\lim_{n\to\infty}Y_{\tau_{\rho_n}}=\lim_{n\to\infty}V_{\tau_{\rho_n}}=\lim_{n\to\infty}V_{\sigma_{\rho_n}}=X_\rho>V_\rho$$

which is a contradiction. So we can find a subset $C\in\mathcal{F}_\rho$ and a positive integer n_1 such that $C\subset B$, $P(C)>0$ and $\tau_{\rho_n}\geq\rho$ on C for every $n\geq n_1$. Let $\zeta_0>0$ be such that $E\big(\sup_{0\leq t\leq T}X_t\mathbf{1}_E\big)\leq\frac{\delta}{2}P(C)$ for every $E\in\mathcal{F}$ provided $P(E)<\zeta_0$. Since the filtration is left continuous we can find a subset $D\in\cup_{n>0}\mathcal{F}_{\rho_n}$ such that $P(D\triangle C)<\zeta_0$. Choose n_2 such that $D\in\mathcal{F}_{\rho_{n_2}}$, and so $D\in\mathcal{F}_{\rho_n}$ for every $n\geq n_2$. Let $n\geq n_0\vee n_1\vee n_2$ then

$$\delta\mathbf{1}_C\leq(V_{\rho_n}-V_\rho)\mathbf{1}_C=(V_{\tau_{\rho_n}\wedge\rho_n}-V_{\tau_{\rho_n}\wedge\rho})\mathbf{1}_C\leq(V_{\tau_{\rho_n}\wedge\rho_n}-V_{\tau_{\rho_n}\wedge\rho})\mathbf{1}_D+\sup_{0\leq t\leq T}X_t\mathbf{1}_{D\triangle C}$$

where the first inequality follows from the fact that $C\subset B$ and the equality follows from the definition of the subset C. Taking the conditional expectation with respect to $\mathcal{F}_{\rho_n}$ in the above inequality, relying on the fact that by Lemma 3.3) the process $V_{\tau_{\rho_n}\wedge(\rho_n+t)}$ is a submartingale and since $D\in\mathcal{F}_{\rho_n}$ if $n\geq n_2$ we obtain that

$$\delta E\big(\mathbf{1}_C|\mathcal{F}_{\rho_n}\big)\leq E\big(\sup_{0\leq t\leq T}X_t\mathbf{1}_{D\triangle C}|\mathcal{F}_{\rho_n}\big).$$

Taking the expectation in this inequality we arrive at

$$\delta P(C)\leq E\big(\sup_{0\leq t\leq T}X_t\mathbf{1}_{C\triangle D}\big)\leq\frac{\delta}{2}P(C)$$

which is a contradiction, and so we have $\lim_{n\to\infty}\sigma_{\rho_n}=\sigma_\rho$ a.s. The claim that $\lim_{n\to\infty}\tau_{\rho_n}=\tau_\rho$ a.s. can be proved similarly by using the supermartingale property of $V_{\sigma_{\rho_n}\wedge(\rho_n+t)}$. □

3.7. **Corollary.** *If a.s. X_t and Y_t have continuous paths then the process V_t is regular.*

Proof. Let $\{\rho_n\}_{n\geq 0}$ be a nondecreasing sequence of stopping times and set $\rho = \lim_{n\to\infty} \rho_n$. From Lemma 3.6, the continuity of the processes X_t, Y_t and the equality $X_{\sigma_\rho} = Y_{\tau_\rho}$ on the set $\sigma_\rho = \tau_\rho < T$ it follows that

$$\lim_{n\to\infty} R(\sigma_{\rho_n}, \tau_{\rho_n}) = R(\sigma_\rho, \tau_\rho) \quad \text{a.s.}$$

Using that $R(\sigma_{\rho_n}, \tau_{\rho_n}) \leq \sup_{0\leq t\leq T} X_t$, the assumption that the latter is integrable and that $V_{\rho_n} = E\big(R(\sigma_{\rho_n}, \tau_{\rho_n})|\mathcal{F}_{\rho_n}\big)$ we obtain from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} E\big(V_{\rho_n}\big) = E\big(V_\rho\big).$$

□

We can use this result to obtain the following

3.8. **Lemma.** *For $0 \leq t < T$ let $V_{\sigma_t\wedge s} = M_s^1 - A_s^1$ and $V_{\tau_t\wedge s} = M_s^2 + A_s^2$ be the Doob–Meyer decompositions of the right continuous supermartingale and the submartingale with respect to the filtration $\{\mathcal{F}_s\}_{t\leq s\leq T}$ where M^i are right continuous martingales and A^i, $i = 1, 2$ are natural increasing processes. Then the processes A^i are a.s. continuous. If the filtration $\mathcal{F}_t$ has the property that every martingale with respect to it has a continuous modification then the supermartingale $V_{\sigma_t\wedge s}$ and the submartingale $V_{\tau_t\wedge s}$ have continuous modifications too.*

Proof. Since V_t is a regular process (see Corollary 3.7) we obtain that so are $V_{\sigma_t\wedge s}$ and $V_{\tau_t\wedge s}$ and the continuity of the processes A^i, $i = 1, 2$ follows from Theorem 4.14, Section 1.4 of [7]. The fact that these super and submartingales have continuous modifications when every martingale with respect to this filtration has a continuous modification is now clear. □

3.9. **Proposition.** *Let $X_t \geq Y_t$ be a.s continuous processes satisfying (3.1) which are adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and assume that this filtration has the property that every martingale with respect to it has an a.s. continuous modification. Then the Dynkin game value process V_t (see (3.2)) has also an a.s. continuous modification.*

Proof. Let V_t be a right continuous modification of the game value process. We saw that this process has a.s RCLL paths and we want to show that in fact it is a.s continuous. Let Ω^* be the subset of all $\omega \in \Omega$ such that V_t is RCLL and that $X_t, Y_t, V_{\sigma_q\wedge s}$ and $V_{\tau_q\wedge s}$ are continuous for any $0 \leq t \leq T$ and each rational $q \leq s \leq T$. From the above it follows that $P(\Omega^*) = 1$. We claim that for every $\omega \in \Omega^*$ the path $V_t(\omega)$ is continuous. Assume that $V_t(\omega)$ is not continuous at some point t_0 then it must have a jump at this point. Let q_i be an increasing sequence with $\lim_{i\to\infty} q_i = t_0$ and assume that $\lim_{i\to\infty} V_{q_i} \neq V_{t_0}$. If $\sigma_{q_i} < t_0$ and $\tau_{q_i} < t_0$ for every i then

$$X_{t_0} = \lim_{i\to\infty} X_{\sigma_{q_i}} = \lim_{i\to\infty} V_{\sigma_{q_i}} = \lim_{i\to\infty} V_{\tau_{q_i}} = Y_{t_0},$$

and so $\lim_{i\to\infty} V_{q_i} = V_{t_0}$ which we assumed not to be the case. So without loss of generality assume that $\sigma_{q_i}(\omega) \geq t_0$ for some i. Since q_i is rational and $\omega \in \Omega^*$ the path $V_{\sigma_{q_i}(\omega)\wedge t}(\omega)$ is continuous for every $q_i \leq t$. In particular, it is continuous for $q_i < t_0$, and so

$$\lim_{t\to t_0^-} V_t(\omega) = \lim_{t\to t_0^-} V_{\sigma_{q_i}(\omega)\wedge t}(\omega) = V_{\sigma_{q_i}(\omega)\wedge t_0}(\omega) = V_{t_0}(\omega)$$

yielding the continuity of $V_{t_0}(\omega)$. □

4. Multi stopping Dynkin games

Our setup for a multi stopping Dynkin game consists of a sequence of payoff processes $\{X_i, Y_i\}_{i=1}^l$ and of a delay time $\delta > 0$ between successive stoppings so that the first player is supposed to pay to the second one the amount $X_i(t)$ or $Y_i(t)$ if the i-th stopping occurs at time t and it is done by the first or by the second (or by both) player, respectively. We assume that $X_i \geq Y_i$ and these processes are supposed to be nonnegative, continuous, adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and for some $p>1$,

$$E\big(\sup_{0\leq t\leq T} X_i(t)\big)^p < \infty \tag{4.1}$$

whenever $1\leq i\leq l$.

Consider the following processes defined inductively by

$$X_t^{(1)} = X_l(t),\ Y_t^{(1)} = Y_l(t) \quad \text{and} \quad V_t^{(1)} = esssup_{\tau\in\mathcal{T}_t} essinf_{\sigma\in\mathcal{S}_t} E\big(R^{(1)}(\sigma,\tau)|\mathcal{F}_t\big)$$

where

$$R^{(1)}(\sigma,\tau) = X_\sigma^{(1)}\mathbf{1}_{\sigma<\tau} + Y_\tau^{(1)}\mathbf{1}_{\sigma\geq\tau},$$

and for $2\leq i\leq l$,

$$X_t^{(i)} = X_{l-i+1}(t) + E\big(V_{t+\delta}^{i-1}|\mathcal{F}_t\big),\ Y_t^{(i)} = Y_{l-i+1}(t) + E\big(V_{t+\delta}^{i-1}|\mathcal{F}_t\big),$$

$$\text{and} \quad V_t^{(i)} = esssup_{\tau\in\mathcal{T}_{t,T}} essinf_{\sigma\in\mathcal{T}_{t,T}} E\big(R^{(i)}(\sigma,\tau)|\mathcal{F}_t\big)$$

where

$$R^{(i)}(\sigma,\tau) = X_\sigma^{(i)}\mathbf{1}_{\{\sigma<\tau\}} + Y_\tau^{(i)}\mathbf{1}_{\{\sigma\geq\tau\}}.$$

4.1**. Lemma.** *For each $i=1,...,l$ the processes $X^{(i)},Y^{(i)}$ have a.s. continuous paths and they satisfy the condition (4.1) and $X^{(i)}\geq Y^{(i)}\geq 0$.*

Proof. The fact that $X^{(i)}\geq Y^{(i)}\geq 0$ is obvious from the definition. The proof that they satisfy condition (4.1) is exactly as in Lemma 1 from [1]. It remains to show that $X^{(i)}$ and $Y^{(i)}$ have a.s. continuous paths. Since X_i and Y_i have a.s. continuous paths and $V_t^{(1)}$ is a.s. continuous in t by Proposition 3.9 it suffices to show that if V_t is a continuous process satisfying

$$E\sup_{0\leq t\leq T}|V_t|^p < \infty,\ \ p>1 \tag{4.2}$$

then the process $W_\delta(t) = E\big(V_{t+\delta}|\mathcal{F}_t\big)$ has a continuous modification. From (4.2) and the continuity of V_t and of $\mathcal{F}_t$ in t it is not difficult to see that for each given $t\in[0,T]$ the process $W_\delta(t)$ is a.s. continuous at t. Thus it suffices to show that there is a set $\Omega^*\subset\Omega$ such that $P(\Omega^*)=1$ and for each $\omega\in\Omega^*$ and any Cauchy sequence $\{q_n\}$ of rational numbers in the interval $[0,T]$ the sequence $W_\delta(t,\omega)$ is also a Cauchy sequence. Indeed, in this case for each $t\in[0,T]$ we can define the value of $W_\delta(t,\omega)$ to be the limit of $W_\delta(q_i,\omega)$ as $i\to\infty$ where $\{q_i\}_{i=0}^\infty$ is any rational sequence with $\lim_{i\to\infty} q_i = t$. Then on the set Ω^* the process will have continuous paths and from the a.s. continuity of $W_\delta(t)$ at each fixed t this yields a modification. In order to find such set Ω^* we proceed as follows. Set $\varepsilon_n = \sup_{\{0\leq s,t\leq T:|s-t|\leq 1/n\}}|V_s - V_t|$ and $\mathcal{E}_{n,t} = E\big(\varepsilon_n|\mathcal{F}_t\big)$. Using Doob's maximal inequality (see, for instance, [5]) we obtain

$$E\big(\sup_{0\leq t\leq T}\mathcal{E}_{n,t}^p\big) \leq (\frac{p}{1-p})^p E\big(\varepsilon_n^p\big).$$

Since the paths of V_t are a.s. continuous then $\lim_{n\to\infty}\varepsilon_n = 0$ a.s. By (4.2) it follows from the Lebesgue dominated convergence theorem that

$$E\big(\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathcal{E}_{n,t}^p\big) = \lim_{n\to\infty} E\big(\sup_{0\leq t\leq T}\mathcal{E}_{n,t}^p\big) \leq \lim_{n\to\infty}(\frac{p}{1-p})^p E\big(\varepsilon_n^p\big) = 0$$

Hence, $\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathcal{E}_{n,t} = 0$ a.s. It follows that there exists a measurable subset $\Omega^*\subset\Omega$ such that $P(\Omega^*)=1$ and for every $\omega\in\Omega^*$,

(1) $\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathcal{E}_{n,t}(\omega) = 0$;

(2) For any rational numbers q, p, r in $[0, T]$ and each n such that $|p - q| \leq 1/n$,

$$|E\big(V_{p+\delta}|\mathcal{F}_r\big)(\omega) - E\big(V_{q+\delta}|\mathcal{F}_r\big)(\omega)| = |E\big(V_{p+\delta} - V_{q+\delta}|\mathcal{F}_r\big)(\omega)|$$

$$\leq E\big(|V_{p+\delta} - V_{q+\delta}||\mathcal{F}_r\big)(\omega) \leq E\big(\varepsilon_n|\mathcal{F}_r\big)(\omega) \leq \sup_{0\leq t\leq T} E\big(\varepsilon_n|\mathcal{F}_t\big)(\omega) = \sup_{0\leq t\leq T} \mathcal{E}_{n,t}(\omega);$$

(3) For every $n \leq m$,

$$\sup_{0\leq t\leq T} \mathcal{E}_{m,t}(\omega) \leq \sup_{0\leq t\leq T} \mathcal{E}_{n,t}(\omega);$$

