A NONCONVENTIONAL INVARIANCE PRINCIPLE FOR RANDOM FIELDS

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ABSTRACT. In [16] we obtained a nonconventional invariance principle (functional central limit theorem) for sufficiently fast mixing stochastic processes with discrete and continuous time. In this paper we derive a nonconventional invariance principle for sufficiently well mixing random fields.

1. INTRODUCTION

Nonconventional ergodic theorems (see [12]) known also after [3] as polinomial ergodic theorems studied the limits of expressions having the form $1/N\sum_{n=1}^{N}T^{q_1(n)}f_1\cdots T^{q_\ell(n)}f_\ell$ where T is a weakly mixing measure preserving transformation, f_i's are bounded measurable functions and q_i's are polynomials taking on integer values on the integers. Originally, these results were motivated by applications to multiple recurrence for dynamical systems taking functions f_i being indicators of some measurable sets. Later such results were extended to the case when q_i's are polinomials on $\mathbb{Z}^\nu$ (see [17]) and to some $\mathbb{Z}^\nu$ actions (see [2]).

Using the language of probability this kind of results may be called nonconventional laws of large numbers and as a natural follow up we arrived at the invariance principle (functional central limit theorem) in [16] showing convergence in distribution to Gaussian processes for expressions of the form

$$1/\sqrt{N}\sum_{0\leq n\leq Nt}\big(F\big(X(q_1(n)),...,X(q_\ell(n))\big)-\bar{F}\big) \tag{1.1}$$

where $X(n),n\geq 0$ is a sufficiently fast α,ρ or ψ-mixing vector valued process with some moment conditions and stationarity properties, F is a continuous function with polinomial growth and certain regularity properties, $\bar{F}=\int Fd(\mu\times\cdots\times\mu)$, μ is the distribution of each $X(n)$, $q_j(n)=jn$, $j\leq k$ and $q_j,j>k$ are positive functions taking on integer values on integers with some growth conditions which are satisfied, for instance, when q_i's are polynomials of growing degrees.

The goal of this paper is to prove an invariance principle type result when $n\in\mathbb{Z}^\nu$ is multidimensional. This can be done either by considering functions $q_i:\mathbb{Z}^\nu\to\mathbb{Z}_+$ with $X(n),n\geq 0$ being again a vector valued stochastic process or, more generally, considering maps $q_i:\mathbb{Z}^\nu\to\mathbb{Z}^\nu$ taking now $X(n),n\in\mathbb{Z}^\nu$ to be a vector valued

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random field which will be our setup in this paper. Namely, for $t=(t_1,...,t_\nu)\in[0,1]^\nu$ and a positive integer N we consider expressions of the form

$$\xi_N(t)=N^{-\nu/2}\sum_{n=(n_1,...,n_\nu):0\le n_i\le Nt_i\,\forall i}\big(F\big(X(q_1(n)),...,X(q_\ell(n))\big)-\bar F\big) \tag{1.2}$$

where $X(n),n\in\mathbb{Z}^\nu$ is a sufficiently well mixing vector valued random field, with some moment conditions and stationarity properties, F and $\bar F$ are similar to above, $q_j(n)=jn$, $j\le k$ and $q_i:\mathbb{Z}^\nu\to\mathbb{Z}^\nu$, $i=k+1,...,\ell$ map $\mathbb{Z}^\nu_+=\{n=(n_1,...,n_\nu)\in\mathbb{Z}^\nu:\,n_i\ge 0,\,i=1,...,\nu\}$ into itself. Assuming some growth conditions of $|q_i|$, $i>k$ in $|n|$ we will show that the random field $\xi_N(t)$ converges in distribution to a Gaussian random field on $[0,1]^\nu$.

In [16] we were able to obtain the latter result for one dimensional n relying on martingale approximations and martingale limit theorems but for random fields this machinery is not readily available. Still, we are able to combine some of mixingale technique from [18] and [19] together with an appropriate grouping of summands in (1.2) in order to obtain both convergence of finite dimensional distributions and the tightness of infinite dimensional ones. Other known methods which work successfully when proving limit theorems for random fields (see, for instance, [5], [7], [8] and [20]) rely one way or another on characteristic functions (or other devices based on weak dependence) which are hard to deal with in the nonconventional setup as demonstrated in [14] in view of the strong dependence of the summands in (1.2) on the far away members of the random field. For specific lattice models with sufficiently good mixing properties to fit our setup we refer the reader to [1] and references there.

