AN INTRODUCTION TO FINANCIAL MATHEMATICS

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ABSTRACT.

1. LECTURE 6: FUNDAMENTAL THEOREMS OF ASSET PRICING

1.1. **Arbitrage.** We return to the general discrete time financial market described in Lecture 4 which consists of a probability space with a filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\geq 0}, P)$ and of d-stocks $S = (S^1, ..., S^d)$, $S^i = (S^i_n)_{n\geq 0}$ and a bond $B = (B_n)_{n\geq 0}$ which will be called a general (B, S) financial market. Here

$$B_n = B_0 \prod_{1\leq k\leq n} (1 + r_k) \text{ and } S^i_n = S^i_0 \prod_{1\leq k\leq n} (1 + \rho^i_k)$$

with a predictable sequence $r_k \geq 0$, $k = 1, 2, ...$ and adapted sequences ρ^i_k, $-1 < \rho^i_k$, $k = 1, 2, ...$ We recall that a pair $\pi = (\beta, \gamma)$ of predictable sequences of random variables $\beta = \{\beta_n\}_{n\geq 0}$ and $\gamma = \{\gamma^1_n, ..., \gamma^d_n\}_{n\geq 0}$ is called a self-financing trading strategy if

$$\Delta X^\pi_n = X^\pi_n - X^\pi_{n-1} = \beta_n \Delta B_n + \sum_{i=1}^{d} \gamma^i_n \Delta S^i_n = \beta_n \Delta B_n + (\gamma_n, \Delta S_n).$$

where $X^\pi_n = \beta_n B_n + \sum_{i=1}^d \gamma^i_n S^i_n = \beta_n B_n + (\gamma_n, S_n)$ is the portfolio value at time n corresponding to the strategy π.

1.1. **Definition.** (arbitrage)

A self-financing trading strategy π provides an arbitrage opportunity at time N if $X^\pi_0 = 0$, $X^\pi_N \geq 0$ a.s. and $P\{X^\pi_N > 0\} > 0$ (equivalently, $EX^\pi_N > 0$). A financial market has no arbitrage opportunities

1) if $X^\pi_0 = 0$ and $X^\pi_N \geq 0$ implies that $X^\pi_N = 0$ a.s.;

2) (in the weak sense) if $X^\pi_0 = 0$ and $X^\pi_n \geq 0$ for all $n = 1, 2, ..., N$ implies that $X^\pi_N = 0$ a.s.;

3) (in the strong sense) if $X^\pi_0 = 0$ and $X^\pi_N \geq 0$ implies that $X^\pi_n = 0$ a.s. for all $n = 1, 2, ..., N$.

1.2. **Remark.** For studying arbitrage related issues we are talking about events of the form $\{X^\pi_n > 0\}$, $\{X^\pi_n \geq 0\}$ and $\{X^\pi_n = 0\}$, and so we can deal instead with adjusted quantities $\tilde{X}^\pi_n = \frac{X^\pi_n}{B_n}$, $\tilde{S}_n = \frac{S_n}{B_n}$ and $\tilde{B}_n = 1$ (i.e., in fact, we can assume that the bond interest rate is zero).

1.3. **Theorem.** *(First fundamental theorem of asset pricing) The general (B, S) financial market defined above has no arbitrage opportunities (in the sense of 1))*

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if and only if there exists a martingale measure (i.e. a measure $\tilde P$ equivalent to P such that $\tilde S_n = \frac{S_n}{B_n}$, $n \geq 0$ is a martingal with respect to it).

Proof. a) Suppose that a martingale measure $\tilde P$ exists, π is a self-financing strategy and X_n^π is the value of the corresponding portfolio. Then $\tilde X_n^\pi = \frac{X_n^\pi}{B_n}$, $n \geq 0$ is a martingale with respect to $\tilde P$. Hence, if $X_0^\pi = 0$ a.s. then $0 = E\tilde X_0^\pi = E\tilde X_N^\pi$, and so if $\tilde X_N^\pi \geq 0$ a.s. then $\tilde X_N^\pi = 0$, i.e. there is no arbitrage opportunities.

b) The other direction: "no arbitrage opportunities implies existence of a martingale measure", is more difficult to prove. The proof here will follow mostly [3] (see also [1], [2] and [4]). We can assume here that the initial value X_0^π of the porfolio is zero and then the adjusted value of the portfolio $\tilde X_n^\pi = \frac{X_n^\pi}{B_n}$ at time n is given by the formula

$$\tilde X_n^\pi = \sum_{k=1}^n (\gamma_k,\, \Delta\tilde S_k)$$

where $\pi = (\beta_k, \gamma_k)_{k=1}^N$ is a self-financing trading strategy and $\tilde S_k = \frac{S_k}{B_k}$. No arbitrage means that if $\tilde X_N^\pi \geq 0$ a.s. then $\tilde X_N^\pi = 0$ a.s.

Let M_N be the set of all random variables ξ having the form

$$\xi = \sum_{k=1}^N (\gamma_k,\, \Delta\tilde S_k)$$

for all predictable d-dimensional sequences γ_k, $k = 1, ..., N$ and denote by L^0_+ the set of all non-negative $\mathcal{F}_N$-measurable random variables. Then the no arbitrage assumption is equivalent to

$$M_N \cap L^0_+ = \{0\}. \tag{1.1}$$

Indeed, clearly, (1.1) implies that there is no arbitrage. In the other direction, if there is no arbitrage then there exists no self-financing trading strategy $\pi = (\beta_k, \gamma_k)_{k=1}^N$ such that $\tilde X_N^\pi \geq 0$ and $P\{\tilde X_N^\pi > 0\} > 0$ since, recall, $\tilde X_0^\pi = 0$. But if (1.1) does not hold true then there exists a predictable sequence γ_k, $k = 1, ..., N$ such that

$$\xi = \sum_{k=1}^N (\gamma_k,\, \Delta\tilde S_k) \geq 0 \text{ and } P\{\xi > 0\} > 0.$$

Now define $\beta_1 = -(\gamma_1, \tilde S_0)$ and recursively,

$$\beta_{k+1} = (\beta_k + (\gamma_k, \tilde S_k) - (\gamma_{k+1}, \tilde S_k)),\ k = 1, ..., N-1.$$

Then $\pi = (\beta_k, \gamma_k)_{k=1}^N$ is a self-financing trading strategy and $\tilde X_N^\pi = \xi$ which contradicts the no arbitrage assumption.