(4) For every rational $q \in [0, T]$ the path $E\big(V_{q+\delta}|\mathcal{F}_t\big)(\omega)$ is continuous in t;

Let $\{q_m\}$ be a Cauchy sequence of rational numbers from $[0, T]$ and let $\epsilon > 0$. By (1) for every $\omega \in \Omega^*$ there exists $n_0 = n_0(\omega)$ such that

$$\sup_{0\leq t\leq T} \mathcal{E}_{n,t}(\omega) \leq \epsilon/3$$

provided $n \geq n_0$. Let m_0 be such that $|q_k - q_l| \leq 1/n_0$ for every $k, l \geq m_0$. By (3) there is $n_1 = n_1(\omega)$ such that

$$\big|E\big(V_{q_{m_0}+\delta}|\mathcal{F}_s\big)(\omega) - E\big(V_{q_{m_0}+\delta}|\mathcal{F}_t\big)(\omega)\big| \leq \epsilon/3$$

provided $0 \leq s \leq t \leq T$ and $|s-t| \leq 1/n_1$. Let $m_1 \geq m_0$ be such that $|q_k - q_l| \leq 1/n_1$ for every $k, l > m_1$. If $k, l > m_1$ then from the properties above it is easy to see that

$$\big|W_\delta(q_l) - W_\delta(q_k)\big| \leq \big|W_\delta(q_l) - E\big(V_{q_{m_0}+\delta}|\mathcal{F}_{q_l}\big)\big|$$
$$+\big|E\big(V_{q_{m_0}+\delta}|\mathcal{F}_{q_l}\big) - E\big(V_{q_{m_0}+\delta}|\mathcal{F}_{q_k}\big)\big| + \big|E\big(V_{q_{m_0}+\delta}|\mathcal{F}_{q_k}\big) - W_\delta(q_k)\big|$$
$$\leq 2\sup\nolimits_{0\leq t\leq T} \mathcal{E}_{n_0,t} + \big|E\big(V_{q_{m_0}+\delta}|\mathcal{F}_{q_l}\big) - E\big(V_{q_{m_0}+\delta}|\mathcal{F}_{q_k}\big)\big| \leq \epsilon,$$

and so the the sequence $E\big(V_{q_n+\delta}|\mathcal{F}_{q_n}\big)$ is also a Cauchy sequence and the lemma is proved. □

Next we discuss the game value of a multi stopping Dynkin game. In order to simplify the notations we consider nondiscounted payoffs which always can be achieved by appropriate normalization. Let $\mathbf{s}, \mathbf{t} \in \mathcal{S}_T$ and set

$$H(\mathbf{s}, \mathbf{t}) = E\big(\sum_{i=1}^{l} R_i(\sigma_i, \tau_i)\big)$$

where

$$R_i(\sigma, \tau) = X_i(\sigma)\mathbf{1}_{\{\sigma<\tau\}} + Y_i(\tau)\mathbf{1}_{\{\sigma\geq\tau\}}$$

and $((\sigma_1, ..., \sigma_l)(\tau_1, ..., \tau_l)) = F(\mathbf{s}, \mathbf{t})$ (see Definition 2.1 and below it). As in Theorem 2.5 we define two special stopping time strategies $\mathbf{s}^*, \mathbf{t}^*$ by

$$\mathbf{s}^*(1, 0) = \inf\{0 \leq t : V_t^{(l)} = X_t^{(l)}\} \wedge T,\ \mathbf{t}^*(1, 0) = \inf\{0 \leq t : V_t^{(l)} = Y_t^{(l)}\} \tag{4.3}$$

and for $1 < i \leq l$,

$$\mathbf{s}^*(i, \sigma) = \inf\{\sigma + \delta \leq t : V_t^{(l-i+1)} = X_t^{(l-i+1)}\} \wedge T \quad \text{and}$$
$$\mathbf{t}^*(i, \sigma) = \inf\{\sigma + \delta \leq t : V_t^{(l-i+1)} = Y_t^{(l-i+1)}\}.$$

The main result of this section is the following.

4.2. **Proposition.** *For every* $\mathbf{s}, \mathbf{t} \in \mathcal{S}_T$*,*

$$H(\mathbf{s}^*, \mathbf{t}) \leq H(\mathbf{s}^*, \mathbf{t}^*) \leq H(\mathbf{s}, \mathbf{t}^*), \tag{4.4}$$

i.e. the multi stopping Dynkin game possesses a saddle point $< \mathbf{s}^*, \mathbf{t}^* >$*, and so it has a value which is equal to* $H(\mathbf{s}^*, \mathbf{t}^*)$*.*

In order to prove Proposition 4.2 we need the following key lemma.

4.3. **Lemma.** *Let $\mathbf{s},\mathbf{t}\in\mathcal{S}_T$ and set*

$$F(\mathbf{s}^*,\mathbf{t})=((\sigma_1^*,..,\sigma_l^*),(\tau_1,...,\tau_l)),\quad F(\mathbf{s},\mathbf{t}^*)=((\sigma_1,..,\sigma_l),(\tau_1^*,...,\tau_l^*)).$$

Then

$$E\big(R^{(i-1)}(\sigma^*_{l-i+2},\tau_{l-i+2})+R_{l-i+1}(\sigma^*_{l-i+1},\tau_{l-i+1})\big)\le E\big(R^{(i)}(\sigma^*_{l-i+1},\tau_{l-i+1})\big) \tag{4.5}$$

and

$$E\big(R^{(i-1)}(\sigma_{l-i+2},\tau^*_{l-i+2})+R_{l-i+1}(\sigma_{l-i+1},\tau^*_{l-i+1})]\ge E\big(R^{(i)}(\sigma_{l-i+1},\tau^*_{l-i+1})\big) \tag{4.6}$$

Proof. We prove only the first inequality since the second one follows in a similar way. From the definitions of $X^{(i)},Y^{(i)}$ and $R^{(i)}$ we obtain

$$\begin{aligned}R^{(i)}(\sigma^*_{l-i+1},\tau_{l-i+1})&=\mathbf{1}_{\{\sigma^*_{l-i+1}<\tau_{l-i+1}\}}X^{(i)}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}}+\mathbf{1}_{\{\sigma^*_{l-i+1}\ge\tau_{l-i+1}\}}Y^{(i)}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}}\\
&=\mathbf{1}_{\{\sigma^*_{l-i+1}<\tau_{l-i+1}\}}\big(X_{l-i+1}(\sigma^*_{l-i+1}\wedge\tau_{l-i+1})+E\big(V^{(i-1)}_{(\sigma^*_{l-i+1}\wedge\tau_{l-i+1})+\delta}|\mathcal{F}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}}\big)\big)\\
&\quad+\mathbf{1}_{\{\sigma^*_{l-i+1}\ge\tau_{l-i+1}\}}\big(Y_{[l-i+1],\sigma^*_{l-i+1}\wedge\tau_{l-i+1}}+E\big(V^{(i-1)}_{(\sigma^*_{l-i+1}\wedge\tau_{l-i+1})+\delta}|\mathcal{F}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}}\big)\big)\\
&=R_{l-i+1}(\sigma^*_{l-i+1},\tau_{l-i+1})+E\big(V^{(i-1)}_{(\sigma^*_{l-i+1}\wedge\tau_{l-i+1})+\delta}|\mathcal{F}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}}\big),\end{aligned}$$

and so

$$E\big(R^{(i)}(\sigma^*_{l-i+1},\tau_{l-i+1})]=E\big(R_{l-i+1}(\sigma^*_{l-i+1},\tau_{l-i+1})\big)+E\big(V^{(i-1)}_{(\sigma^*_{l-i+1}\wedge\tau_{l-i+1})+\delta}\big). \tag{4.7}$$

On the other hand, from the definition of σ^*_{l-i+2} (see (4.3)) we obtain

$$\begin{aligned}R^{(i-1)}(\sigma^*_{l-i+2},\tau_{l-i+2})&=\mathbf{1}_{\{\sigma^*_{l-i+2}<\tau_{l-i+2}\}}X^{(i-1)}_{\sigma^*_{l-i+2}}+\mathbf{1}_{\{\sigma^*_{l-i+2}\ge\tau_{l-i+2}\}}Y^{(i-1)}_{\tau_{l-i+2}}\\
&\le\mathbf{1}_{\{\sigma^*_{l-i+2}<\tau_{l-i+2}\}}V^{(i-1)}_{\sigma^*_{l-i+2}}+\mathbf{1}_{\{\sigma^*_{l-i+2}\ge\tau_{l-i+2}\}}V^{(i-1)}_{\tau_{l-i+2}}=V^{(i-1)}_{\sigma^*_{l-i+2}\wedge\tau_{l-i+2}}.\end{aligned}$$

Since $V^{(i-1)}_{\sigma^*_{l-i+2}\wedge(\sigma^*_{l-i+1}\wedge\tau_{l-i+1}+\delta+t)}$ is a supermartingale with respect to $\{\mathcal{F}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}+\delta+t}\}_{t\ge0}$ (see Corollary 3.3) it follows that

$$E\big(R^{(i-1)}(\sigma^*_{l-i+2},\tau_{l-i+2})\big)\le E\big(E\big(V^{(i-1)}_{\sigma^*_{l-i+2}\wedge\tau_{l-i+2}}|\mathcal{F}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}+\delta}\big)\big)\le E\big(V^{(i-1)}_{\sigma^*_{l-i+1}\wedge\tau_{l-i+1}+\delta}\big) \tag{4.8}$$

which together with (4.7) yields (4.5). □

For the special case $\mathbf{s}=\mathbf{s}^*$ and $\mathbf{t}=\mathbf{t}^*$ set

$$F(\mathbf{s}^*,\mathbf{t}^*)=((\sigma_1^*,..,\sigma_l^*),(\tau_1^*,..,\tau_l^*)).$$

Then from (4.5) and (4.6) we obtain for every $1<i\le l$ that

$$E\big(R^{(i-1)}(\sigma^*_{l-i+2},\tau^*_{l-i+2})+R_{l-i+1}(\sigma^*_{l-i+1},\tau^*_{l-i+1})\big)=E\big(R^{(i)}(\sigma^*_{l-i+1},\tau^*_{l-i+1})\big). \tag{4.9}$$

Proof of Proposition 4.2. We prove only the left hand side inequality in (4.4) while the second inequality can be proved in a similar way. Let $\mathbf{s}\in\mathcal{S}_T$ and set

$$F(\mathbf{s}^*,\mathbf{t})=((\sigma_1(\mathbf{s}^*,\mathbf{t}),..,\sigma_l(\mathbf{s}^*,\mathbf{t})),(\tau_1(\mathbf{s}^*,\mathbf{t}),..,\tau_l(\mathbf{s}^*,\mathbf{t})))\quad\text{and}$$
$$F(\mathbf{s}^*,\mathbf{t}^*)=((\sigma_1(\mathbf{s}^*,\mathbf{t}^*),..,\sigma_l(\mathbf{s}^*,\mathbf{t}^*)),(\tau_1(\mathbf{s}^*,\mathbf{t}^*),..,\tau_l(\mathbf{s}^*,\mathbf{t}^*))).$$

From Lemma 4.3 we obtain that for every $i>1$,

$$\begin{aligned}&E\big(R^{(i-1)}(\sigma_{l-i+2}(\mathbf{s}^*,\mathbf{t}),\tau_{l-i+2}(\mathbf{s}^*,\mathbf{t}))+\sum_{j=1}^{l-i+1}R_j(\sigma_j(\mathbf{s}^*,\mathbf{t})\tau_j(\mathbf{s}^*,\mathbf{t}))\big)\\
&\le E\big(R^{(i)}(\sigma_{l-i+1}(\mathbf{s}^*,\mathbf{t}),\tau_{l-i+1}(\mathbf{s}^*,\mathbf{t}))+\sum_{j=1}^{l-i}R_j(\sigma_j(\mathbf{s}^*,\mathbf{t})\tau_j(\mathbf{s}^*,\mathbf{t}))\big)\end{aligned}$$

and for $(\mathbf{s}^*, \mathbf{t}^*)$,

$$E\big(R^{(i-1)}(\sigma_{l-i+2}(\mathbf{s}^*,\mathbf{t}^*),\tau_{l-i+2}(\mathbf{s}^*,\mathbf{t}^*)) + \sum_{j=1}^{l-i+1} R_j(\sigma_j(\mathbf{s}^*,\mathbf{t}^*)\tau_j(\mathbf{s}^*,\mathbf{t}^*))\big)$$
$$= E\big(R^{(i)}(\sigma_{l-i+1}(\mathbf{s}^*,\mathbf{t}^*),\tau_{l-i+1}(\mathbf{s}^*,\mathbf{t}^*)) + \sum_{j=1}^{l-i} R_j(\sigma_j(\mathbf{s}^*,\mathbf{t}^*)\tau_j(\mathbf{s}^*,\mathbf{t}^*))\big).$$

Hence,

$$H(\mathbf{s}^*,\mathbf{t}) = E\big(\sum_{i=1}^{l} R_j(\sigma_1(\mathbf{s}^*,\mathbf{t}),\tau_1(\mathbf{s}^*,\mathbf{t}))\big) \le E\big(R^{(l)}(\sigma_1(\mathbf{s}^*,\mathbf{t}),\tau_1(\mathbf{s}^*,\mathbf{t}))\big) \tag{4.10}$$

and for $(\mathbf{s}^*, \mathbf{t}^*)$,

$$H(\mathbf{s}^*,\mathbf{t}^*) = E\big(R^{(l)}(\sigma_1(\mathbf{s}^*,\mathbf{t}^*),\tau(\mathbf{s}^*,\mathbf{t}^*))\big). \tag{4.11}$$

