# Course no. 80573: Probability and stochastic processes, 2004-2005

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**Problem set 1:**
**Kolmogorov theorems, Markov processes, Brownian motion**

## 1 Gaussian processes

A finite family of random variables $\xi_1, ..., \xi_n$ is called a (nondegenerate) Gaussian family or a (nondegenerate $n$-dimensional) Gaussian random vector if there exists a symmetric positively definite matrix $A = (a_{ij})$ (i.e. $\sum_{i,j=1}^{n} a_{ij}x_i x_j > 0$ provided $\sum_{i=1}^{n} |x_i| > 0$) and a vector $a = (a_1, ..., a_n)$ such that

$$P\{(\xi_1, ..., \xi_n) \in \Gamma\} = \int_\Gamma \frac{1}{(2\pi)^{n/2}(\det A)^{1/2}} e^{-\frac{1}{2}\left(A^{-1}(x-a),(x-a)\right)} dx, \qquad (1)$$

where $(\cdot, \cdot)$ denotes the scalar (inner) product of vectors. More generally, A finite family of random variables $\xi_1, ..., \xi_n$ is called a Gaussian family or a Gaussian random vector (may be degenerate) if for any constants $b_1, ..., b_n$ the sum $\sum_{i=1}^{n} b_i x_i$ is a normal (Gaussian) random variable with a mean $\mu$ and a variance $\sigma^2$ which could be zero. In particular, a constant (i.e. nonrandom) vector is a (degenerate) Gaussian vector (which is excluded by the first definition which introduces nondegenerate Gaussian random vectors). We call also a Gaussian random vector degenerate if it has linearly dependent (with constant coefficients) components. An infinite family of random variables is called Gaussian if any finite subfamily is Gaussian. In particular, $\xi_t$, $t \in T \subset \mathbb{R}$ is a Gaussian process if it is a Gaussian family.

**Problems.**

1. Prove that if (1) holds true then

$$E\xi_i = a_i \text{ and } \mathrm{Cov}(\xi_i, \xi_j) = E(\xi_i - a_i)(\xi_j - a_j) = a_{ij}.$$

2. Prove that if $\xi = (\xi_1, ..., \xi_n)$ is a Gaussian (degenerate or not) random vector and $L$ is a nonrandom invertible $n \times n$ matrix then $L\xi$ is a Gaussian (degenerate or not) random vector.

3. Let $a(t,s)$ be a function on $\mathbb{R}_+ \times \mathbb{R}_+ = [0,\infty)$ such that $a(s,t) = a(t,s)$ and for any $t_1, ..., t_n, s_1, ..., s_n \in \mathbb{R}_+$ and $x_1, ..., x_n \in \mathbb{R}$ with $\sum_{i=1}^n |x_i| > 0$ we have

$$\sum_{i,j=1}^n a(t_i, s_j) x_i x_j > 0.$$

Prove (using Kolmogorov's theorem) that there exists a Gaussian process $\xi_t$, $t \geq 0$ such that $E\xi_t = 0$, $\forall t \geq 0$ and $E\xi_t\xi_s = a(t,s) \, \forall t, s \geq 0$.

4. Prove that if $a(t,s) = \min(t,s) - ts$ then there exists a Gaussian process $\xi_t$, $1 > t > 0$ with almost surely continuous (even Hölder continuous) paths which satisfies $E\xi_t = 0$, $\forall 1 > t > 0$ and $E\xi_t\xi_s = a(t,s) \, \forall 1 > t, s > 0$.

5. Suppose that $a(t,s) = b(\min(t,s)) > 0$ where $b(u)$ is a positive strictly increasing function. Prove that if $\xi_t$, $t \geq 0$ is the Gaussian process constructed with such $a(t,s)$ then for any $0 \leq t_1 < t_2 < \cdots < t_n$ the increments $\xi_{t_2} - \xi_{t_1}, \xi_{t_3} - \xi_{t_2}, \cdots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent. Show that if $b(u) = u$ we obtain the Brownian motion.