Set

$$K_N = M_N - L^0_+ = \{\eta = \xi - \zeta : \xi \in M_N,\, \zeta \in L^0_+\}$$

which is a convex cone. Then (1.1) is equivalent to

$$K_N \cap L^0_+ = \{0\}.$$

We will show in Appendix to this lecture that K_N is closed with respect to the convergence in probability which is the same as to be closed with respect to the almost sure convergence since any convergent in probability sequence has a subsequence converging almost surely.

Set $K_N^1 = K_N \cap L^1$ where $L^1 = L^1(\Omega, \mathcal{F}_N, P)$ is the space of all integrable $\mathcal{F}_N$-measurable random variables. Then K_N^1 is a convex cone in L^1. Since, as we will show this in Appendix, K_N is closed with respect to the convergence in probability, it is also closed with respect to the stronger convergence in L^1, and so K_N^1 is closed with respect to the latter convergence. Then relying on the Hahn-Banach theorem and related functional analysis results we will derive in Appendix that any $x \neq 0$ in the set $L_+^1 = L_+^1(\Omega, \mathcal{F}_N, P)$ of all non-negative integrable $\mathcal{F}_N$-measurable random variables can be separated from K_N^1 in the sense that for any $x \in L_+^1$, $x \neq 0$ there exists $z_x \in L^\infty = L^\infty(\Omega, \mathcal{F}_N, P)$ such that

$$Ez_x\xi < Ez_x x \tag{1.2}$$

for all $\xi \in K_N^1$.

Relying on the above assertion we observe that $Ez_x x > 0$ for any $x \in L_+^1$, $x \neq 0$ since $0 \in K_N^1$, and so, since K_N^1 is a cone, i.e. (1.2) being true for some $\xi \in K_N^1$ must remain true for any $a\xi$ where $a \geq 0$ is a constant, we obtain that $Ez_x\xi \leq 0$. Moreover, since K_N^1 contains all integrable negative random variables, $z_x \geq 0$ a.s. as for otherwise we could take a random variable ξ which is negative precisely where z_x is negative being zero elsewhere which would give $Ez_x\xi > 0$ leading to the contradiction. By a version of the Halmos-Savage theorem, whose proof is provided below, the family of measures $\{z_x P\}$ contains a countable equivalent subfamily $\{z_{x_i}P,\, i \in \mathbb{N}\}$ (i.e. both families have the same null sets). Set $\rho = \sum_i 2^{-i}\frac{z_{x_i}}{\|z_{x_i}\|_\infty}$ and $\tilde{x} = \mathbb{I}_{\{\rho=0\}}$. Then $Ez_{x_i}\tilde{x} = 0$ for all i, and so $Ez_x\tilde{x} = 0$ for all $x \in L_+^1$. Hence, $\tilde{x} = 0$ a.s. since for otherwise, $Ez_{\tilde{x}}\tilde{x} > 0$ by the above. It follows that $\rho > 0$ a.s. and the probability measure $\tilde{P} = c\rho P$ with $c = 1/E\rho$ is equivalent to P, $\frac{d\tilde{P}}{dP} \in L^\infty$ and $E_{\tilde{P}}\xi \leq 0$ for all $\xi \in K_N$.

Replacing, if necessary, P by the equivalent probability

$$P' = Ce^{-\sum_{1\leq k\leq N}|S_k|}P$$

we can assume without loss of generality that $S_1, S_2, ..., S_N$ are P-integrable, and so they are also $\tilde{P}$-integrable. Taking $\xi = \xi_k = \pm(\gamma_k, \Delta\tilde{S}_k)$ with bounded predictable $\gamma_1, ..., \gamma_N$ we conclude that $E_{\tilde{P}}\xi \leq 0$ implies that, in fact, $E_{\tilde{P}}\xi = 0$, and so

$$0 = E_{\tilde{P}}(\gamma_k,\, E(\Delta\tilde{S}_k|\mathcal{F}_{k-1})).$$

Since this holds true for any bounded $\mathcal{F}_{k-1}$-measurable d-dimensional random vector γ_k then $E_{\tilde{P}}(\Delta\tilde{S}_k|\mathcal{F}_{k-1}) = 0$ a.s. Thus, $\tilde{P}$ is a martingale measure.

It remains to provide the Halmos-Savage theorem argument while the more technically heavy closedness and separation arguments needed for (1.2) we leave for Appendix. Consider the family $\{yP\}$ where y's are finite convex combinations of z_x's. As always, $\operatorname{esssup}_y \mathbb{I}_{\{y>0\}}$ can be attained on a sequence $\mathbb{I}_{\{\tilde{y}_k>0\}}$, $k = 1, 2, ...$ and taking $y_k = \frac{1}{k}\sum_{i=1}^k \tilde{y}_i$ we obtain that this essential supremum is attained on the increasing sequence $\mathbb{I}_{\{y_k>0\}}$, $k = 1, 2, ...$ Clearly, $\{y_k P\}$ is a countable equivalent subfamily of $\{yP\}$ and each y_k has the form $y_k = m_k^{-1}\sum_{j=1}^{l_k} q_{j,k} z_{x_{j,k}}$ where $x_{j,k}$ are all different, $q_{j,k} \geq 0$ and $\sum_{j=1}^{l_k} q_{j,k} = m_k$. Now, the countable family $\{z_{x_{j,k}}\}$ is an equivalent subfamily of $\{z_x P\}$, as required. This completes the proof of both Halmos-Savage theorem and the first fundamental theorem of asset pricing keeping in mind that the proof of the separation assertion (1.2) will be given in Appendix. $\square$

1.2. **Counterexamples.** (i) In the CRR market we saw that a martingale measure exists and it is unique;

(ii) If $d=\infty$ (infinitely many stocks) then the theorem is not valid since there exists a market without arbitrage opportunities and without martingale measures (see [4], p.415).

(iii) If $N=\infty$ (infinite horizon, perpetual market securities) then the theorem is not valid again since in this case there exists a market with arbitrage opportunities and with a martingale measure (see [4]).