Observe that from the definitions of $\mathbf{s}^*$ and $\mathbf{t}^*$ for every $\mathbf{t} \in \mathcal{S}_T$ the inequality

$$E\big(R^{(l)}(\sigma_1(\mathbf{s}^*,\mathbf{t}),\tau_1(\mathbf{s}^*,\mathbf{t}))\big) \le E\big(R^{(l)}(\sigma_1(\mathbf{s}^*,\mathbf{t}^*),\tau_1(\mathbf{s}^*,\mathbf{t}^*))\big) \tag{4.12}$$

is just the case of the usual Dynkin game and so we have the equality

$$V_0^{(l)} = E\big(R^{(l)}(\sigma_1(\mathbf{s}^*,\mathbf{t}^*),\tau_1(\mathbf{s}^*,\mathbf{t}^*))\big). \tag{4.13}$$

Note that $\sigma_1(\mathbf{s}^*,\mathbf{t}) = \mathbf{s}^*(1,0)$ does not depend on $\mathbf{t}$ and similarly $\tau_1(\mathbf{s},\mathbf{t}^*)$ does not depend on $\mathbf{s}$. From (4.10) , (4.11) and (4.12) we obtain that

$$H(\mathbf{s}^*,\mathbf{t}) \le E\big(R^{(l)}(\sigma_1(\mathbf{s}^*,\mathbf{t}),\tau_1(\mathbf{s}^*,\mathbf{t}))\big) \le E\big(R^{(l)}(\sigma_1(\mathbf{s}^*,\mathbf{t}^*),\tau_1(\mathbf{s}^*,\mathbf{t}^*))\big) = H(\mathbf{s}^*,\mathbf{t}^*). \tag{4.14}$$

□

5. Doob–Meyer decomposition

Let $X_t \ge Y_t \ge 0$ be two a.s. continuous processes satisfying (3.1) and adapted to the filtration $\{\mathcal{F}_t\}_{0\le t}$ which satisfies the conditions of Proposition 3.9. Let V_t be the game value process with a time horizon $T > 0$ given by (3.2) and let $\delta \ge 0$. Since by Corollary 3.2 for each $0 \le t \le T$ the process $V_{\sigma_{t+\delta}\wedge s}$ in s is a supermartingale with respect to $\{\mathcal{F}_s\}_{s\ge t+\delta}$ it admits a Doob–Meyer decomposition (see [5] or [7]),

$$V_{\sigma_{t+\delta}\wedge s} = M_s(t) - A_s(t), \quad \forall s \ge t+\delta \tag{5.1}$$

where $M_s(t)$ and $A_s(t)$ are a martingale and an increasing process, respectively, satisfying

$$M_{t+\delta}(t) = V_{t+\delta}, \quad \text{and so} \quad A_{t+\delta}(t) = 0.$$

For $t \le s \le t+\delta$ set also

$$M_s(t) = E\big(V_{t+\delta}|\mathcal{F}_t\big). \tag{5.2}$$

In view of properties of the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ we can assume that for every t the process $M_s(t)$, $s \ge t$ is an a.s continuous modification of the martingale above. In view of Proposition 3.9 we can assume also that the game value process V_t is a.s. continuous and for every $t \ge 0$ we can take the a.s. continuous modification of the increasing process $A_s(t)$, $s \ge t+\delta$, as well.

5.1**. Lemma.** *There is a subset Ω^* with $P(\Omega^*) = 1$ such that for every $\omega \in \Omega^*$ and every non decreasing sequence of rational numbers $\{q_i\}_{i=1}^{\infty}$ the limit*

$$\lim_{i\to\infty} M_{q_i+r}(q_i)(\omega)\mathbf{1}_{\{q_i+r\le\sigma_{q_i+\delta}(\omega)\}} \tag{5.3}$$

exist for all $r > \delta$.

Proof. Let $p > q$. Then

$$\big((M_s(q) - A_{p+\delta}(q)) - (A_r(q) - A_{p+\delta}(q))\big)\mathbf{1}_{\{\sigma_{q+\delta}\geq p+\delta\}} = \big(M_s(p) - A_s(p)\big)\mathbf{1}_{\{\sigma_{q+\delta}\geq p+\delta\}}$$

for every $s \geq p+\delta$. Indeed, the left hand side is equal to $V_{\sigma_{q+\delta}\wedge r}\mathbf{1}_{\{\sigma_{q+\delta}\geq p+\delta\}}$ and by (3.6) this process is equal to $V_{\sigma_{p+\delta}\wedge r}\mathbf{1}_{\{\sigma_{q+\delta}\geq p+\delta\}}$ which is the right hand side. Furthermore, both sides are the Doob decompositions of the same supermartingale and since we take their a.s. continuous modifications then by uniqueness they are indistinguishable (see [7]). We conclude that for every $p > q$ there is a subset $\Omega(p,q)$ with $P(\Omega(p,q)) = 1$ such that for every $\omega \in \{\sigma_{q+\delta} \geq p+\delta\} \cap \Omega(p,q)$,

$$A_s(q)(\omega) - A_{p+\delta}(q)(\omega) = A_s(p)(\omega) \quad \text{and} \quad M_s(q)(\omega) - A_{p+\delta}(q)(\omega) = M_s(p)(\omega),\ \forall s \geq p+\delta. \tag{5.4}$$

Hence, we can find a subset Ω^* with $P(\Omega^*) = 1$ such that every $\omega \in \Omega^*$,

(1) $V_t(\omega)$,$X_t(\omega)$ are continuous in t;
(2) $M_{q+r}(q)(\omega)$ and $A_{q+r}(q)(\omega)$ are continuous in r for every rational number q;
(3) For every two rational numbers $p > q$ the equalities (5.4) hold true.

Let $\{q_i\}_{i=1}^{\infty}$ be non decreasing sequence of rational numbers with a limit $t < T$. Fix $r > \delta$ and let $\omega \in \Omega^*$. Assume, first, that $t + r > \sigma_{t+\delta}(\omega)$. Then we can find $i_0 = i_0(\omega)$ such that $q_i + r > \sigma_{t+\delta} \geq \sigma_{q_i+\delta}$ for every $i \geq i_0$. It follows that $M_{q_i+r}(q_i)\mathbf{1}_{\{q_i+r\leq\sigma_{q_i+\delta}\}} = 0$ for every $i \geq i_0$, and so the limit in (5.3) exists.

Next, assume that $t + r \leq \sigma_{t+\delta}(\omega)$ and let $\epsilon > 0$. Since $r > \delta$ and $V_t(\omega), X_t(\omega)$ are continuous we obtain, from the definition of σ_t that

$$\lim_{i\to\infty} \sigma_{q_i+\delta} = \sigma_{t+\delta} \geq t + r > t + \delta.$$

This means that there exists $i_0 = i_0(\omega)$ such that for every $i, j \geq i_0$,

$$\sigma_{q_i+\delta} > t + \delta \geq q_j + \delta, \tag{5.5}$$

and so by (3.6) we obtain that $\sigma_{q_i+\delta} = \sigma_{t+\delta}$. Since $\sigma_{t+\delta} \geq t + r \geq q_i + r$ it follows also that for every $i \geq i_0$,

$$\sigma_{q_i+\delta} \geq q_i + r. \tag{5.6}$$

From (5.6) we see that for $i \geq i_0$,

$$M_{q_i+r}(q_i)(\omega) = M_{q_i+r}(q_i)(\omega)\mathbf{1}_{\{q_i+r\leq\sigma_{q_i+\delta}(\omega)\}}.$$

By (5.5) we can use the condition (3) above to obtain that for every $j \geq i \geq i_0$,

$$A_{q_j+\delta}(q_{i_0}) - A_{q_i+\delta}(q_{i_0}) = A_{q_j+\delta}(q_i).$$

Using the condition (2) we also get that

$$\lim_{i<j\to\infty} A_{q_j+\delta}(q_i) = \lim_{i<j\to\infty} \big(A_{q_j+\delta}(q_{i_0}) - A_{q_i+\delta}(q_{i_0})\big) = 0.$$

Let $i_1 = i_1(\omega)$ be such that

$$A_{q_i+\delta}(q_{i_1})(\omega) < \epsilon/3, \quad \forall i \geq i_1.$$

By the condition (2) above there is $i_2 = i_2(\omega)$ such that

$$|M_{q_i+r}(q_{i_1}) - M_{q_j+r}(q_{i_1})| < \epsilon/3 \quad \forall i, j \geq i_2.$$

Finally, let i_3 be such that $q_{i_3} + r \geq t + \delta$. Then, for every $i, j \geq i_1 \vee i_2 \vee i_3$,

$$|M_{q_i+r}(q_i) - M_{q_j+r}(q_j)| \leq |M_{q_i+r}(q_i) - M_{q_i+r}(q_{i_1})| + |M_{q_i+r}(q_{i_1}) - M_{q_j+r}(q_{i_1})|$$
$$+|M_{q_j+r}(q_{i_1}) - M_{q_j+r}(q_j)| = A_{q_i+\delta}(q_{i_1}) + |M_{q_i+r}(q_{i_1}) - M_{q_j+r}(q_{i_1})| + A_{q_j+\delta}(q_{i_1}) < \epsilon.$$

□

5.2. **Corollary.** *For any $0 \le t \le T$ and $\delta < r \le T - t$ there exists a random variable $\hat{M}_{t+r}(t)$ such that*

$$\hat{M}_{t+r}(t) = M_{t+r}(t)\mathbf{1}_{\{t+r \le \sigma_{t+\delta}\}} \qquad a.s.$$

Furthermore, for any ω from Ω^* found in Lemma 5.1 and every $\delta < r$ the function $\omega_r(t) = \hat{M}_{t+r}(t)(\omega)$ is left continuous. Moreover, let $\{t_i\}_{i=1}^{\infty}$ and $\{r_i\}_{i=1}^{\infty}$ be two non decreasing sequences of real numbers with limits t and r, respectively, such that $r_i > \delta$ for every $i \ge 1$. Then for each $\omega \in \Omega^*$,

$$\lim_{i\to\infty} \hat{M}_{t_i+r_i}(t_i)(\omega) = \hat{M}_{t+r}(t)(\omega). \tag{5.7}$$

Proof. By Lemma 5.1 for any $\omega \in \Omega^*$ we can define

$$\omega_r(t) = \hat{M}_{t+r}(t)(\omega) = \lim_{i\to\infty} M_{q_i+r}(q_i)(\omega)\mathbf{1}_{\{q_i+r \le \sigma_{q_i+\delta}\}},$$

where $\{q_i\}_{i=0}^{\infty}$ is some non decreasing sequence of rational numbers with the limit t, and the function $\omega_r(t)$ is left continuous. Add t to the set of the rational numbers and let Ω^* satisfies conditions (1),(2) and (3) above with respect to this extended set of numbers. Then

$$\hat{M}_{t+r}(t) = \lim_{i\to\infty} M_{q_i+r}(q_i)\mathbf{1}_{\{t+r \le \sigma_{q_i+\delta}\}} = M_{t+r}\mathbf{1}_{\{r+t \le \sigma_{t+r}\}}, \quad \forall \omega \in \Omega_t^*$$

and $P(\Omega_t^*) = 1$ so the first equality of the corollary holds true.

In order to prove (5.7) it suffices to show that $\lim_{i\to\infty} \hat{M}_{q_i+r_i}(q_i)(\omega) = \hat{M}_{t+r}(t)(\omega)$ for each t, r and any $\omega \in \Omega^*$ where $\{q_i\}_{i=0}^{\infty}$ is a nondecreasing sequence of rational numbers with the limit t. From the definition of $\hat{M}_{t+r}(t)$ we see that for any $\omega \in \Omega^*$, each $\delta < r$ and every rational number q,

$$\hat{M}_{q+r}(q)(\omega) = M_{q+r}(q)(\omega)\mathbf{1}_{\{q+r \le \sigma_{q+\delta}(\omega)\}}. \tag{5.8}$$

Let $\omega \in \Omega^*$ and $\epsilon > 0$. Assume, first, that $\sigma_{t+\delta}(\omega) < t + r$. As in Lemma 5.1 we can find $i_0 = i_0(\omega)$ such that $\sigma_{q_i+r_i} < q_i + r_i$ for every $i \ge i_0$. Hence, $\hat{M}_{q_i+r_i}(q_i)(\omega) = 0$, and so the limit (5.7) exist in this case. Next, assume that $\sigma_{t+r}(\omega) \ge t + r$. Then, again, similarly to Lemma 5.1 we can find $i_0 = i_0(\omega)$ such that for every $i \ge i_0$ and each $j \ge 1$,

$$\sigma_{q_i+\delta} \ge q_i + r \ge q_i + r_j,$$

and so using (5.8) it follows that $\hat{M}_{q_i+r_j}(q_i)(\omega) = M_{q_i+r_j}(q_i)(\omega)$ for each such i, j and ω. As in Lemma 5.1 we can find $i_1 = i_1(\omega) \ge i_0$ such that $A_{q_i+\delta}(q_{i_1})(\omega) < \epsilon/3$ for every $i \ge i_1$, and so from the condition (3) for Ω^* we see that for every $i \ge i_1$ and each $j \ge 1$,

$$|M_{q_i+r_j}(q_i) - M_{q_i+r_j}(q_{i_1})| \le A_{q_i+\delta}(q_{i_1}) < \epsilon/3. \tag{5.9}$$

Note that the inequality (5.9) remains true if we replace r_j by r. Taking the limit with respect to i in (5.9) and using the continuity of $M_r(q_{i_0})$ together with the definition of $\hat{M}_{t+r_j}(t)$ we obtain that for every $j \ge 1$,

$$|\hat{M}_{t+r_j}(t) - M_{t+r_j}(q_{i_1})| \le \epsilon/3.$$

Using again the continuity of $M_r(q_{i_1})$ we can find $i_2 = i_2(\omega)$ and $j_0 = j_0(\omega)$ such that

$$|M_{q_i+r_j}(q_{i_1}) - M_{t+r}(q_{i_1})| < \epsilon/3 \quad \forall i \ge i_2, \ \forall j \ge j_0.$$

We conclude, using (5.8), that for any $i \ge i_0 \vee i_1 \vee i_2$ and $j \ge j_0$,

$$|\hat{M}_{q_i+r_j}(q_i) - \hat{M}_{t+r}(t)| = |M_{q_i+r_j}(q_i) - \hat{M}_{t+r}(t)|$$

$$= |M_{q_i+r_j}(q_i) - M_{q_i+r_j}(q_{i_1})| + |M_{q_i+r_j}(q_{i_1}) - \hat{M}_{t+r}(q_{i_1})| + |M_{t+r}(q_{i_1}) - \hat{M}_{t+r}(t)| < \epsilon.$$

We get that (5.7) holds true when $\{t_i\}_{i=0}^{\infty}$ is a rational sequence and the general case follows from left continuity of $\hat{M}$. □

Note that (5.7) can be written as

$$\lim_{i\to\infty}\hat{M}_{r_i}(t_i)=\hat{M}_r(t) \tag{5.10}$$

where $\{r_i\}$ and $\{t_i\}$ are non decreasing sequences of real numbers such that $r_i > t_i+\delta$. We now extend the definition of $\hat{M}_r(t)$ for $r=t+\delta$ by setting

$$\hat{M}_{t+\delta}(t)=V_{t+\delta}.$$

Observe that the limit (5.10) cannot be extended to the case where $r=t+\delta$ and $r_i\ge t_i+\delta$ since in this case $\hat{M}_{r_i}(t_i)=0$ if $r_i>\sigma_{t_i+\delta}$ for every large i, though in general $V_{t+\delta}\neq 0$. On the other hand, assume also that $\sigma_{t+\delta}>t+\delta$ then as in Lemma 5.1 there exists i_0 such that $\sigma_{t_i+\delta}=\sigma_{t+\delta}>t+\delta\ge t_i+\delta$ for every $i>i_0$ and using arguments similar to Lemma 5.1 and Corollary 5.2 we obtain (5.10) for this case too.

