Course no. 80573: Probability and stochastic processes, 2004-2005

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Problem set 2:
Martingales, Markov chains, Brownian motion, Stopping times and strong Markov property, Laws of large numbers and the central limit theorem, Large deviations, Law of iterated logarithm

1 Martingales

1. Let $B_1(t), B_2(t), B_3(t)$ be 3 independent Brownian motions. Prove that $X(t) = \big(B_1^2(t) + B_2^2(t) + B_3^2(t)\big)^{-1/2}$, $t \geq \delta > 0$ is a supermartingale and using this show that $B_1^2(t) + B_2^2(t) + B_3^2(t) \to \infty$ almost surely as $t \to \infty$.

2. Let $X = (X_n, \mathcal{F}_n)$ be a supermartingale. Show that
$$P\{\max_{1\leq j\leq n} |X_j| \geq a\} \leq Ca^{-1} \max_{1\leq j\leq n} E|X_j|$$
where $C \leq 3$ (the constant C can be taken to be 1 if X is a martingale or if X does not change sign).

3. Let $\Omega = [0, 1]$, $\mathcal{F}$ be the Borel σ-algebra on $[0, 1]$, let P denotes the Lebesgue measure and let f be a Borel measurable function on $[0, 1]$ with $\int |f|dP < \infty$. Put
$$f_n(x) = 2^n \int_{k2^{-n}}^{(k+1)2^{-n}} f(y)dy, \;\; k2^{-n} \leq x < (k + 1)2^{-n}.$$
Show that $f_n(x) \to f(x)$ P-almost surely.

2 Markov chains

4. A thinker who owns r umbrellas travels back and forths between home and office, taking along an umbrella (if there is one at hand) in rain (probability

of which is p) but not in shine (probability $1-p$). Let the state be the number of umbrellas at hand, irrespective of whether the thinker is at home or at work. Set up the transition matrix of the corresponding Markov chain and find the stationary probabilities. Find the steady-state (stationary) probability of his getting wet, and show that five umbrellas will protect him with probability, at least, 0.95 against any climate (any p).

5. A deck of n cards is shuffled so that a card is chosen at random (with probability $1/n$) and it is put to the bottom of the deck. Show that if the number of shuffles tends to infinity the probability of each particular card order tends to $\frac{1}{n!}$.

3 Brownian motion

6. Suppose that $(W_t,\, t \geq 0)$ is a stochastic process having independent increments whose distribution depends only on the time difference, i.e. if $s_i < t_i \leq s_{i+1} < t_{i+1},\ i = 1, ..., k-1$ then $W_{t_i} - W_{s_i},\ i = 1, ..., k$ are independent and the distribution of $W_{t_i} - W_{s_i}$ depends only on $t_i - s_i$. Show that if $EW_1^2 = 1$ and for any $c > 0$ the process $\tilde{W}_t(\omega) = c^{-1} W_{c^2 t}(\omega)$ has the same finite dimensional distributions as W_t then these are finite dimensional distributions of the Brownian motion.

7. Let $B(t)$ be the Brownian motion starting at 0. Show that the process $B^*(t)$ defined by $B^*(t) = tB(1/t)$ if $t > 0$ and $B^*(0) = 0$ is also the Brownian motion.

4 Stopping times and strong Markov property

8. Prove the strong Markov property of the Poisson process.

9. Let B_t be the Brownian motion starting at zero and set $\tau = \inf\{t \geq 0 : B_t = a + bt\}$ where $a > 0$. Use the exponential martingale $\exp(\alpha B_t - \alpha^2 t/2)$ to compute $E \exp(-\lambda\tau)$.

5 Laws of large numbers and the central limit theorem

10. Let $X_1, X_2, ...$ be i.i.d. random variables with $EX_i = 0$ and $EX_i^2 = 1$. Show that

$$\frac{\sum_{k=1}^n X_k}{\left(\sum_{k=1}^n X_k^2\right)^{1/2}}$$

converges in distribution as $n \to \infty$ to a standard normal random variable.

11. Let $X_1, X_2, ...$ be be i.i.d. random variables with $EX_i = 0$ and $EX_i^2 = 1$. Let $a_n \to \infty$ as $n \to \infty$ be a sequence of positive integers and let N_n be a sequence of positive integer valued random variables such that $N_n a_n^{-1} \to 1$ in probability as $n \to \infty$. Show that $S_{N_n} a_n^{-1/2}$ converge in distribution as $n \to \infty$ to a standard normal random variable.

6 Large deviations

12. Denote by H_n the number of heads and by T_n the number of tails obtained after a fair coin is tossed n times. Set $S_n = H_n - T_n$. Show that

$$\lim_{n\to\infty} P\{S_n > an\}^{1/n} = \left((1+a)^{(1+a)}(1-a)^{(1-a)}\right)^{-1/2}$$

provided $0 < a < 1$. What is the answer if $a \geq 1$.

13. Set

$$T_n = \sum_{k:|k-\frac{1}{2}n|>\frac{1}{2}an} \begin{pmatrix} n \\ k \end{pmatrix}$$

for $0 < a < 1$ and

$$S_n = \sum_{k>n(1+a)} \frac{n^k}{k!}$$

for $a > 0$. Find $\lim_{n\to\infty} T_n^{1/n}$ and $\lim_{n\to\infty} S_n^{1/n}$.

7 Law of iterated logarithm type results

14. Let $\xi_1, \xi_2, ...$ be a sequence of i.i.d. random variables with standard normal distribution. Show that

$$P\{\limsup_{n\to\infty} \frac{\xi_n}{\sqrt{2\ln n}} = 1\} = 1.$$

15. Let $\xi_1, \xi_2, ...$ be a sequence of i.i.d. random variables with the Poisson law with parameter $\lambda > 0$. Show that $(\forall\lambda)$,

$$P\{\limsup_{n\to\infty} \frac{\xi_n \ln\ln n}{\ln n} = 1\} = 1.$$