1.3. **Complete and incomplete markets.**

1.4. **Definition.** A financial market defined above on a probability space $(\Omega,\mathcal{F},P)$ with a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ and a horizon N is called complete (or N-complete) if for any $\mathcal{F}_N$-measurable payoff function (claim) $f_N=f_N(\omega)$ there exists a self-financing trading strategy π with an initial capital x so that the corresponding portfolio value satisfies $X_0^\pi=x$ and $X_N^\pi=f_N$ i.e. any (European) contingent claim is replicable (or attainable). Otherwise the market is called incomplete.

Denote by $\mathcal{P}_N(P)$ the set of all martingale measures for the above maket with a horizon N.

1.5. **Theorem.** *(Second fundamental theorem of asset pricing). A financial market defined above without opportunity of arbitrage and with* $d,N<\infty$ *is complete if and only if* $\mathcal{P}_N(P)$ *consists of one measure only.*

Proof. a) Assume that the market is complete. Let $\Gamma\in\mathcal{F}_N$ and define $f_N(\omega)=\mathbb{I}_\Gamma(\omega)$. By completness of the market there exists a self-financing trading strategy π and an initial capital x such that $X_0^\pi=x$ and $X_N^\pi=f_N$. If there exist two martingale measures P_1 and P_2 then $\frac{X_n^\pi}{B_n}$, $n\geq 0$ is a martingale with respect to both P_1 and P_2. Then for $i=1,2$,

$$\frac{x}{B_0}=\frac{X_0^\pi}{B_0}=E_{P_i}\frac{X_N^\pi}{B_N}=E_{P_i}\frac{\mathbb{I}_\Gamma}{B_N}=\int_\Gamma B_N^{-1}dP_i.$$

Hence, $\int_\Gamma B_N^{-1}dP_1=\int_\Gamma B_N^{-1}dP_2$ for any $\Gamma\in\mathcal{F}_N$ and since $B_N>0$ we obtain that $P_1=P_2$. Indeed, if $\mu(\Gamma)=\int_\Gamma B_N^{-1}dP_1=\int_\Gamma B_N^{-1}dP_2$ then $P_1(\Gamma)=\int_\Gamma B_Nd\mu=P_2(\Gamma)$ for any $\Gamma\in\mathcal{F}_N$.

b) It is more difficult to prove the other direction: if there exists only one martingale measure then any payoff function (claim) is attainable or, equivalently, if some $\mathcal{F}_N$-measurable claim cannot be replicated by a self-financing trading strategy then there exists at least two martingale measures. Since the market is without arbitrage then by the first fundamental theorem of asset pricing at least one martingale measure Q exists.

Let L_N be the set of all random variables ξ having the form

$$\xi=\lambda+\sum_{k=1}^N(\gamma_k,\,\Delta\tilde{S}_k)$$

where λ is a real number and γ_k, $k=1,...,N$ is a predictable d-dimensional sequence. If some $\mathcal{F}_N$-measurable claim cannot be replicated then there exists an $\mathcal{F}_N$-measurable random variable $H_N\notin L_N$. As in the proof of the first fundamental theorem of asset pricing we can assume without loss of generality that H_N and $\tilde{S}_k$, $k=0,1,...,N$ are integrable with respect to P since for otherwise we can take the equivalent measure $e^{-|H_N|+\sum_{1\le\le N}|\tilde{S}_k|}P$ and proceed with the proof for the latter measure. By the construction in the proof of the first fundamental theorem of asset pricing we may assume that the Radon-Nikodim derivative $\frac{dQ}{dP}$ is bounded, and so H_N and $\tilde{S}_k$, $k = 0, 1, ..., N$ are integrable with respect to Q, as well.

We will see in Appendix that L_N is closed with respect to the convergence in probability, and so it is also closed with respect to the L^1-covergence. By the above we can assume without loss of generality that $L_N \subset L^1(\Omega, \mathcal{F}_N, Q)$ and since $H_N \notin L_N$ is also assumed to be Q-integrable it will follow by the separation theorem in Appendix that there exists $z \in L^\infty(\Omega, \mathcal{F}_N, Q)$ which separates L_N and H_N, i.e. that

$$E_Q z H_N > E_Q z\ell \tag{1.3}$$

for any $\ell \in L_N$. Since L_N is a subspace, in particular, if $\ell \in L_N$ then $a\ell \in L_N$ for any constant $a \in \mathbb{R}$, then we must have $E_Q z\ell = 0$ for all $\ell \in L_N$.

Since $\gamma_k \equiv 0$ for all k is a predictable sequence then $1 \in L_N$, and so

$$E_Q z \cdot 1 = E_Q z = \int z dQ = 0.$$

Set

$$g = 1 + \frac{z}{2\|z\|_\infty} \ge \frac{1}{2} > 0,$$

where $\|\cdot\|_\infty$ is the L^∞-norm, and

$$\tilde{Q}(\Gamma) = \int_\Gamma g dQ$$

for any $\Gamma \in \mathcal{F}_N$. The Radon-Nikodim derivative $g = \frac{d\tilde{Q}}{dQ}$ is bounded away from zero and infinity, and so integrable random variables with respect to Q and $\tilde{Q}$ are the same.

Next,

$$\tilde{Q}(\Omega) = E_Q 1 + \frac{E_Q z}{2\|z\|_\infty} = 1,$$

and so $\tilde{Q}$ is an equivalent probability measure. Observe that taking $\lambda = 0$ we see that

$$\sum_{k=1}^N (\gamma_k, \Delta\tilde{S}_k) \in L_N$$

for any predictable sequence γ_k, $k = 1, ..., N$. Hence, if γ_k, $k = 1, ..., N$ is predictable and bounded then

$$\begin{aligned} E_{\tilde{Q}}\big(\textstyle\sum_{k=1}^N (\gamma_k, \Delta\tilde{S}_k)\big) &= E_Q\big((1 + \tfrac{z}{2\|z\|_\infty})\textstyle\sum_{k=1}^N (\gamma_k, \Delta\tilde{S}_k)\big) \\ &= E_Q\big(\textstyle\sum_{k=1}^N (\gamma_k, \Delta\tilde{S}_k)\big) = \textstyle\sum_{k=1}^N E_Q(\gamma_k, \Delta\tilde{S}_k). \end{aligned}$$

Since Q is a martingale measure,

$$E_Q(\gamma_k, \Delta\tilde{S}_k) = E_Q(\gamma_k,\ E_Q(\Delta\tilde{S}_k|\mathcal{F}_{k-1})) = 0.$$