Let τ,ρ be two stopping times such that $\tau+\delta\le\rho\le\sigma_{\tau+\delta}$. Then from Corollary 5.2 and the above,

$$\hat{M}_{\rho(\omega)}(\tau(\omega))=V_{\tau+\delta}\mathbf{1}_{\{\tau+\delta=\sigma_{\tau+\delta}\}}+\lim_{n\to\infty}\mathbf{1}_{\{\tau+\delta<\sigma_{\tau+\delta}\}}\sum_{k=0}^{2^n-1}\hat{M}_{\rho_{n_k}(\omega)}(n_k)\mathbf{1}_{\{n_k\le\tau(\omega)<n_{k+1}\}}\quad\text{a.s.} \tag{5.11}$$

where $n_k=\frac{k}{2^n}T$ and $\rho_{n_k}=\rho\wedge\sigma_{n_k+\delta}$. Note that we take ρ_{n_k} and not just ρ because in this case $\rho_{n_k}\le\sigma_{n_k+\delta}$, and so

$$\hat{M}_{\rho_{n_k}}(n_k)=M_{\rho_{n_k}}(n_k).$$

Since for every n_k the process $M_r(n_k)$ continuous it is, in particular, progressively measurable, and so the function $\hat{M}_{\rho_{n_k}}(n_k)$ is measurable (see [7]). We conclude that the sum in the right hand side of (5.11) is measurable for every n, and so $\hat{M}_\rho(\tau)$ is measurable too.

5.3. **Lemma.** *Let $0\le\tau,\rho,\rho'\le T$ be stopping times such that $\tau+\delta\le\rho'\le\rho\le\sigma_{\tau+\delta}$. Then the random variables $\hat{M}_\rho(\tau)$ and $\hat{M}_{\rho'}(\tau)$ are integrable and satisfy*

$$E\big(\hat{M}_\rho(\tau)|\mathcal{F}_{\rho'}\big)=\hat{M}_{\rho'}(\tau). \tag{5.12}$$

Proof. First, note that $\{n_k\le\tau<n_{k+1}\}\in\mathcal{F}_\tau\subset\mathcal{F}_{\rho'}$ since $\tau\le\rho'$. Next, the case $\rho'>\rho_{n_k}$ can happened only if $\rho\ge\rho'>\sigma_{n_k+\delta}$, and so in this case $\rho'_{n_k}=\rho_{n_k}=\sigma_{n_k+\delta}$. By these two facts and taking into account that $M_r(n_k)$ is a martingale we obtain for every n and $0\le k\le n$ that

$$\begin{aligned}E\big(M_{\rho_{n_k}}(n_k)\mathbf{1}_{\{n_k\le\tau<n_{k+1}\}}|\mathcal{F}_{\rho'}\big)&=E\big(M_{\rho_{n_k}}(n_k)|\mathcal{F}_{\rho'}\big)\mathbf{1}_{\{n_k\le\tau<n_{k+1}\}}\\&=\big(E\big(M_{\rho_{n_k}}(n_k)|\mathcal{F}_{\rho'}\big)\mathbf{1}_{\{\rho'\le\rho_{n_k}\}}+E\big(M_{\rho_{n_k}}(n_k)|\mathcal{F}_{\rho'}\big)\mathbf{1}_{\{\rho'>\rho_{n_k}\}}\big)\mathbf{1}_{\{n_k\le\tau<n_{k+1}\}}\\&=\big(M_{\rho'}(n_k)\mathbf{1}_{\{\rho'\le\rho_{n_k}\}}+M_{\rho_{n_k}}(n_k)\mathbf{1}_{\{\rho'>\rho_{n_k}\}}\big)\mathbf{1}_{\{n_k\le\tau<n_{k+1}\}}=M_{\rho'_{n_k}}(n_k)\mathbf{1}_{\{n_k\le\tau<n_{k+1}\}}.\end{aligned} \tag{5.13}$$

Next, for any $m\ge0$ and $n>0$ set

$$\mathsf{A}_m=\bigcup_{k=0}^{2^m-1}\{n_k\le\tau\le n_{k+1}\}\cap\{\sigma_{n_k+\delta}>\tau+\delta\}\quad\text{and}\quad N_n^m=\Big(\sum_{k=0}^{2^n-1}M_{\rho_{n_k}}(n_k)\mathbf{1}_{\{n_k\le\tau\le n_{k+1}\}}\Big)\mathbf{1}_{\mathsf{A}_m}$$

Then by (5.6) it follows that $\bigcup_{m\ge0}\mathsf{A}_m=\{\sigma_{\tau+\delta}>\tau+\delta\}$ and $\mathsf{A}_m\in\mathcal{F}_{\tau+\delta}$ for every $m\ge0$. We claim that for every $n\ge m$,

$$N_n^m\ge N_{n+1}^m.$$

Indeed, fix some $\omega\in\mathsf{A}_m$ and assume that $\omega\in\{n_{k_0}\le\tau<n_{k_0+1}\}$. Then

$$\text{either}\quad\omega\in\{\frac{k_0}{2^n}T\le\tau<\frac{2k_0+1}{2^{n+1}}T\}\quad\text{or}\quad\omega\in\{\frac{2k_0+1}{2^{n+1}}T\le\tau<\frac{2(k_0+1)}{2^{n+1}}T\}.$$

In the first case it is easy to see that $N_n^m(\omega)=N_{n+1}^m(\omega)=M_{\rho_{n_{k_0}}}(n_{k_0})$. In the second case note that since $\sigma_{n_{k_0}}>\tau+\delta\ge(n+1)_{2k_0+1}+\delta$ we can use (5.4) to obtain

$$M_{\rho_{n_{k_0}}}(n_k)-A_{(n+1)_{2k_0+1}}(n_k)=M_{\rho_{n_{k_0}}}((n+1)_{2k_0+1}).$$

We also get $\sigma_{n_{k_0}+\delta} = \sigma_{(n+1)_{2k_0+1}+\delta} = \sigma_{\tau+\delta}$, and so $\rho_{n_{k_0}} = \rho_{(n+1)_{2k_0+1}}$. From the above and the fact that $A_r(t)$ is non negative it follows that

$$N_n^m(\omega) = M_{\rho_{n_{k_0}}}(n_k) \geq M_{\rho_{n_{k_0}}}((n+1)_{2k_0}) = M_{\rho_{(n+1)_{2k_0}}}((n+1)_{2k_0+1}) = N_{n+1}^m(\omega).$$

Since $M_\rho(t)$ is integrable in view of (4.1) then N_n^m is also integrable for every $m \geq 0$ and $n \geq 0$. Since for each m the sequence N_n^m is nonincreasing in $n \geq m$ we can use the Lebesgue bounded convergence theorem for conditional expectations to get that for each m and every σ-algebra $\mathcal{G}$,

$$E\big(\lim_{n\to\infty} N_n^m|\mathcal{G}\big) = \lim_{n\to\infty} E\big(N_n^m|\mathcal{G}\big).$$

Since $\mathsf{A}_m \subset \{\sigma_{\tau+\delta} > \rho+\delta\}$ for every m we see that

$$\lim_{n\to\infty} N_n^m = \lim_{n\to\infty}\Big(\sum_{k=0}^{2^n-1} M_{\rho_{n_k}}(n_k)\mathbf{1}_{\{n_k\leq\tau\leq n_{k+1}\}}\Big)\mathbf{1}_{\mathsf{A}_m} = \hat{M}_\rho(\tau)\mathbf{1}_{\mathsf{A}_m}$$

and since $\mathsf{A}_m \in \mathcal{F}_{\tau+\delta} \subset \mathcal{F}_{\rho'}$ for every m we can use (5.14) to obtain that

$$E\big(\hat{M}_\rho(\tau)|\mathcal{F}_{\rho'}\big)\mathbf{1}_{\mathsf{A}_m} = E\big(\hat{M}_\rho(\tau)\mathbf{1}_{\mathsf{A}_m}|\mathcal{F}_{\rho'}\big) = E\big(\lim_{n\to\infty} N_n^m|\mathcal{F}_{\rho'}\big) = \lim_{n\to\infty} E\big(N_n^m|\mathcal{F}_{\rho'}\big)$$
$$= \lim_{n\to\infty} E\Big(\big(\textstyle\sum_{k=0}^{2^n-1} M_{\rho_{n_k}}(n_k)\mathbf{1}_{\{n_k\leq\tau\leq n_{k+1}\}}\big)\mathbf{1}_{\mathsf{A}_m}|\mathcal{F}_{\rho'}\Big) = \lim_{n\to\infty} E\Big(\big(\textstyle\sum_{k=0}^{2^n-1}$$
$$M_{\rho_{n_k}}(n_k)\mathbf{1}_{\{n_k\leq\tau\leq n_{k+1}\}}\big)|\mathcal{F}_{\rho'}\Big)\mathbf{1}_{\mathsf{A}_m} = \lim_{n\to\infty}\big(\textstyle\sum_{k=0}^{2^n-1} M_{\rho'_{n_k}}(n_k)\mathbf{1}_{\{n_k\leq\tau\leq n_{k+1}\}}\big)\mathbf{1}_{\mathsf{A}_m} = \hat{M}_{\rho'}(\tau)\mathbf{1}_{\mathsf{A}_m}.$$

Now letting m to ∞ we obtain that

$$E\big(\hat{M}_\rho(\tau)|\mathcal{F}_{\rho'}\big)\mathbf{1}_{\{\sigma_{\tau+\delta}>\tau+\delta\}} = \hat{M}_{\rho'}(\tau)\mathbf{1}_{\sigma_{\{\tau+\delta}>\tau+\delta\}}$$

and (5.12) follows. Note that if we take $\rho' = \tau+\delta$ then

$$E\big(\hat{M}_\rho(\tau)|\mathcal{F}_{\tau+\delta}\big) = V_{\tau+\delta},$$

and so $\hat{M}_\rho(\tau)$ is also integrable by (4.1). □

Next, we extend the definition of $M_r(t)$ for smaller r setting $\hat{M}_r(t) = E\big(V_{t+\delta}|\mathcal{F}_r\big)$ for any $0 \leq t \leq T$ and $t \leq r \leq t+\delta$ and in the case $r = t+\delta$ we choose the continuous modification of the process $\hat{M}_t(t) = E\big(V_{t+\delta}|\mathcal{F}_t\big)$ (see Lemma 4.1). It is easy to see that for every stopping time $0 \leq \tau \leq T$,

$$E\big(\hat{M}_{\tau+\delta}|\mathcal{F}_\tau\big) = E\big(V_{\tau+\delta}|\mathcal{F}_\tau\big) = \hat{M}_\tau \tag{5.14}$$

We summarize this section in the following

5.4. **Corollary.** *Let $M_r(t)$ $t \geq 0$; $r \geq t$ be as in (5.1) and (5.2). Then for any t there exists a modification of this process such that*

(i) $M_\rho(0)$ *is integrable for every* $0 \leq \rho \leq \sigma_0$ *and*

$$E\big(M_\rho(0)|\mathcal{F}_{\rho'}\big) = M_{\rho'}(0)$$

whenever $0 \leq \rho' \leq \rho \leq \sigma_0$*;*

(ii) *Let* τ *be some stopping time then* $M_\rho(\tau)$ *is integrable for every* $\tau+\delta \leq \rho \leq \sigma_\tau$ *and*

$$E\big(M_\rho(\tau)|\mathcal{F}_{\rho'}\big) = M_{\rho'}(\tau)$$

whenever $\tau+\delta \leq \rho' \leq \rho \leq \sigma_{\tau+\delta}$*;*

(iii) *Let* τ *be some stopping time then*

$$E\big(M_{\tau+\delta}(\tau)|\mathcal{F}_\tau\big) = M_\tau(\tau).$$

6. Markovian case

Let $X_t \geq Y_t \geq 0$ be two continuous processes adapted with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the same conditions as in the previous section. Before dealing with the Markovian case we derive a general result concerning the game value process. For every $0 \leq s \leq t$ set

$$V_{s,t} = esssup_{s\leq\tau\leq t} essinf_{s\leq\sigma\leq t} E\big(R(\sigma,\tau)|\mathcal{F}_s\big).$$

Then by (3.3),

$$V_{s,t} = E\big(R(\sigma_{s,t},\tau_{s,t})|\mathcal{F}_s\big)$$

where

$$\sigma_{s,t} = \inf\{s \leq r : X_r = V_{s,r}\} \wedge t, \qquad \tau_{s,t} = \inf\{s \leq r : Y_r = V_{s,r}\}. \tag{6.1}$$

For $0 \leq t \leq s$,

$$\{\sigma_{0,t} < \tau_{0,s}\} = \{\sigma_{0,t}\wedge s < \tau_{0,s}\} \quad and \quad \{\sigma_{0,t} \geq \tau_{0,s}\} = \{\sigma_{0,t}\wedge s \geq \tau_{0,s}\},$$

and so

$$V_{0,t} - V_{0,s} = E\big(R(\sigma_{0,t},\tau_{0,t})\big) - E\big(R(\sigma_{0,s},\tau_{0,s})\big) \geq E\big(R(\sigma_{0,t},\tau_{0,s})\big) - E\big(R(\sigma_{0,t}\wedge s,\tau_{0,s})\big) = 0.$$

Hence, $V_{0,t}$ is nondecreasing in t.