# 2 Brownian motion

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Define the one dimensional Brownian motion $B(t)$, $t \geq 0$ as a Markov process on $\mathbb{R} = (-\infty, \infty)$ with the transition probability

$$P(t, x, \Gamma) = \int_\Gamma p(t, y - x) dy$$

where $p(t,z) = (2t\pi)^{-1/2} e^{-\frac{z^2}{2t}}$.

**Problems.**

6. Prove the Chapman-Kolmogorov equality

$$\int p(t, y - x) p(s, z - y) dy = p(s + t, z - x).$$

7. Assume that $B(0) = 0$ and prove that for $0 < t_1 < t_2 < \cdots < t_n$,

$$B(t_1),\ B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$$

are independent random variables. (Hint: represent this vector as a linear transformation of the vector $B(t_1), ..., B(t_n)$.

8. Prove that $B(t) - B(s)$ is the Gaussian (normal) random variable with mean zero and the variance $t - s$.

9. Prove that with probability one $B(t)$, $t \geq 0$ is not Hölder continuous for any exponent $\gamma > 1/2$.

10*. Prove that there exists no modification $\tilde{B}$ of $B$ (i.e. $P\{B(t) = \tilde{B}(t)\} = 1$ for each $t$) such that $\tilde{B}(t)$ is Hölder continuous also with the exponent $\gamma = 1/2$. In fact, it is possible to show that with probability one there exist times $t = t(\omega)$ depending on $\omega$ where $B(t,\omega)$ is Hölder continuous with the exponent $\gamma = 1/2$ (which is not true for $\gamma > 1/2$.

# 3 Poisson process

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Define the Poisson process $N(t)$, $t \geq 0$ as a Markov chain (process) on $\mathbb{N} = \{0, 1, ...\}$ with the transition probability

$$P(t, n, \Gamma) = \sum_{m \in \Gamma} p(t, m - n)$$

where $p(t, m-n) = e^{-\lambda t} \frac{(\lambda t)^{m-n}}{(m-n)!}$ if $m \geq n$ and $p(t, m-n) = 0$ if $m < n$.

**Problems.**

11. Prove the Chapman-Kolmogorov equality

$$\sum_n p(t, n-k) p(s, m-n) = p(t+s, m-k).$$

12. Assume that $N(0) = 0$ and prove that for $0 < t_1 < t_2 < ... < t_n$,

$$N(t_1),\ N(t_2) - N(t_1), ..., N(t_n) - N(t_{n-1})$$

are independent random variables.

# 4 Cauchy process.

Define the Cauchy process $\xi(t)$, $t \geq 0$ as a Markov process on $\mathbb{R}$ with the transition probability

$$P(t, x, \Gamma) = \int_\Gamma p(t, y - x) dy$$

where $p(t, z) = \frac{t}{\pi(t^2 + z^2)}$.

**Problems.**

13*. Prove the Chapman-Kolmogorov equality

$$\int p(t, y - x) p(s, z - y) dy = p(t + s, z - x).$$

(Hint: it is easier to integrate here using the complex variable integration).

14. Assume that $\xi(0) = 0$ and prove that for $0 < t_1 < t_2 < ... < t_n$,

$$\xi(t_1),\ \xi(t_2) - \xi(t_1), ..., \xi(t_n) - \xi(t_{n-1})$$

are independent random variables.

# 5 Stationary distribution

15. Let $\xi_n$, $n = 0, 1, 2, ...$ be a Markov chain on the state space $\mathbb{N} = 0, 1, 2, ...$ with the transition probability matrix $P = (p_{ij},\ i, j \in \mathbb{N})$. Prove that if

$$p_{ij} \geq a_j p_{kj}$$

for all $i, j, k \in \mathbb{N}$ and some positive $a_j$ then there exists a unique stationary distribution $\eta$ which is an infinite probability vector satisfying $\eta P = \eta$.