Thus,

$$E_{\tilde{Q}}\Big(\sum_{k=1}^N (\gamma_k, \Delta\tilde{S}_k)\Big) = 0$$

for any bounded predictable sequence $\gamma_k = (\gamma_k^{(1)}, ..., \gamma_k^{(d)})$, $k = 1, ..., N$. Taking $\gamma_k^{(i)} = \mathbb{I}_\Gamma$ for an arbitrary $\Gamma \in \mathcal{F}_{k-1}$, $\gamma_k^{(j)} = 0$ for $j \neq i$ and $\gamma_l \equiv 0$ for all $l \neq k$ we obtain

$$E_{\tilde{Q}} \mathbb{I}_\Gamma \Delta \tilde{S}_k^{(i)} = \int_\Gamma (\tilde{S}_k^{(i)} - \tilde{S}_{k-1}^{(i)}) d\tilde{Q} = 0.$$

Since this holds true for any $\Gamma \in \mathcal{F}_{k-1}$ and $i = 1, ..., d$ we obtain by the definition of the conditional expectation that

$$E_{\tilde{Q}}(\tilde{S}_k^{(i)} | \mathcal{F}_{k-1}) = \tilde{S}_{k-1}^{(i)}$$

for all $i = 1, ..., d$. This being true for all $k = 1, ..., N$ yields that $\tilde{Q}$ is a martingale measure different from Q. This completes the proof of the second fundamental theorem of asset pricing keeping in mind that the separation assertion (1.3) will be established in Appendix. □

1.4. **Appendix 1: Finite sample space.** Before considering the general case we will deal with the simpler situation where $\Omega = \{\omega_1, ..., \omega_m\}$ is a finite sample space, and so each random variable ξ on Ω can be described by the vector $(\xi(\omega_1), ..., \xi(\omega_m))$. Then the separation results we relied upon in the first and the second fundamental theorems of asset pricing can be reduced to the following separation hyperplane theorem.

1.6. **Theorem.** *Let $F \subset \mathbb{R}^m$ be a non-empty compact convex set and L be a non-empty linear subspace of $\mathbb{R}^m$ such that $L \cap F = \emptyset$. Then there exists an $(m-1)$-dimensional hyperplane $H \supset L$ such that for some $z \in \mathbb{R}^m \setminus \{0\}$,*

$$H = \{x \in \mathbb{R}^m : (x, z) = 0\} \quad \textit{and} \quad (y, z) > 0 \quad \forall y \in F.$$

Proof. Set

$$G = F - L = \{x \in \mathbb{R}^m : x = f - \ell,\ f \in F,\ \ell \in L\}.$$

Then $0 \notin G$, $G \neq \emptyset$ and it is easy to see that G is a closed convex set. Let $B = \{x \in \mathbb{R}^m : |x| \leq r\}$ be the ball centered at 0 with the radius r large enough so that $B \cap G \neq \emptyset$. Then $B \cap G$ is a closed and bounded subset of $\mathbb{R}^m$, and so it is compact. Thus, $g(x) = |x|$ attains its minimum in $B \cap G$ at a point $z \in B \cap G$. By the construction, $z \neq 0$ and $|z| \leq r$. Clearly, $|x| \geq |z|$ for any $z \in G$ since $|x| = g(x) \geq g(z) = |z|$ for any $x \in B \cap G$ and $|x| > r$ when $x \in G \setminus B$. Since $\lambda x + (1 - \lambda) z \in G$ for any $\lambda \in (0, 1)$ and $x \in G$ by convexity of G we obtain

$$|\lambda x + (1 - \lambda) z|^2 \geq |z|^2 \quad \forall x \in G,\ \lambda \in (0, 1).$$

Hence, $2(1-\lambda)(x, z) - 2|z|^2 + \lambda(|x|^2 + |z|^2) \geq 0$. Letting $\lambda \to 0$ we obtain $(x, z) \geq |z|^2$ which means $((f - \ell), z) \geq |z|^2$ for all $f \in F$ and $\ell \in L$, and so $(\ell, z) \leq (f, z) - |z|^2$. Since this holds true for any vector ℓ in a linear space L, in particular, for $a\ell$, $a \in \mathbb{R}$, we must have $(\ell, z) = 0$ for each $\ell \in L$ and $(f, z) \geq |z|^2 > 0$. This completes the proof of the theorem defining $H = \{x \in \mathbb{R}^m : (x, z) = 0\}$. □

In the application of this result to the first fundamental theorem of asset pricing we recall that the adjusted stock values $\tilde{S}_n = \frac{S_n}{B_n}$ and the coefficients γ_n, $n = 1, ..., N$ are d-dimensional random vectors and since $\Omega = \{\omega_1, ..., \omega_m\}$ we can view now the set M_N defined at the beginning of the proof as a linear subspace of $\mathbb{R}^m$ containing (non-random) vectors of the form $\xi = \sum_{k=1}^N (\gamma_k, \Delta \tilde{S}_k)$ where $\xi =$

$(\xi(\omega_1),...,\xi(\omega_m))$ and $(\gamma_k,\Delta\tilde S_k) = \big((\gamma_k(\omega_1),\Delta\tilde S_k(\omega_1)),...,(\gamma_k(\omega_m),\Delta\tilde S_k(\omega_m))\big)$. Let

$$F=\{\zeta=(\zeta_1,...,\zeta_m)\in\mathbb{R}^m:\,\zeta_i\geq 0,\,\sum_{i=1}^m\zeta_i=1\}$$

then assuming no arbitrage $M_N\cap F=0$. Hence, by the separation hyperplane theorem there exists $z\in\mathbb{R}^m\setminus\{0\}$ such that $\langle\xi,z\rangle=0$ for any $\xi\in M_N$ and $\langle\zeta,z\rangle>0$ for any $\zeta\in F$, where $\langle x,y\rangle=\sum_{i=1}^m x_iy_i$ for $x=(x_1,...,x_m)$, $y=(y_1,...,y_m)\in\mathbb{R}^m$. Since $\zeta^{(i)}=(\zeta_1,...,\zeta_m)\in F$ when $\zeta_i=1$ and $\zeta_j=0$ for $j\neq i$ we obtain that $z=(z_1,...,z_m)$ has $z_j>0$ for all $j=1,...,m$. Set

$$\tilde P(\{\omega+i\})=\frac{z_i}{\sum_{j=1}^m z_j},\,i=1,...,m.$$

Then for any $\xi\in M_N$,

$$E_{\tilde P}\xi=(\sum_{i=1}^m z_i)^{-1}\langle\xi,z\rangle=0.$$

In particular, this holds true for any $\xi=(\gamma_k,\Delta\tilde S_k)$ with an arbitrary bounded γ_k measurable with respect to $\mathcal{F}_{k-1}$, and so

$$0=E_{\tilde P}(\gamma_k,\Delta\tilde S_k)=E_{\tilde P}(\gamma_k,E_{\tilde P}(\Delta\tilde S_k|\mathcal{F}_{k-1})).$$