For two stopping times σ,τ and $\alpha > 0$ define

$$R^\alpha(\sigma,\tau) = (X_\sigma+\alpha)\mathbf{1}_{\sigma<\tau} + Y_\tau\mathbf{1}_{\{\tau\leq\sigma\}}$$

and denote by $V^\alpha_{0,t}$ the value of the Dynkin game where we replace X_t by $X_t+\alpha$. We also define $\sigma^\alpha_{0,t},\tau^\alpha_{0,t}$ to be the corresponding optimal stopping times with respect to this game.

6.1**. Lemma.** *For every $t \geq 0$,*

$$V_{0,t} \leq V^\alpha_{0,t} \leq V_{0,t}+\alpha,$$

and so in the sup norm,

$$\lim_{n\to\infty} V^{1/n}_{0,t} = V_{0,t}.$$

Proof. We have

$$V^\alpha_{0,t} - V_{0,t} \geq E\big(R^\alpha(\sigma_{0,t},\tau^\alpha_{0,t})\big) - E\big(R(\sigma_{0,t},\tau^\alpha_{0,t})\big) = E\big(\alpha\mathbf{1}_{\{\sigma_{0,t}<\tau^\alpha_{0,s}\}}\big) \geq 0$$

and

$$V_{0,t} - V^\alpha_{0,t} \geq E\big(R(\sigma^\alpha_{0,t},\tau_{0,t})\big) - E\big(R^\alpha(\sigma^\alpha_{0,t},\tau_{0,t})\big) = E\big(-\alpha\mathbf{1}_{\{\sigma^\alpha_{0,t}<\tau_{0,t}\}}\big) \geq -\alpha.$$

□

6.2**. Lemma.** *The function $V_{0,t}$ is continuous in t.*

Proof. Assume, first, that $X_t > Y_t$ for all t. Let $s \leq t$ and set

$$\sigma^t_{0,s} = \inf\{0 \leq r \leq s : X_r = V_{r,s}\} \wedge t.$$

Since $X_s > Y_s = V_{s,s}$,

$$\{\sigma^t_{0,t} < t\} = \{\sigma_{0,s} < s\}, \{\sigma^t_{0,s} = t\} = \{\sigma_{0,s} = s\},$$

and so for every stopping time $0 \leq \tau \leq t$,

$$\{\sigma^t_{0,s} < \tau\} = \{\sigma_{0,s} < \tau\wedge s\},$$

$$\{\sigma^t_{0,s} \geq \tau\} \cap \{\sigma^t_{0,s} < t\} = \{\sigma_{0,s} \geq \tau\} \cap \{\sigma^t_{0,s} < t\} = \{\sigma_{0,s} \geq \tau\wedge s\} \cap \{\sigma^t_{0,s} < t\}$$

and

$$\{\sigma^t_{0,s} \geq \tau\} \cap \{\sigma^t_{0,s} = t\} = \{\sigma_{0,s} \geq \tau\wedge s\} \cap \{\sigma^t_{0,s} < t\} = \{\sigma_{0,s} \geq \tau\wedge s\} \cap \{\sigma^t_{0,s} < t\}.$$

Hence,

$$\{\sigma^t_{0,s} < \tau\} = \{\sigma_{0,s} \geq \tau\wedge s\}.$$

Let

(6.2) $$\varepsilon_t(\zeta) = \sup_{\{0\le r,r'\le t:|r-r'|<\zeta\}} \max\big(|X_r - X_{r'}|, |Y_r - Y_{r'}|\big)$$

then

$$V_{0,s} - V_{0,t} \ge E\big(R(\sigma_{0,s}, \tau_{0,t}\wedge s)\big) - E\big(R(\sigma^t_{0,s}, \tau_{0,t})\big)$$
$$\ge E\big((Y_{\tau_{0,t}\wedge s} - Y_{\tau_{0,t}})\mathbf{1}_{\{\sigma_{0,s}\ge\tau_{0,t}\wedge s\}}\big) \ge -E\big(\varepsilon_t(t-s)\big).$$

Since $V_{0,t}$ is nondecreasing in t we see that

(6.3) $$|V_{0,s} - V_{0,t}| \le E\big(\varepsilon_t(t-s)\big)$$

By our assumption X_r, Y_r are a.s continuous and satisfy (3.1), and so by the Lebesgue bounded convergence theorem

$$\lim_{s\to t}|V_{0,s} - V_{0,t}| = 0.$$

In order to remove the assumption $X_t > Y_t$ we apply, first, the above proof to $V^{1/n}_{0,t}$, $n = 1, 2, ...$ (where this assumption is satisfied) obtaining their continuity in t which according to Lemma 6.1 yields $V_{0,t}$ as a limit of continuous functions in the sup norm making it continuous in t, as well. □

6.3. ***Remark***. Using the same argument as in Lemma 6.2 we can also easily get that for every $0 \le r \le s \le t$,

(6.4) $$|V_{r,s} - V_{r,t}| \le E\big(\varepsilon_t(s-t)|\mathcal{F}_r\big) \quad \text{a.s.}$$

Next we turn to Dynkin games in discrete time. Let $0 \le t$ and let n be a positive integer. Define $\mathcal{T}^{(n),t}$ to be the set of all stopping times $0 \le \sigma \le s$ that get values in the set $\{\frac{tk}{n} : k = 0, 1, .., n\}$. For every $k \le m \le n$ define

$$\mathcal{T}^{(n),t}_{k,m} = \{\sigma \in \mathcal{T}^{(n),t} : \frac{ks}{n} \le \sigma \le \frac{mt}{n}\}$$

and

$$V^{(n)}_{k,m} = esssup_{\sigma\in\mathcal{T}^{(n),t}_{k,m}} essinf_{\tau\in\mathcal{T}^{(n),t}_{k,m}} E\big(R(\sigma,\tau)|\mathcal{F}_{\frac{kt}{n}}\big).$$

Then we have the following (see [6]),

6.4. **Lemma.** *For every* $0 \le k \le m \le n$,

$$V^{(n),t}_{k,m} = E\big(R(\sigma_{k,m}, \tau_{k,m})|\mathcal{F}_{\frac{ks}{n}}\big)$$

and also for every $\sigma, \tau \in \mathcal{T}^{(n),t}_{k,m}$,

$$E\big(R(\sigma_{k,m}, \tau)|\mathcal{F}_{\frac{kt}{n}}\big) \le E\big(R(\sigma_{k,m}, \tau_{k,m})|\mathcal{F}_{\frac{kt}{n}}\big) \le E\big(R(\sigma, \tau_{k,m})|\mathcal{F}_{\frac{kt}{n}}\big)$$

where

$$\sigma_{k,m} = \inf\{\frac{kt}{n} \le \frac{lt}{n} \le \frac{mt}{n} : X_{\frac{lt}{n}} = V^{(n),t}_{l,m}\} \wedge \frac{mt}{n} \text{ and } \tau_{k,m} = \inf\{\frac{kt}{n} \le \frac{lt}{n} \le \frac{mt}{n} : Y_{\frac{lt}{n}} = V^{(n),t}_{l,m}\}.$$

For each stopping time $\tau \in \mathcal{T}$ we define $d_n(\tau) = \sum_{k=0}^{n-1} \frac{(k+1)t}{n}\mathbf{1}_{\{\frac{(k)t}{n}<\tau\le\frac{(k+1)t}{n}\}}$. From Proposition 3.2 in [6] it follows that for any stopping times $0 \le \sigma, \tau \le t$,

(6.5) $$R(\sigma, d_n(\tau)) + \epsilon_t(\frac{t}{n}) \ge R(d_n(\sigma), d_n(\tau))) \ge R(d_n(\sigma), \tau) - \epsilon_t(\frac{t}{n})$$

where $\varepsilon_t(\zeta)$ is defined in (6.2). We fix a horizon $t > 0$ and n and write $V^{(n)}$ for $V^{(n),t}$. Set also $t_k = \frac{kt}{n}$ and for every $0 \le s \le t$ denote

(6.6) $$k_s = \max_{0\le k\le n}\{\frac{kt}{n} \le s\} \quad \text{and} \quad n(s) = \frac{tk_s}{n} \quad \text{so that} \quad n(s) = t_{k_s}.$$

6.5. **Lemma.** *For every* $0 \le k \le m \le n$,

$$|V_{t_k,t_m} - V^{(n)}_{k,m}| \le E\big(\varepsilon_{t_m}(\frac{t}{n})|\mathcal{F}_{t_k}\big)$$

Proof. For $\sigma_{k,m}$ and $\tau_{k,m}$ defined in Lemma 6.4 and σ_{t_k,t_m} and τ_{t_k,t_m} defined in (6.1) we obtain from (6.5), (3.3) and Lemma 6.4 that

$$V_{t_k,t_m} - V^{(n)}_{k,m} \le E\big(R(\sigma_{k,m},\tau_{t_k,t_m})|\mathcal{F}_{t_k}\big) - E\big(R(\sigma_{k,m},d_n(\tau_{t_k,t_m}))|\mathcal{F}_{t_k}\big) \le E\big(\varepsilon_{t_m}(\frac{t}{n})|\mathcal{F}_{t_k}\big).$$

On the other hand,

$$V_{t_k,t_m} - V^{(n)}_{k,m} \ge E\big(R(\sigma_{t_k,t_m},\tau_{k,m})|\mathcal{F}_{t_k}\big) - E\big(R(d_n(\sigma_{t_k,t_m}),\tau_{k,m})|\mathcal{F}_{t_k}\big) \ge -E\big(\varepsilon_{t_m}(\frac{t}{n})|\mathcal{F}_{t_k}\big).$$

□

Now, let $0 \le r \le t$ then for every $l \le n$,

$$E\big(\epsilon_t(\frac{t}{n})|\mathcal{F}_{n(r)}\big) \le E\big(\epsilon_t(\frac{t}{l})|\mathcal{F}_{n(r)}\big),$$

and so

$$\limsup_{n\to\infty} E\big(\epsilon_t(\frac{t}{n})|\mathcal{F}_{n(r)}\big) \le \limsup_{n\to\infty} E\big(\epsilon_t(\frac{t}{l})|\mathcal{F}_{n(r)}\big).$$

Since $\lim_{n\to\infty} n(r) = r$ we obtain by the properties of the filtration that a.s.,

$$\limsup_{n\to\infty} E\big(\epsilon_t(\frac{t}{l})|\mathcal{F}_{n(r)}\big) = \lim_{n\to\infty} E\big(\epsilon_t(\frac{t}{l})|\mathcal{F}_{n(r)}\big) = E\big(\epsilon_t(\frac{t}{l})|\mathcal{F}_{r}\big)$$

Using the Lebesgue bounded convergence theorem it follows that a.s.,

$$\lim_{n\to\infty} E\big(\epsilon_t(\frac{t}{n})|\mathcal{F}_{n(r)}\big) = 0 \tag{6.7}$$

This yields the following

6.6. **Corollary.** *Let* $0 \le r \le s \le t$ *then a.s.,*

$$\lim_{n\to\infty} V^{(n)}_{k_r,k_s} = V_{r,s}$$

Proof. By the triangle inequality

$$|V^{(n)}_{k_r,k_s} - V_{r,s}| \le |V^{(n)}_{k_r,k_s} - V_{n(r),n(s)}| + |V_{n(r),n(s)} - V_{n(r),s}| + |V_{n(r),s} - V_{r,s}|.$$

Using (6.6) we obtain from Lemma 6.5 that a.s.,

$$|V^{(n)}_{k_r,k_s} - V_{n(r),n(s)}| \le E\big(\varepsilon_{n(s)}(\frac{t}{n})|\mathcal{F}_{n(r)}\big) \le E\big(\varepsilon_{t}(\frac{t}{n})|\mathcal{F}_{n(r)}\big),$$

and so from (6.7) it follows that this term tends to 0 a.s. as n tends to ∞. For the second term we use (6.4) in Remark 6.3 and (6.2) to obtain

$$|V_{n(r),n(s)} - V_{n(r),s}| \le E\big(\varepsilon_t(\frac{t}{n})|\mathcal{F}_{n(r)}\big).$$

Next, using Proposition 3.9 we obtain that for every fixed s the process $V_{r,s}$ is continuous in r, and so the right hand side of the above formula tends to 0, as well. □

Next, let S_t, $t \in [0,T]$ be a continuous Markov process with respect to $\{\mathcal{F}_t\}_{t\ge 0}$ with the state space $\mathbb{R}^m$ and a Markov family $\{P_x\}_{x\in\mathbb{R}^m}$. Let P be the probability on $\mathcal{F}$ with expectation E such that for any bounded measurable function μ on $\mathbb{R}^m$,

$$E\big(\mu(S^x_t)\big) = E_x\big(\mu(S_t)\big)$$

where E_x is the expectation with respect to P_x and S_t^x is the process S_t starting at x. We assume that there is $C_S > 0$ such that for every stopping time $\sigma \le T$,

$$E|S_\sigma^x - S_\sigma^y| \le C_S|x-y| \tag{6.8}$$

which is trivially satisfied when S_t is given by (2.2). A Markovian Dynkin game is determined by two payoff functions $g(x) \ge f(x) \ge 0$ on the state space $\mathbb{R}^m$ so that

$$X_t^x = g(S_t^x) \quad \text{and} \quad Y_t^x = f(S_t^x).$$

We assume that the functions g, f are Lipschitz continuous with Lipschitz constants C_g and C_f, respectively, and, in addition, for every $0 \le t \le s$,

$$E\big(sup_{0\le t\le s} g(S_t^x)\big) < \infty. \tag{6.9}$$

Set

$$\varepsilon_s^x(\zeta) = \sup_{\{0\le t,t'\le s: |t-t'|\le\zeta\}} \max(|g(S_t^x) - g(S_{t'}^x)|, |f(S_t^x) - f(S_{t'}^x)|).$$

6.7. **Lemma.** *Let $0 \le s \le t$ then for every $\zeta > 0$ a.s.,*

$$E_{[S_s^x]}\big(\varepsilon_{t-s}(\alpha)\big) \le E_x\big(\varepsilon_t(\alpha)|\mathcal{F}_s\big)$$

Proof. Define $D^{(m)} = \{\frac{k}{2^m} : k = 0, .., 2^m\}$ and $D_t^{(m)} = \{t \cdot d : d \in D^{(m)}\}$. Set

$$\varepsilon_t^{x,(m)}(\zeta) = \max_{\{d,d'\in D_t^{(m)}: |d-d'|\le\zeta\}} \max(|g(S_d^x) - g(S_{d'}^x)|, |f(S_d^x) - f(S_{d'}^x)|).$$

It is easy to see that for every ζ and x a.s.,

$$\lim_{m\to\infty} \varepsilon_t^{x,(m)}(\zeta) = \varepsilon_t^x(\zeta).$$

Using the Lebesgue bounded convergence theorem we obtain

$$\lim_{m\to\infty} E_x\big(\varepsilon_{t-s}^{(m)}(\zeta)\big) = E_x\big(\varepsilon_{t-s}(\zeta)\big),$$

and so we see that for almost all ω,

$$\lim_{m\to\infty} E_{[S_s^x(\omega)]}\big(\varepsilon_{t-s}^{(m)}(\zeta)\big) = E_{[S_s^x(\omega)]}\big(\varepsilon_{t-s}(\zeta)\big).$$

On the other hand, by the Markov property it follows that

$$E_{[S_s^x]}\big(\varepsilon_{t-s}^{(m)}(\zeta)\big) = E_{[S_t^x]}\big(\max_{\{d,d'\in D_{t-s}^{(m)}: |d-d'|\le\zeta\}} \max(|g(S_d^x) - g(S_{d'}^x)|, |f(S_d^x) - f(S_{d'}^x)|)\big)$$
$$= E_x\big(\max_{\{d,d'\in D_{t-s}^{(m)}: |d-d'|\le\zeta\}} \max(|g(S_{t+d}^x) - g(S_{s+d'}^x)|, |f(S_{s+d}^x) - f(S_{s+d'}^x)|)|\mathcal{F}_s\big) \le E_x\big(\varepsilon_t(\zeta)|\mathcal{F}_s\big).$$

Since this inequality holds true for every m the result follows. □

Let $V_{s,t}(x)$ and $V_{k,m}^{(n)}(x)$ be the same as $V_{s,t}$ and $V_{k,m}^{(n)}$ but defined using the expectations E_x in place of E. If $0 \le s \le t$ then by (6.3),

$$|V_{0,s}(x) - V_{0,t}(x)| \le E_x\big(\varepsilon_t(t-s)\big).$$

Using Lemma 6.7 we obtain that for every $0 \le r \le s$,

$$|V_{0,s}(S_r^x) - V_{0,t}(S_r^x)| \le E_{[S_x^r]}\big(\varepsilon_t(t-s)\big) \le E_x\big(\varepsilon_{t+r}(t-s)|\mathcal{F}_r\big).$$

Under (6.9) we can use the Lebesgue bounded convergence theorem for the above conditional expectation to obtain that

$$\lim_{s\to t} V_{0,s}(S_r^x) = V_{0,t}(S_r^x) \quad \text{a.s.} \tag{6.10}$$

We have also the following property

6.8. **Lemma.** *There exists a constant $C>0$ such that for every $0\le m\le n$,*

$$|V^{(n)}_{0,m}(x)-V^{(n)}_{0,m}(y)|<C|x-y|.$$

Proof. Let

$$R^x(\sigma,\tau)=g(S^x_\sigma)\mathbf{1}_{\{\sigma<\tau\}}+f(S^x_\sigma)\mathbf{1}_{\{\sigma\ge\tau\}}.$$

From Lemma 6.4 we see that

$$V^{(n)}_{0,m}(x)-V^{(n)}_{0,m}(y)=E_x\big(R(\sigma_{0,m},\tau_{0,m})\big)-E_y[R(\sigma_{0,m},\tau_{0,m})\big)\le E\big(R^x(\sigma^y_{0,m},\tau^x_{0,m})\big)$$
$$-E\big(R^y(\sigma^y_{0,m},\tau^x_{0,m})\big)\le E\big(|g(S^x_{\sigma^y_{0,m}})-g(S^y_{\sigma^y_{0,m}})|\big)+E\big(|f(S^x_{\tau^x_{0,m}})-f(S^y_{\sigma^x_{0,m}})|\big)$$
$$C_gE\big(|S^x_{\sigma^y_{0,m}}-S^y_{\sigma^y_{0,m}}|\big)+C_fE\big(|S^x_{\tau^x_{0,m}}-S^y_{\tau^x_{0,m}}|\big)\le C_gC_S|x-y|+C_fC_S|x-y|\le C|x-y|.$$

□

Before formulating the main result of this section we recall that by [3] in the discrete case we have that

$$V^{(n)}_{k,m}(x)=V^{(n)}_{0,m-k}(S^x_{t_k}) \tag{6.11}$$

6.9. **Proposition.** *For every $s\le t$,*

$$V_{s,t}(x)=V_{0,t-s}(S^x_s)\quad a.s. \tag{6.12}$$

Proof. Using (6.11) we see that for every n,

$$|V_{s,t}(x)-V_{0,t-s}(S^x_s)|\le|V_{s,t}(x)-V^{(n),t}_{k_s,k_t}(x)|+|V^{(n)}_{0,k_t-k_s}(S^x_{n(s)})-V^{(n)}_{0,k_t-k_s}(S^x_s)|$$
$$+|V^{(n)}_{0,k_t-k_s}(S^x_t)-V_{0,t_{k_t-k_s}}(S^x_t)|+|V_{0,t_{k_t-k_s}}(S^x_t)-V_{0,t-s}(S^x_t)|.$$

From Corollary 6.6 we obtain that the first term above tends a.s. to 0 as n tends to ∞. From Lemma 6.8 it follows that

$$|V^{(n)}_{0,k_t-k_s}(S^x_{n(s)})-V^{(n)}_{0,k_t-k_s}(S^x_s)|\le C|S^x_{n(s)}-S^x_s|,$$

and so since the process is continuous the second term also tends a.s. to 0. For the third term we use Lemma 6.5 and Lemma 6.7 to obtain

$$|V^{(n)}_{0,k_t-k_s}(S^x_s)-V_{0,t_{k_t-k_s}}(S^x_s)|\le E_{[S^x_s]}\big(\varepsilon_{t_{k_t-k_s}}(\frac{t}{n})\big)\le E_x\big(\varepsilon_{t_{k_t-k_s}+s}(\frac{t}{n})|\mathcal{F}_s\big)$$

where the right hand side tends a.s. to 0 by the Lebesgue bounded convergence theorem. Next, $t_{k_t-k_s}=\frac{(n-k_s)t}{n}=t-n(s)$, and so $t_{k_t-k_s}$ tends to $t-s$ as n tends to ∞, and so by (6.10) the last term tends a.s. to 0, as well. □

Let now S^x_t, $t\in[0,T]$ be the price of m stocks in question at time t given by (2.2) provided their initial prices were represented by a vector x. Assume that the game option payoffs are given by $X^x_t=g(S^x_t)$ and $Y^x_t=f(S^x_t)$. Then if at time u the stocks prices are x then the game option value function (see [6]) is given by

$$F(x,u)=\sup_{0\le\tau\le T-u}\inf_{0\le\sigma\le T-u}E\big(e^{-r\sigma\wedge\tau}\big(g(S^x_\sigma)\mathbf{1}_{\{\sigma<\tau\}}+f(S^x_\tau)\mathbf{1}_{\{\sigma\ge\tau\}}\big)\big).$$

In order to study this function we need to extend our discussion on Markovian Dynkin games to the non homogenious case. Thus, we extend the state space to $\mathcal{E}=\mathbb{R}^m_+\times\mathbb{R}_+$ and consider $\bar{S}^{[x,u]}_t=(S^x_t,u+t)$ which is a Markov process on this state space with a Markov family $\{P_{[x,u]}\}_{[x,u]\in\mathcal{E}}$. We introduce a pair of function $g(x,u),f(x,u)$ on $\mathcal{E}$ which satisfy the following conditions

(i) for every u the functions f,g are Lipschitz continuous in x with a constant that does not depend on u;
(ii) For every x these functions are continuous in u;

(iii) For every $[x,u]\in E$ and $t\geq 0$,

$$E_{[x,u]}\big(\sup_{0\leq s\leq t} g(\bar S_s)\big)<\infty.$$

We define $V_{s,t}(x,u)$ exactly as $V_{s,t}$ but with respect to $P_{[x,u]}$ in place of P. So, in particular, for every $[x,u]\in\mathcal{E}$ we have a pair of optimal stopping times $<\sigma^{[x,u]}_{s,t},\tau^{[x,u]}_{s,t}>$ which are defined exactly as $<\sigma_{s,t},\tau_{s,t}>$ with respect to $P_{[x,u]}$.

6.10**. Lemma.** *Assume that $g(x,u)$ and $f(x,u)$ satisfy the conditions (i)–(iii) above then so does $V_{0,t}(x,u)$.*

Proof. Fix $t\geq 0$ and write $\sigma^{[x,u]}$ and $\tau^{[x,u]}$ in place of $\sigma^{[x,u]}_{0,t}$ and $\tau^{[x,u]}_{0,t}$. The proof of condition (i) goes on exactly as the proof of Lemma 6.8 and we note that the Lipschitz constant of $V_{0,t}(x,u)$ in x does not depend on t. For condition (ii) we observe that

$$\begin{aligned}V_{0,t}(x,u)-V_{0,t}(x,w)&\leq E_{[x,u]}\big(R(\sigma^{[x,w]},\tau^{[x,u]})\big)-E_{[x,w]}[R(\sigma^{[x,w]},\tau^{[x,u]})]\\ &=E\big(\big(g(S^x_{\sigma^{[x,w]}},\sigma^{[x,w]}+u)-g(S^x_{\sigma^{[x,w]}},\sigma^{[x,w]}+w)\big)\mathbf{1}_{\{\sigma^{[x,w]}<\tau^{[x,u]}\}}\big)\\ &+E\big(\big(f(S^x_{\tau^{[x,u]}},\tau^{[x,u]}+u)-f(S^x_{\tau^{[x,u]}},\tau^{[x,u]}+w)\big)\mathbf{1}_{\{\sigma^{[x,w]}\geq\tau^{[x,u]}\}}\big)\\ +E\big(|g(S^x_{\sigma^{[x,w]}},\sigma^{[x,w]}+u)-g(S^x_{\sigma^{[x,w]}},\sigma^{[x,w]}+w)|\big)&+E\big(|f(S^x_{\tau^{[x,u]}},\tau^{[x,u]}+u)-f(S^x_{\tau^{[x,u]}},\tau^{[x,u]}+w)|\big)\end{aligned}$$

Exchanging u and w and using the Lebesgue bounded convergence theorem in view of (iii) and relying on (ii) we obtain that $V_{0,s}(x,u)$ is continuous in u for any x. □

For every $t\leq T-u$ set $\hat V_t(x,u)=V_{t,T-u}(x,u)$. Since the Lipschitz constant does not depend on t we see from Lemma 6.10 that the function $\hat V_0(x,u)=V_{0,T-u}(x,u)$ satisfies conditions (i) and (ii), as well. By (6.12) we obtain that

$$\hat V_0(\bar S^{[x,u]}_t)=\hat V_0(S^x_t,u+t)=V_{0,T-u-t}(\bar S^{[x,u]}_t)=V_{t,T-u}(x,u)=\hat V_t(x,u)\tag{6.13}$$

Observe also that by the definition,

$$\hat V_0(x,u)=\sup_{0\leq\tau\leq T-u}\inf_{0\leq\sigma\leq T-u}E_{[x,u]}\big(g(\bar S_\sigma)\mathbf{1}_{\{\sigma<\tau\}}+f(\bar S_\tau)\mathbf{1}_{\{\sigma\geq\tau\}}\big).\tag{6.14}$$

Next, we show that in the Markovian case the optimal stopping times can be described explicitly. In order to do so define the following regions in the state space $\mathcal{E}$,

$$\mathcal{C}_s=\{(x,u):\hat V_0(x,u)=g(x,u)\}\quad\text{and}\quad\mathcal{C}_b=\{(x,u):\hat V_0(x,u)=f(x,u)\}.$$

Since the functions in brackets are continuous these regions are closed sets. Now we can write the optimal stopping times in the form

$$\begin{aligned}\sigma^{[x,u]}_{s,T-u}&=\inf\{s\leq t:V_{t,T-u}(x,u)=g(\bar S^{[x,u]}_t)\}\wedge T\\ &=\inf\{s\leq t:\hat V_0(\bar S^{[x,u]}_t)=g(\bar S^{[x,u]}_t)\}\wedge T=\inf\{s\leq t:\bar S^{[x,u]}_t\in\mathcal{C}_s\}\wedge T\end{aligned}$$

and, similarly,

$$\tau^{[x,u]}_{s,T-u}=\inf\{s\leq t:\bar S^{[x,u]}_t\in\mathcal{C}_b\}.$$

We conclude that

$$\hat V_0(x,u)=E_{[x,u]}\big(R(\rho(\mathcal{C}_s),\rho(\mathcal{C}_b)\big)\tag{6.15}$$

where $\rho(\mathcal{C})=\inf\{t\geq 0:\bar S_t\in\mathcal{C}\}$ for every $\mathcal{C}\subset E$.