It follows that $E_{\tilde P}(\Delta\tilde S_k|\mathcal{F}_{k-1})=0$ for $k=1,2,...,N$ implying that $\tilde P$ is a martingale measure, completing the proof of the first fundamental theorem of asset pricing for the case of a finite sample space Ω.

For the second fundamental theorem of asset pricing in this case we apply the separating hyperplane theorem directly taking L as a subspace of $\mathbb{R}^m$ containing vectors of the form $(\lambda+\sum_{k=1}^N(\gamma_k,\Delta\tilde S_k)(\omega_i),\,i=1,...,m)$ with $\lambda\in\mathbb{R}$ and predictable sequences $(\gamma_k,\,k=1,...,N)$. Choosing F to be the single vector $H_N=(H_N(\omega_1),...,H_N(\omega_m))$ which is a non replicable claim, we find $z\in\mathbb{R}^m$ such that $\langle z,H_N\rangle>\langle z,\ell\rangle$ for any $\ell\in L$ and proceed with the proof as in the main part of the lecture.

1.5. **Appendix 2: A closedness theorem.**

1.7. **Theorem.** *(i) The set of random variables* $M_N=\{\xi:\,\xi=\sum_{k=1}^N(\gamma_k,\Delta\tilde S_k),\,\gamma_k,\,k=1,...,N\}$ *is closed with respect to convergence in probability;*

(ii) The set of random variables $L_N=\{\lambda+\xi:\,\xi\in M_N,\,\lambda\in\mathbb{R}\}$ *is closed with respect to convergence in probability;*

(iii) Under the no arbitrage condition the set of random variables $K_N=M_N-L^0_+=\{\eta=\xi-\zeta:\,\xi\in M_N,\,\zeta\in L^0_+\}$, *where* L^0_+ *is the set of all non-negative* $\mathcal{F}_N$-*measurable random variables, is closed with respect to convergence in probability.*

Proof. We will, first, prove (iii) following [3] indicating in appropriate places that (i) follows by simplifying the same proof. At the end we will add arguments which will yield (ii). Observe, that if Ω is a finite sample space then M_N and L_N can be viewed as linear subspaces of a finite dimensional Euclidean space, and so they are closed, while K_N is a convex cone there which, as can be easily seen, is also closed. The proof of the general case we begin with the following result from [3].

1.8. **Lemma.** *For any sequence η^n, $n\geq 1$ of random d-dimensional vectors such that $\eta=\liminf_{n\to\infty}|\eta^n|<\infty$ we can choose another sequence $\tilde\eta^k$, $k\geq 1$ of random d-dimensional vectors such that for any ω the sequence $\tilde\eta^k(\omega)$, $k\geq 1$ is a convergent subsequence of $\eta^n(\omega)$, $n\geq 1$ (though for different ω's the subsequences may be different) and if $k(\omega)=n$ whenever $\tilde\eta^{k(\omega)}(\omega)=\eta^n(\omega)$ then $k=k(\omega)$ is measurable.*

Proof. Set $\tau_0=0$ and recursively,

$$\tau_k=\min\{n\,\tau_{k-1}:\,||\eta^n|-\eta|\leq k^{-1}\}.$$

Let $\tilde\eta_0^k=\eta^{\tau_k}$ then $\sup_k|\tilde\eta_0^k|<\infty$. Next, let $\tilde\eta_0^k=(\tilde\eta_0^k,\tilde\eta_{0,2}^k,...,\tilde\eta_{0,d}^k)$ and set $\tilde\eta_{0,1}=\liminf_{k\to\infty}\tilde\eta_{0,1}^k$. Define $\tau_0'=0$ and recursively, $\tau_k'=\min\{n>\tau_{k-1}':\,|\tilde\eta_{0,1}-\tilde\eta_{0,1}|\leq k^{-1}\}$. Now, set $\tilde\eta_1^k=\tilde\eta_0^{\tau_k'}$. Clearly, $\tilde\eta_1^k(\omega)$, $k\geq 1$ is a subsequence of $\eta^n(\omega)$, $n\geq 1$ such that its first component $\tilde\eta_{1,1}^k(\omega)$ converges. Furthermore, if $k(\omega)=n$ when $\tilde\eta_1^{k(\omega)}(\omega)=\eta^n(\omega)$ then $k=k(\omega)$ is measurable. Next, we apply this procedure to the second component of $\tilde\eta_1^k$ choosing a subsequence $\tilde\eta_2^k$ where both first and second components converge. Repeating this d times in total we obtain a sequence $\tilde\eta^k$, $k\geq 1$ with required properties. □

Next, we return to the proof of closedness of K_N with respect to convergence in probability assuming that there is no arbitrage. We proceed by induction taking, first, $N=1$. Since from every convergent in probability sequence we can choose an a.s. convergent subsequence we can take

$$(\gamma_1^n,\,\Delta\tilde S_1)-r^n\to\zeta\quad\text{a.s. as }\;n\to\infty$$

where γ_1^n is $\mathcal F_0$-measurable and $r^n\in L_+^0$. It suffices to find $\mathcal F_0$-measurable random variables $\tilde\gamma_1^k$ and $\tilde r^k\in L_+^0$ such that $\tilde\gamma_1^k\to\tilde\gamma_1$ a.s. and $(\tilde\gamma_1^k,\Delta\tilde S_1)-\tilde r^k\to\zeta$ a.s. as $k\to\infty$ since then $\tilde r^k\to(\tilde\gamma_1,\Delta\tilde S_1)-\zeta\in L_+^0$, and so $\zeta\in K_1$. Clearly, if $\Omega_1,...,\Omega_l\in\mathcal F_0$ is a finite partition of Ω then it suffices to construct the required sequence $\tilde\gamma_1^k$ and $\tilde r^k$ on each Ω_j separately. Observe that in proving closedness of M_N we have $r^n\equiv 0$ for all n, and so we can view this as $r^n\to 0$ as $n\to\infty$ which holds true automatically, i.e. in this case we do not have to deal with $\tilde r^k$'s at all.