The last thing we do in this section is the following calculation which will be needed for the discounted case. Assume that $g(x,u),f(x,u)$ satisfy the condition (i),(ii) and (iii) above. Set

$$\begin{aligned}&\tilde g(x,u)=e^{-ru}g(x,u),\ \tilde f(x,u)=e^{-ru}f(x,u),\\ &V^d_{s,t}(x,u)=E_{[x,u]}\big(e^{-r\sigma\wedge\tau}\big(g(\bar S_\sigma)\mathbf{1}_{\{\sigma<\tau\}}+f(\bar S_\tau)\mathbf{1}_{\{\sigma\geq\tau\}}\big)|\mathcal{F}_s\big)\quad\text{and}\\ &\tilde V_{s,t}(x,u)=E_{[x,u]}\big(\tilde g(\bar S_\sigma)\mathbf{1}_{\{\sigma<\tau\}}+\tilde f(\bar S_\tau)\mathbf{1}_{\{\sigma\geq\tau\}}|\mathcal{F}_s\big).\end{aligned}$$

Then it is easy to see that $\tilde{V}_{s,t}(x,u) = e^{-ru}V^d_{s,t}(x,u)$. From (6.12) for $\tilde{V}$ we obtain that $\tilde{V}_{s,t}(x,u) = \tilde{V}_{0,t-s}(\bar{S}_t)$, and so

$$V^d_{s,t}(x,u) = e^{-rs}V^d_{0,t-s}(\bar{S}_s). \tag{6.16}$$

7. Perfect hedge and fair price

In this section we return to the setting of Section 2 and give the proof of Theorem 2.5. For every $1 \le i \le l$ and $0 \le t < T$ let $\{M^{(i)}_r(t)\}^T_{r=t}$ be the martingale in the Doob–Meyer decomposition

$$V^{(i)}_{\sigma_{t+\delta}\wedge s} = M^{(i)}_s(t) + A^{(i)}_s(t) \quad \forall s \ge t+\delta$$

(see Sections 3 and 4) where $V_t(i)$ is defined by (2.10) and (2.11). Relying on Corollary 5.4 we choose $M^{(i)}_r(t)\ t \ge 0\ ;\ r \ge t$ to be already a modification satisfying the conditions of that corollary. For every $x \ge 0$ we define a portfolio strategy $\pi^x = \pi^x(t_1,...,t_i)$ with $\pi^x_0 = x$ by induction in i. For the empty sequence ϕ we consider the martingale $M^l_t(0)\mathbf{1}_{\{x\ge V^l_0\}}$ (with respect to the probability measure $\tilde{P}$). By the martingale representation theorem (see [7], Section 3.4) there exists a progressively measurable process $\alpha_t(\phi) = (\alpha_{1,t}(\phi),...,\alpha_{m,t}(\phi))$, $t \ge 0$ such that $\int_0^T \alpha_{i,t}(\phi)^2 dt < \infty$ for every $1 \le i \le m$ and

$$M^{(l)}_t(0) = M^{(l)}_0(0) + \int_0^t \alpha_s(\phi)\cdot d\tilde{W}_s. \tag{7.1}$$

Let ι be the inverse matrix of κ and define

$$\gamma^{\pi^x}_{i,t}(\phi) = \tilde{S}^{-1}_{i,t}[\iota^t\alpha]_i \tag{7.2}$$

where $[v]_i$ is the i-th coordinate of a vector v. Set

$$\beta^{\pi^x}_t(\phi) = \big(M^{(l)}_t(0) - \gamma^{\pi^x}_t(\phi)\cdot S_t\big)/B_0.$$

We claim that the pair $(\beta^{\pi^x}_t(\phi), \gamma^{\pi^x}_t(\phi))$ represents a self financing portfolio strategy. Indeed, let

$$Z^{\pi^x}_t(\phi) = \beta^{\pi^x}_t(\phi)B_t + \gamma^{\pi^x}_t(\phi)\cdot S_t = e^{rt}M^{(l)}_t(0).$$

Then by (7.2) and (2.1),

$$\begin{aligned} dZ^{\pi^x}_t(\phi) = de^{rt}M^{(l)}_t(0) = rZ^{\pi^x}_t(\phi)dt + e^{rt}\alpha_t\cdot d\tilde{W} = \beta^{\pi^x}_t(\phi)dB_t \\ +r\gamma^{\pi^x}_t(\phi)S_t dt + (\iota^t\alpha_t)\cdot(\kappa\tilde{W}) = \beta^{\pi^x}_t(\phi)dB_t + \textstyle\sum_{i=0}\gamma^{\pi^x}_t(\phi)S_t(rdt + \sum_{j=1}^m \kappa_{i,j}d\tilde{W}) \\ = \beta^{\pi^x}_t(\phi)dB_t + \textstyle\sum_{i=0}\gamma^{\pi^x}_t(\phi)S_t(\mu_i dt + \sum_{j=1}^m \kappa_{i,j}dW) = \beta^{\pi^x}_t(\phi)dB_t + \sum_{i=0}\gamma^{\pi^x}_t(\phi)dS_t. \end{aligned}$$

We assume above that $x \ge V^{(l)}_0 = M^{(l)}_0$, otherwise the martingale is 0 and all the definitions are trivial. Note also that by Definition 2.2, $G_0(\phi) = \pi^x_0 - Z^{\pi^x}_0 = (x - V^{(l)}_0)^+ \ge 0$ for every $0 \le x$. Next, suppose we defined $\pi^x(t_1,..,t_{i-1})$ for every $(t_1,...,t_{i-1}) \in \mathcal{L}$ and let $(t_1,...,t_i) \in \mathcal{L}$. Set

$$A(t_1,...,t_i) = \{Z^{\pi^x}_{t_i}(t_1,...,t_{i-1}) \ge e^{rt_i}V^{(l-i+1)}_{t_i}\},$$

and so $A(t_1,..,t_i) \in \mathcal{F}_{t_i}$. Then, the process $\{M^{(l-i)}_t(t_i)\mathbf{1}_{A(t_1,..,t_i)}\}_{t_i\le t}$ is a martingale with respect to $\{\mathcal{F}_s\}_{t_i\le s}$. As in the case of the empty sequence we can find

$$\alpha_t(t_1,..,t_i) = (\alpha_{1,t}(t_1,..,t_i),...,\alpha_{m,t}(t_1,..,t_i))$$

such that for every $1 \le i \le m$ the process $\alpha_{i,t}(t_1,..,t_i)$ is progressively measurable with respect to $\{\mathcal{F}_s\}_{t_i\le s}$, satisfies $\int_0^T \alpha_{i,t}(t_1,..,t_i)^2 dt < \infty$ and

$$M^{(l-i)}_t(t_i)\mathbf{1}_{A(t_1,..,t_i)} = M^{(l-i)}_{t_i}(t_i)\mathbf{1}_{A(t_1,..,t_i)} + \int_{t_i}^t \alpha_s(t_1,..,t_i)\cdot d\tilde{W}_s. \tag{7.3}$$

We then define

$$\gamma^{\pi^x}_{i,t}(t_1,...,t_i) = \tilde{S}^{-1}_{i,t}[\iota^t\alpha]_i \quad i = 1,..,m$$

and

$$\beta_t^{\pi^x}(t_1,...,t_i) = \big(M_t^{(l-i)}(t_i)\mathbf{1}_{A(t_1,..,t_i)} - \gamma_t^{\pi^x}(t_1,...,t_i)\cdot S_t\big)/B_0.$$

As above we obtain

$$Z_t^{\pi^x}(t_1,...,t_i) = \beta_t^{\pi^x}(t_1,...,t_i)B_t + \gamma_t^{\pi^x}(t_1,...,t_i)\cdot S_t = e^{rt}M_t^{(l-i)}(t_i)\mathbf{1}_{A(t_1,..,t_i)},\quad \forall t\ge t_i$$

$$\text{and}\quad dZ_t^{\pi^x}(t_1,...,t_i) = \beta_t^{\pi^x}(t_1,...,t_i)dB_t + \gamma_t^{\pi^x}(t_1,...,t_i)\cdot dS_t.$$

Hence, π^x is a self financing portfolio. Using the definition of $A(t_1,..,t_i)$ and the above equality we get

$$G_{t_i}^{\pi^x}(t_1,...,t_i) = Z_{t_i}^{\pi^x}(t_1,...,t_{i-1}) - Z_{t_i}^{\pi^x}(t_1,...,t_i)Z_{t_i}^{\pi^x}(t_1,...,t_{i-1})$$
$$-e^{rt_i}E\big(V_{t_i+\delta}^{(l-i)}|\mathcal{F}_{t_i}\big)\mathbf{1}_{A(t_1,..,t_i)} \ge e^{t_i r}\big(V_{t_i}^{(l-i+1)} - E\big(V_{t_i+\delta}^{(l-i)}|\mathcal{F}_{t_i}\big)\big)\mathbf{1}_{A(t_1,..,t_i)}$$
$$\ge e^{t_i r}\big(Y_{t_i}^{(l-i+1)} - E\big(V_{t_i+\delta}^{(l-i)}|\mathcal{F}_{t_i}\big)\big)\mathbf{1}_{A(t_1,..,t_i)} \ge Y_{i-1}(t_i)\mathbf{1}_{A(t_1,..,t_i)} \ge 0,$$

and so π^x is a portfolio strategy.

7.1**. Lemma.** *For every x the pair (π^x, g^*) is a hedge and if $x\ge V_0^{(l)}$ then this pair is a perfect hedge.*

Proof. If $x < V_0^{(l)}$ then by the definition $Z_t^{\pi^x}(t_1,...,t_i) = 0$ and $G_0(\phi) = x$ and so (π^x, g) is a hedge in a trivial way so we may assume that $x \ge V_0^{(l)}$. Let $\mathbf{t}\in\mathcal{S}_T$ be some stopping strategy and set

$$F(\mathbf{s}^*,\mathbf{t}) = ((\sigma_1^*,...,\sigma_l^*),(\tau_1,....,\tau_l)).$$

We prove by induction that for every $1\le i\le m$,

$$Z^{\pi^x}_{\sigma_i^*\wedge\tau_i}(\sigma_1^*\wedge\tau_1,...,\sigma_{i-1}^*\wedge\tau_{i-1}) \ge e^{r\sigma_i^*\wedge\tau_i}V^{(l-i+1)}_{\sigma_i^*\wedge\tau_i} \tag{7.4}$$

and that $(\pi,\mathbf{s}^*)$ satisfy conditions (i)–(iv) of Definition 2.3. If $i=1$ then $Z_t^{\pi} = e^{rt}M_t^{(l)}(0)$ taking into account that $x\ge V_0^{(l)}$ and since $M_t^{(l)}(0)$ is a martingale $(\pi^x,\mathbf{s}^*)$ satisfies (i). In particular,

$$Z^{\pi^x}_{\sigma_1^*\wedge\tau_1}(\phi) = e^{r\sigma_1^*\wedge\tau_1}M^{(l)}_{\sigma_1^*\wedge\tau_1}(0) \ge e^{r\sigma_1^*\wedge\tau_1}V^{(l)}_{\sigma_1^*\wedge\tau_1},$$

and so

$$\tilde{P}(A(\sigma_1^*\wedge\tau_1)) = 1.$$

Hence, using property (i) of $M_r^{(l-1)}(t)$ from Corollary 5.4 it follows that

$$Z^{\pi^x}_{\sigma_1^*\wedge\tau_1}(\sigma_1^*\wedge\tau_1) = e^{r\sigma_1^*\wedge\tau_1}M^{(l-1)}_{\sigma_1^*\wedge\tau_1}(\sigma_1^*\wedge\tau_1) = e^{r\sigma_1^*\wedge\tau_1}E\big(V^{(l-1)}_{\sigma_1^*\wedge\tau_1+\delta}|\mathcal{F}_{\sigma_1^*\wedge\tau_1}\big).$$