Let $\gamma_1=\liminf_{n\to\infty}|\gamma_1^n|$. On the set $\Omega_1=\{\gamma_1<\infty\}$ we can use the above lemma to construct $\mathcal F_0$-measurable $\tilde\gamma_1^k$ such that $\tilde\gamma_1^k(\omega)$ is a convergent subsequence of $\gamma_1^n(\omega)$ for every $\omega\in\Omega_1$ and if $\tilde\gamma_1^k(\omega)=\gamma_1^n(\omega)$ then we take $\tilde r^k(\omega)=r^n(\omega)$. If $P(\Omega_1)=1$ then we are done and if not we consider $\Omega_2=\{\gamma_1=\infty\}$. On Ω_2 set $\eta_1^n=\frac{\gamma_1^n}{|\gamma_1^n|}$ and $h_1^n=\frac{r_1^n}{|\gamma_1^n|}$ and observe that $(\eta_1^n,\Delta\tilde S_1)-h_1^n\to 0$ a.s. By the above lemma we obtain a $\mathcal F_0$-measurable $\tilde\eta_1^k$ such that $\tilde\eta_1^k(\omega)$ is a convergent subsequence of $\eta_1^n(\omega)$ for any ω. Denoting the limit by $\tilde\eta_1$, we obtain that $(\tilde\eta_1,\Delta\tilde S_1)=\tilde h_1$ where $\tilde h_1$ is non-negative, and so by the no arbitrage assumption $K_N\cap L_+^0=\{0\}$ we have $(\tilde\eta_1,\Delta\tilde S_1)=0$. Observe that since $r_1^n=0$ for all n in the proof of closedness of M_N we obtain $(\tilde\eta_1,\Delta\tilde S_1)=0$ automatically without any need in the no arbitrage assumption.

Next, if $d=1$ then $\Delta\tilde S_1=0$ a.s. since $|\tilde\eta_1|=1$. But in this case $-r^n\to\zeta$ a.s. as $n\to\infty$, and so $-\zeta\in L_+^0$, and so $-\zeta\in L_+^0$, whence $\zeta=0-(-\zeta)\in K_1$. Now, suppose that $d>1$. Then we proceed by induction decreasing the dimension on each step. Suppose that we constructed the required sequences when the dimension does not exceed $d-1$. Now let our random vectors have the dimension d. Since $|\tilde\eta_1|=1$ we can partition Ω_2 into d disjoint subsets $\Omega_2^i\in\mathcal F_0$, $i=1,...,d$ such that the i-th component $\tilde{\eta}_1^{(i)}$ of $\tilde{\eta}_1$ does not vanish on Ω_2^i. For each i define $\bar{\gamma}_1^n$ on Ω_2^i by $\bar{\gamma}_1^n = \gamma_1^n - \frac{\gamma_1^{ni}}{\tilde{\eta}_1^{(i)}}\tilde{\eta}_1$ where γ_1^{ni} is the i-th component of the d-dimensional random vector γ_1^n. Then

$$(\bar{\gamma}_1^n, \Delta\tilde{S}_1) = \sum_{j=1}^{d}(\gamma_1^{nj} - \frac{\tilde{\eta}_1^{(j)}\gamma_1^{ni}}{\tilde{\eta}_1^{(i)}})\Delta\tilde{S}_1^j = (\gamma_1^n, \Delta\tilde{S}_1)$$

since $(\tilde{\eta}_1, \Delta\tilde{S}_1) = 0$ as proved above. Now on each Ω_2^i we have, essentially, $(d-1)$-dimensional vectors $\bar{\gamma}_1^n$ and $\Delta\tilde{S}_1$ since $\bar{\gamma}_1^{ni} = 0$ on Ω_2^i, and so the i-th component $\Delta\tilde{S}_1^i$ of $\Delta\tilde{S}_1$ plays no role in the scalar product $(\bar{\gamma}_1^n, \Delta\tilde{S}_1)$, as well. In view of the above equality

$$(\bar{\gamma}_1^n, \Delta\tilde{S}_1) - r^n \to \zeta \quad \text{a.s. as } n \to \infty,$$

and so we decreased the dimension accomplishing the induction step.

Thus, we derived the closedness assertion for M_N and K_N when $N = 1$ and in order to complete the proof for any N we assume the validity of the assertion for $N-1$ and show that it remains true for N. Thus, let

$$\sum_{j=1}^{N-1}(\gamma_j^n, \Delta\tilde{S}_j) - r^n \to \zeta \quad \text{a.s. as } n \to \infty$$

where γ_j^n are $\mathcal{F}_{j-1}$-measurable and $r^n \in L_+^0$. Consider $(\gamma_N^n, \Delta\tilde{S}_N)$ and set $\gamma_N = \liminf_{n\to\infty}|\gamma_N^n|$. On the set $\Omega_1 = \{\gamma_N < \infty\}$ by the above lemma we can choose $\mathcal{F}_{N-1}$-measurable $\tilde{\gamma}_N^k$ such that $\tilde{\gamma}_N^k(\omega)$ is a convergent subsequence of $\gamma_N^n(\omega)$ for every ω. Then $\tilde{\gamma}_N^k \to \eta$ a.s. as $k \to \infty$ where η is $\mathcal{F}_{N-1}$-measurable, and so

$$\sum_{j=1}^{N-1}(\gamma_j^k, \Delta\tilde{S}_j) + (\tilde{\gamma}_N^k, \Delta\tilde{S}_N) \to \zeta + (\eta, \Delta S_N) \text{ a.s. as } k \to \infty.$$

Observe that $(\eta, \Delta\tilde{S}_N) \in K_N$ since we can take zero coefficients in $\Delta\tilde{S}_j$ for $j = 1, ..., N-1$. Furthermore, $\zeta \in K_N$ since $\zeta \in K_{N-1}$ by the induction hypothesis and $\zeta = \zeta + (0, \Delta\tilde{S}_N) \in K_N$. Hence, $\zeta + (\eta, \Delta\tilde{S}_N) \in K_N$.