We conclude that

$$G^{\pi^x}_{\sigma_1^*\wedge\tau_1}(\phi) = Z^{\pi^x}_{\sigma_1^*\wedge\tau_1}(\phi) - Z^{\pi^x}_{\sigma_1^*\wedge\tau_1}(\sigma_1^*\wedge\tau_1) \ge e^{r\sigma_1^*\wedge\tau_1}\big(V^{(l)}_{\sigma_1^*\wedge\tau_1} - E\big(V^{(l-1)}_{\sigma_1^*\wedge\tau_1+\delta}|\mathcal{F}_{\sigma_1^*\wedge\tau_1}\big)\big)$$
$$\ge e^{r\sigma_1^*\wedge\tau_1}\big(R^{(l)}(\sigma_1^*,\tau_1) - E\big(V^{(l-1)}_{\sigma_1^*\wedge\tau_1+\delta}|\mathcal{F}_{\sigma_1^*\wedge\tau_1}\big)\big) = R_1(\sigma_1^*,\tau_1)$$

which gives us (iv) of Definition 2.3 for the case $i=1$. Next, assume that (7.4) holds true for $1\le i<l$. Then, by (7.4),

$$\tilde{P}(A(\sigma_1^*\wedge\tau_1,...,\sigma_i^*\wedge\tau_i)) = 1$$

which together with the definition of π^x yields that

$$Z_\rho^{\pi^x}(\sigma_1^*\wedge\tau_1,...,\sigma_i^*\wedge\tau_i) = e^{r\rho}M_\rho^{(l-i)}(\sigma_i^*\wedge\tau_i) \ge e^{r\rho}V^{(l-i)}_{\sigma_{i+1}^*\wedge\rho}$$

for every stopping time ρ such that $\sigma_i^*\wedge\tau_i+\delta\le\rho$. In particular, this is true for $\rho = \sigma_{i+1}^*\wedge\tau_{i+1}$ obtaining (7.4) for $i+1$, and so

$$Z_\rho^{\pi^x}(\sigma_1^*\wedge\tau_1,...,\sigma_{i+1}\wedge\tau_{i+1}) = e^{r\rho}M_\rho^{(l-i-1)}(\sigma_{i+1}\wedge\tau_{i+1}).$$

Also, since $M_r^{(l-i-1)}(t)$ satisfies (ii) and (iii) of Corollary 5.4 we see that $(\pi^x, \mathbf{s}^*)$ satisfies (ii) and (iii) of Definition 2.3 for $i+1$, as well. Hence,

$$G^{\pi^x}_{\sigma^*_{i+1}\wedge\tau_{i+1}}(\sigma^*_1\wedge\tau_1,...,\sigma_{i+1}\wedge\tau_{i+1}) = Z^{\pi^x}_{\sigma^*_{i+1}\wedge\tau_{i+1}}(\sigma^*_1\wedge\tau_1,...,\sigma_i\wedge\tau_i)$$
$$-Z^{\pi^x}_{\sigma^*_{i+1}\wedge\tau_{i+1}}(\sigma^*_1\wedge\tau_1,...,\sigma_{i+1}\wedge\tau_{i+1}) \geq e^{r\sigma^*_{i+1}\wedge\tau_{i+1}}\big(V^{(l-i)}_{\sigma^*_{i+1}\wedge\tau_{i+1}} - E\big(V^{(l-i-1)}_{\sigma^*_{i+1}\wedge\tau_{i+1}+\delta}|\mathcal{F}_{\sigma^*_{i+1}\wedge\tau_{i+1}}\big)\big)$$
$$\geq e^{r\sigma^*_{i+1}\wedge\tau_{i+1}}\big(R^{(l-i)}(\sigma^*_{i+1},\tau_{i+1}) - E\big(V^{(l-i-1)}_{\sigma^*_{i+1}\wedge\tau_{i+1}+\delta}|\mathcal{F}_{\sigma^*_{i+1}\wedge\tau_{i+1}}\big)\big) = R_{i+1}(\sigma^*_{i+1},\tau_{i+1}).$$

Thus, (iv) of Definition 2.3 holds true for i+1, as well. Note that we use the inequality $V^{(l-i+1)}_{\sigma^*_i\wedge\tau_i} \geq R^{(l-i+1)}(\sigma^*_i,\tau_i)$ which holds true by the definition of σ^*_i and the fact that $Y^{(i)}_t \leq V^{(i)}_t \leq X^{(i)}_t$ for every $1 \leq i \leq l$. □

7.2. **Lemma.** *Let* $(\pi, \mathbf{s})$ *be a perfect hedge. Then* $\pi_0 \geq V^{(l)}_0$

Proof. Let $\mathbf{t}^*$ be the optimal strategy for the buyer (see Section 4) and set

$$F(\mathbf{s},\mathbf{t}^*) = ((\sigma_1,...,\sigma_l),(\tau^*_1,...,\tau^*_l)).$$

First, we show by backward induction in $1 \leq i \leq l$ that

$$e^{-r\sigma_i\wedge\tau^*_i} Z^{\pi}_{\sigma_i\wedge\tau^*_i}(\tau_1\wedge\tau^*_1,...,\sigma_{i-1}\wedge\tau^*_{i-1}) \geq R^{(l-i+1)}(\sigma_i,\tau_i). \tag{7.5}$$

For $i = l$ by the definition of a perfect hedge,

$$Z^{\pi}_{\sigma_l\wedge\tau^*_l}(\sigma_1\wedge\tau^*_1,..,\sigma_{l-1}\wedge\tau^*_{l-1})$$
$$= G^{\pi}_{\sigma_L\wedge\tau^*_l}(\sigma_1\wedge\tau^*_1,...,\sigma_l\wedge\tau^*_l) \geq R_l(\sigma_i,\tau^*_i) = e^{\sigma_l\wedge\tau^*_l r}R^{(1)}(\sigma_l,\tau^*_l).$$

Next, assume that (7.5) holds true for $i+1$. Taking the conditional expectation with respect to $\mathcal{F}_{\sigma_i\wedge\tau^*_i+\delta}$ in (7.5) and using the property (ii) of Definition 2.3 for $(\pi, \mathbf{s})$ we obtain that

$$e^{-r(\sigma_i\wedge\tau^*_i+\delta)}Z^{\pi}_{\sigma_i\wedge\tau^*_i+\delta}(\sigma_1\wedge\tau^*_1,...,\sigma_i\wedge\tau^*_i))) \geq E\big(R^{(l-i)}(\sigma_{i+1},\tau^*_{i+1})|\mathcal{F}_{\sigma_i\wedge\tau^*_i+\delta}\big)$$
$$\geq E\big(V^{(l-i)}_{\sigma_{i+1}\wedge\tau^*_{i+1}}|\mathcal{F}_{\sigma_i\wedge\tau^*_i+\delta}\big) \geq V^{(l-i)}_{\sigma_i\wedge\tau^*_i+\delta}.$$

Observe that the second inequality here holds true since $R^{(l-i+1)}(\sigma_i,\tau^*_i) \geq V^{(l-i+1)}_{\sigma_i\wedge\tau^*_i}$ for every $1 \leq i \leq l$ (see the definition of τ^* in Section 4). The last inequality above holds true in view of the submartingale property of $\{V^{(l-i)}_{s\wedge\tau^*_{i+1}}\}_{\sigma_i\wedge\tau^*_i+\delta\leq s}$ (see Lemma 3.3).

Using the property (iii) of Definition 2.3 for the hedge $(\pi, \mathbf{s})$ we get

$$e^{-r\sigma_i\wedge\tau_i}Z^{\pi}_{\sigma_i\wedge\tau^*_i}(\sigma_1\wedge\tau^*_1,...,\sigma_i\wedge\tau^*_i) \geq E\big(V^{(l-i)}_{\sigma_i\wedge\tau^*_i+\delta}|\mathcal{F}_{\sigma_i\wedge\tau^*_i}\big).$$

Since (π, g) is a perfect hedge we can use property (iv) of Definition 2.3 to obtain

$$e^{-r\sigma_i\wedge\tau_i}Z^{\pi}_{\sigma_i\wedge\tau^*_i}(\sigma_1\wedge\tau^*_1,...,\sigma_{i-1}\wedge\tau^*_{i-1}) = e^{-r\sigma_i\wedge\tau_i}G^{\pi}_{\sigma_i\wedge\tau^*_i}(\sigma_1\wedge\tau^*_1,...,\sigma_i\wedge\tau^*_i)$$
$$+e^{-r\sigma_i\wedge\tau_i}Z^{\pi}_{\sigma_i\wedge\tau^*_i}(\sigma_1\wedge\tau^*_1,...,\sigma_i\wedge\tau^*_i) \geq e^{-r\sigma_i\wedge\tau_i}R_i(\sigma_i,\tau^*_i) + E\big(V^{(l-i)}_{\sigma_i\wedge\tau^*_i+\delta}|\mathcal{F}_{\sigma_i\wedge\tau^*_i}\big) = R^{(l-i+1)}(\sigma_i,\tau^*_i)$$

which is just (7.5) for i. We conclude that (7.5) holds true for every $1 \leq i \leq l$. In particular, if we consider the inequality (7.5) for $i = 1$ and use the property (i) of Definition 2.3 with the same argument as above then

$$\pi_0 \geq Z^{\pi}_0(\phi) = E\big(e^{-r\sigma_1\wedge\tau^*_1}Z^{\pi}_{\sigma_1\wedge\tau^*_1}\big) \geq E\big(R^l(\sigma_1,\tau^*_1)\big) \geq E\big(V^{(l)}_{\sigma_1\wedge\tau^*_1}\big) \geq V^{(l)}_0$$

yielding the result of the lemma. □

We can now complete the proof of Theorem 2.5. From Lemma 7.1 and Lemma 7.2 we see that the fair price V^* for the swing game option (X_i, Y_i, δ) is given by $V_0^{(l)}$. The relations (2.13), (2.14) and (2.16) hold true by Proposition 4.2 and (4.13). Now, the existence of a perfect hedge with the initial capital $V_x^{(l)}$ follows from Lemma 7.1. □

Next, we consider the Markovian case and complete the proof of Theorem 2.6. First, we prove by induction that for every $1 \leq i \leq n$: the functions $g^{(i)}(x,u), f^{(i)}(x,u)$ satisfy conditions (i) and (ii) appearing before Theorem 2.6, the equalities (2.23) hold true and for every $0 \leq s \leq t \leq T$,

$$V_{s,t}^{(i)}(x,u) = e^{-rs} V_{0,t-s}^{(i)}(S_s^{[x,u]}). \tag{7.6}$$

Note that if (2.23) holds true for some i then by the definition of $V_{s,t}^{(i)}(x,u)$ (see (2.19)) we obtain that $V_{s,t}^{(i)}(x,t)$ is of the form of $V_{s,t}^d(x,u)$ (see (6.16) and above) and (7.6) is just (6.16). Hence, we have only to check the first two assertions above. For $i = 1$ we see from the definitions that $g^{(1)}$ and $f^{(1)}$ satisfy (i) and (ii) before Theorem 2.6 and also (2.23) holds true. Suppose now that all the three assertions above hold true for some $i \geq 1$. Assume first that $t \leq T - u - \delta$, i.e. $\delta \leq T - t - u$, and so using the Markov property, Definitions 2.20, 2.22 and the equality (7.6) we obtain that

$$e^{-rt} g^{(i)}(\bar{S}_t^{[x,u]}) = e^{-rt} g_{l-i+1}(S_t^{[x,u]}) + e^{-r(t+\delta)} E_{[\bar{S}_t^{[x,u]}]}\big(\hat{V}_0^{(i-1)}(\bar{S}_{t+\delta}^{[x,u]})\big) = e^{-rt} X_{l-i+1}^{[x,u]}(t)$$
$$+E\big(e^{-r(t+\delta)} V_{0,T-u-t-\delta}^{(i-1)}(\bar{S}_{t+\delta}^{[x,u]})|\mathcal{F}_t\big) = e^{-rt} X_{l-i+1}^{[x,u]}(t) + E_{[x,u]}\big(V_{t+\delta,T-u}^{(i-1)}(x,u)|\mathcal{F}_t\big) = X_t^{(i),[x,u]}.$$

For the case $T - u - \delta < t \leq T - u$ we have $T - u - t < \delta$, and so

$$e^{-rt} g^{(i)}(\bar{S}_t^{[x,u]}) = e^{-rt} g_{l-i+1}(\bar{S}_t^{[x,u]}) + e^{-r(T-u-t)} E_{[\bar{S}_t^{[x,u]}]}\big(\hat{V}_{T-u-t}^{(i-1)}(\bar{S}_{T-u-t}^{[x,u]})\big)$$
$$= e^{-rt} X_{l-i+1}^{[x,u]}(t) + E\big(e^{-r(T-u)} \hat{V}_{T-u}^{(i-1)}(\bar{S}_{T-u}^{[x,u]})|\mathcal{F}_t\big) = e^{-rt} X_{l-i+1}^{[x,u]}(t)$$
$$+E_{[x,u]}\big(V_{0,0}^{(i-1)}(\bar{S}_{T-u}^{[x,u]})|\mathcal{F}_t\big) = e^{-rt} X_{l-i+1}^{[x,u]}(t) + E_{[x,u]}\big(V_{T-u,T-u}^{(i-1)}(\bar{S}_{T-u}^{[x,u]})|\mathcal{F}_t\big) = X_t^{(i),[x,u]}.$$

Thus (2.23) holds true for $g^{(i)}$ and the equality for $f^{(i)}$ can be proved in a similar way. Since by the induction hypothesis $f^{(i-1)}(x,u)$ and $g^{(i-1)}(x,u)$ satisfy the conditions (i) and (ii) in question so does $\hat{V}^{(i-1)}(x,u)$ in view of Lemma 6.10 and it is easy to see that in this case the function $e^{-r\delta\wedge(T-u)} E_{[x,u]}\big(\hat{V}^{(i-1)}(\bar{S}_{\delta\wedge(T-u)})\big)$ also satisfy these conditions. By the definition of $f^{(i)}(x,u)$ and $g^{(i)}(x,u)$ we conclude that they satisfy these conditions, as well.

Next, for every $1 \leq i \leq l$ let $\mathcal{C}_s^{(i)}$ and $\mathcal{C}_b^{(i)}$ be the domains defined in Theorem 2.6. Then by (6.15) the equality (2.24) holds true, and so (see [9]) for each i the function $v(x,u) = \hat{V}^{(i)}(x,u)$ is a solution of the parabolic free boundary problem (2.25) which exists according to [4]. □

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