On $\Omega_2 = \{\gamma_N = \infty\}$ we proceed as in the case $N = 1$ replacing γ_N^n by $\bar{\gamma}_N^n$ which is a random vector having less nonzero components than γ_N^n on elements of a finite partition of Ω_2. Proceeding in the same way we eliminate γ_N^n complitely on elements of a certain finite partition where we reduce the situation to the sum $\sum_{j=1}^{N-1}(\gamma_j^n, \Delta\tilde{S}_j)$ which completes the proof of the assertions 9i) and (iii) of the theorem in view of the induction hypothesis.

It remains to provide arguments which yield the assertion (ii) of the theorem. Let

$$\lambda_n + \xi_n \to \zeta \text{ a.s. as } n \to \infty$$

where $\lambda_n \in \mathbb{R}$ and $\xi_n \in M_N$, $n = 1, 2, ...$ If $1 \in M_N$ then $\lambda_n + \xi_n \in M_N$, and so $\zeta \in M_N$ since we proved that M_N is closed. Now suppose that $1 \notin M_N$. If $\liminf_{n\to\infty}|\lambda_n| = 0$ then we can choose a subsequence $\lambda_{n_i} \to 0$ as $i \to \infty$ and then $\xi_{n_i} \to \zeta$ as $i \to \infty$ which means that $\zeta \in L_N$ since M_N is closed. Now, suppose that $\liminf_{n\to\infty}|\lambda_n| > 0$. Then there exist $a > 0$ and n_0 such that $|\lambda_n \geq a$ when $n \geq n_0$. Now, for any $n \geq n_0$ we define $\beta_n = -\frac{\xi_n}{\lambda_n}$ and since M_N is a linear space $\beta_n \in M_N$.

For each $\varepsilon > 0$ set $\Gamma_{n,\varepsilon} = \{\omega : |\beta_n(\omega) - 1| \geq \varepsilon\}$. If $P(\Gamma_{n,\varepsilon}) \to 0$ then we can choose a subsequence n_i such that $\sum_i P(\Gamma_{n_i,\varepsilon}) < \infty$, and so by the Borel-Cantelli lemma there exists a random $N_\varepsilon < \infty$ a.s. such that $|\beta_n(\omega) - 1| < \varepsilon$ for all $n \geq n_\varepsilon(\omega)$. If this holds true for any $\varepsilon > 0$ then taking a sequence $\varepsilon_k \to 0$ as $k \to \infty$ we can use the diagonal procedure to obtain a subsequence n_j such that $\beta_{n_j} \to 1$ a.s. But $\beta_{n_j} \in M_N$, $1 \notin M_N$ and M_N is closed, whence this cannot happen. It follows that there exist $\varepsilon, \delta > 0$ and a subsequence $n_j \to \infty$ as $j \to \infty$ such that

$$P(\Gamma_{n_j,\varepsilon}) \geq \delta \text{ for all } j.$$

Set $\Omega_D = \{\omega : |\zeta(\omega)| \leq D\}$ and $\Omega_{D,m} = \{\omega : |\lambda_n + \xi_n(\omega)| \leq 2D \text{ for all } n \geq m\}$. Then we can choose D and m so that

$$P(\Omega_{D,m}) > 1 - \delta.$$

But for all $n_j \geq m$ and $\omega \in \Omega_{D,m} \cap \Gamma_{n_j,\varepsilon} \neq \emptyset$,

$$2D \geq |\lambda_{n_j} + \xi_{n_j}(\omega)| = |\lambda_{n_j}||1 - \beta_{n_j}(\omega)| \geq \varepsilon|\lambda_{n_j}|,$$

i.e. $|\lambda_{n_j}| \leq 2D\varepsilon^{-1}$ for all $n_j \geq m$. Hence, we can choose a converging subsequence $\lambda_{n_{j_i}} \to \lambda$ as $i \to \infty$ and then $\xi_{n_{j_i}} \to \xi = \zeta - \lambda \in M_N$ as $i \to \infty$ since M_N is closed. Hence, $\zeta = \lambda + \xi \in L_N$, and so L_N is closed, as well. □

1.6. **Appendix 3: Separation.** The result we relied upon in the proof of both fundamental theorems of asset pricing is the following

1.9. **Theorem.** *(Separation theorem) Let M be a closed convex set in $L^1 = L^1(\Omega, \mathcal{F}, P)$. Then for any $x \in L^1 \setminus M$ there exists $z_x \in L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ such that*

$$Ez_x\xi < Ez_x x$$

for all $\xi \in M$.

Proof. First, observe that when Ω is a finite sample space then the proof reduces to more elementary arguments about a separating hyperplane in the finite dimensional Euclidean space (see [5]). For the general case, recall first that the dual space to $L^1(\Omega, \mathcal{F}, P)$ can be identified with $L^\infty(\Omega, cF, P)$. Indeed, if $z \in L^\infty(\Omega, \mathcal{F}, P)$ then

$$\ell(\zeta) = Ez\zeta, \ \zeta \in L^1(\Omega, \mathcal{F}, P)$$

defines a bounded linear functional on $L^1(\Omega, \mathcal{F}, P)$.

On the other hand, if ℓ is a bounded linear functional on $L^1(\Omega, \mathcal{F}, P)$ then for any $\Gamma \in \mathcal{F}$ we set

$$\mu(\Gamma) = \ell(\mathbb{I}_\Gamma)$$

which defines a finitely additive measure on $\mathcal{F}$, which, in fact, is σ-additive since ℓ is bounded, and it is also absolutely continuous with respect to P. Indeed, if $P(\Gamma) = 0$ then by boundedness of ℓ,

$$\ell(\mathbb{I}_\Gamma) \leq \|\ell\| \int \mathbb{I}_\Gamma dP = 0.$$

Hence, by the Radon-Nikodim theorem there exists a function $g \in L^1$ such that for any $\Gamma \in \mathcal{F}$, $\ell(\mathbb{I}_\Gamma) = \int_\Gamma g dP$. Then

$$|\int_\Gamma g dP| \leq \|\ell\| P(\Gamma),$$

and so $g \le \|\ell\|$ with probability one. It follows that $\ell(\zeta) = \int \zeta g dP$ for any $\zeta \in L^1(\Omega, \mathcal{F}, P)$ with $g \in L^\infty(\Omega, \mathcal{F}, P)$, as required.

Thus, we conclude that in order to prove the separation theorem we have to show that for any $x \in L^1 \setminus M$ there exists a bounded linear functional ℓ_x on L^1 such that

$$\ell_x(\xi) < \ell_x(x) \tag{1.4}$$

for all $\xi \in M$. Without loss of generality we can assume that $0 \in M$ since for otherwise we can take $M - y$ in place of M and $x - y$ in place of x for some $y \in M$. Let $d_x > 0$ be the distance between x and M and let $U_0(r)$ be the open ball centered at 0 with the radius $r < \frac{1}{2}d_x$. Then $U = M + U_0(r)$ and $V_x = x + U_0(r)$ are disjoint open convex sets (neighborhoods) containing M and x, respectively.

Recall, that the Minkovski functional of U is a function on our space L^1 defined by

$$p_U(y) = \inf\{r > 0 : \frac{y}{r} \in U\}.$$

Let $x \notin M$ and U, V_x be as above then the distance between x and $\bar{U}$ is positive, and so it is easy to see that $p_U(x) > 1$. On the one-dimensional space $L_0 = \{\alpha x,\, \alpha \in \mathbb{R}\}$ define the linear functional

$$\ell_0(\alpha x) = \alpha p_U(x)$$

which satisfies

$$\ell_0(\alpha x) \le p_U(\alpha x) \quad \text{for all } \alpha \in \mathbb{R}$$

since

$$p_U(\alpha x) = \alpha p_U(x) \text{ if } \alpha \ge 0 \text{ and } \ell_0(\alpha x) = \alpha \ell_0(x) < 0 \le p_U(\alpha x) \text{ if } \alpha < 0.$$

It is easy to see that the Minkovski functional is nonnegative, convex and positively homogeneous, i.e.

$$p_U \ge 0,\ p_U(y + z) \le p_U(y) + p_U(z) \text{ and } p_U(\alpha y) = \alpha p_U(y) \text{ for all } \alpha > 0.$$

Now we can apply the Hahn-Banach theorem (which will be discussed below) in order to extend the functional ℓ_0 to a linear functional ℓ_x on the whole space L^1 so that

$$\ell(\alpha x) = \ell_0(\alpha x) \text{ for all } \alpha \in \mathbb{R} \text{ and } \ell_x(y) \le p_U(y) \text{ for all } y \in L^1.$$

Then $\ell_x(y) \le p_U(y)$ for $y \in U \supset M$ and $\ell_x(x) = p_U(x) > 1$, and so (1.4) holds true. □

Next, for readers' convenience we will formulate and prove the Hahn-Banach theorem.

1.10. **Theorem.** *(Hahn-Banach) Let p be a nonnegative, convex and positively homogeneous real valued functional defined on a real linear space L and L_0 be a linear subspace of L. If ℓ_0 is a linear functional on L_0 satisfying $\ell_0(x) \le p(x)$ for all $x \in L_0$ then ℓ_0 can be extended to a linear functional ℓ on L such that*

$$\ell(y) = \ell_0(y) \textit{ for all } y \in L_0 \textit{ and } \ell(y) \le p(y) \textit{ for all } y \in L.$$

Proof. If $L_0 = L$ there is nothing to prove and if $L_0 \ne L$ then there exists $z \in L \setminus L_0$, $z \ne 0$. Let L_1 be the linear space generated by L_0 and z, i.e. $L_1 = \{tz + y : t \in \mathbb{R},\, y \in L_0\}$. Set $\ell_1(tz + y) = t\ell_1(z) + \ell_0(y)$, $y \in L_0$ which is a linear functional on L_1 and we only have to specify $\ell_1(z)$ so that

$$t\ell_1(z) + \ell_0(y) \le p(tz + y). \tag{1.5}$$

Since p is positively homogeneous this is equivalent to

$$\ell_1(z) \leq p(\frac{y}{t} + z) - \ell_0(\frac{y}{t}) \text{ if } t > 0 \text{ and } \ell_1(z) \geq -p(-\frac{y}{t} - z) - \ell_0(\frac{y}{t}) \text{ if } t < 0.$$

Since by linearity of ℓ_0 and by convexity of p for any $y_1, y_2 \in L_0$,

$$\ell_0(y_2) - \ell_0(y_1) \leq p(y_2 - y_1) \leq p(y_2 + z) + p(-y_1 - z),$$

we obtain

$$-\ell_0(y_2) + p(y_2 + z) \geq -\ell_0(y_1) - p(-y_1 - z).$$

Set

$$c_1 = \sup_{y \in L_0} (-\ell_0(y) - p(-y - z)) \text{ and } c_2 = \inf_{y \in L_0} (-\ell_0(y) + p(y + z)).$$

By the above, $c_2 \geq c_1$, and so we can set $\ell_1(z) = c$ for some c satisfying $c_2 \geq c \geq c_1$ and then (1.5) will hold true.

If the probability space $(\Omega, \mathcal{F}, P)$ is separable, i.e. the σ-algebra $\mathcal{F}$ is generated by a countable collection of sets, then $L^1(\Omega, \mathcal{F}, P)$ is a separable space. In general, if L is separable we can choose a countable collection $z_1, z_2, ... \in L$ which generates L (i.e. L is the minimal linear space containing $z_1, z_2, ...$) and then we can construct the required functional ℓ by extending it as above successively to the increasing sequence of subspaces $L^{(1)} = \{L_0, z\}$, $L^{(2)} = \{L^{(1)}, z_2\}, ...$ where $L^{(k)}$ is the minimal linear subspace of L containing $L^{(k-1)}$ and z_k, $k = 1, 2, ...$ Then any $y \in L$ belongs to some $L^{(k)}$, and so ℓ will be extended to the whole L with the condition $\ell(y) \leq p(y)$ preserved.

In the non separable case we have to consider the set $\mathcal{T}$ of all extensions ℓ satisfying $\ell \leq p$ with the partial order determined by inclusion of the corresponding linear subspaces where the extensions are defined. Each linearly ordered subset $\mathcal{T}_0$ of $\mathcal{T}$ has an upper bound taking the union of the increasing sequence of corresponding subspaces. By the Zorn lemma there exists a maximal element in $\mathcal{T}$ which is the required extension ℓ of ℓ_0 to the whole L. □

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