2. Preliminaries and main results

Our setup consists of a $\wp$-dimensional random field $\{X(n),\,n\in\mathbb{Z}^\nu\}$ on a probability space $(\Omega,\mathcal{F},P)$ and of a family of σ-algebras $\mathcal{F}_\Gamma\subset\mathcal{F}$, $\Gamma\subset\mathbb{Z}^\nu$ such that $\mathcal{F}_\Gamma\subset\mathcal{F}_\Delta$ if $\Gamma\subset\Delta\subset\mathbb{Z}^\nu$. It is often convenient to measure the dependence between two sub σ-algebras $\mathcal{G},\mathcal{H}\subset\mathcal{F}$ via the quantities

$$\varpi_{q,p}(\mathcal{G},\mathcal{H})=\sup\{\|E\big(g|\mathcal{G}\big)-Eg\|_p:\,g\text{ is }\mathcal{H}-\text{measurable and }\|g\|_q\le 1\} \tag{2.1}$$

where the supremum is taken over real functions and $\|\cdot\|_r$ is the $L^r(\Omega,\mathcal{F},P)$-norm. Then more familiar α,ρ,ϕ and ψ-mixing (dependence) coefficients can be expressed via the formulas (see [6], Ch. 4),

$$\alpha(\mathcal{G},\mathcal{H})=\frac14\varpi_{\infty,1}(\mathcal{G},\mathcal{H}),\;\rho(\mathcal{G},\mathcal{H})=\varpi_{2,2}(\mathcal{G},\mathcal{H})$$
$$\phi(\mathcal{G},\mathcal{H})=\frac12\varpi_{\infty,\infty}(\mathcal{G},\mathcal{H})\text{ and }\psi(\mathcal{G},\mathcal{H})=\varpi_{1,\infty}(\mathcal{G},\mathcal{H}).$$

We set also

$$\varpi_{q,p}(r)=\sup_{\Gamma,\tilde\Gamma:\,dist(\Gamma,\tilde\Gamma)\ge r}\big(|\Gamma\cup\tilde\Gamma|^{-1}\varpi_{q,p}(\mathcal{F}_\Gamma,\mathcal{F}_{\tilde\Gamma})\big) \tag{2.2}$$

where Γ and Δ are finite nonempty subsets of $\mathbb{Z}^\nu$, $dist(\Gamma,\Delta)=\inf_{n\in\Gamma,\tilde n\in\Delta}|n-\tilde n|$ and we write $|\Gamma|$ for cardinality of a set $|\Gamma|$ while, as usual, for numbers or vectors $|\cdot|$ will denote their absolute values or lengths. As shown in [10] imposing decay conditions on dependence coefficients which do not take into account sizes of sets Γ and Δ as in (2.2) would exclude from our setup simple examples of Gibbs random fields. Define also

$$\alpha(l)=\frac{1}{4}\varpi_{\infty,1}(l),\ \rho(l)=\varpi_{2,2}(l),\ \phi(l)=\varpi_{\infty,\infty}(l)\text{ and }\ \psi(l)=\varpi_{1,\infty}(l).$$

Our setup includes also conditions on the approximation rate

$$\beta_p(r)=\sup_{n\in\mathbb{Z}^\nu}\|X(n)-E\big(X(k)|\mathcal{F}_{U_r(n)}\big)\|_p \tag{2.3}$$

where $U_r(n)=\{\tilde n\in\mathbb{Z}^\nu:\,|n-\tilde n|\le r\}$ is the r-neghborhood on n in $\mathbb{Z}^\nu$. Furthermore, we do not require stationarity of the random field $X(n),n\in\mathbb{Z}^\nu$ assuming only that the distribution of $X(n)$ does not depend on n and the joint distribution of $\{X(n),X(n')\}$ depends only on $n-n'$ which we write for further references by

$$X(n)\overset{d}{\sim}\mu\text{ and }(X(n),X(n'))\overset{d}{\sim}\mu_{n-n'}\text{ for all }n,n' \tag{2.4}$$

where $Y\overset{d}{\sim}Z$ means that Y and Z have the same distribution.

Next, let $F=F(x_1,...,x_\ell)$, $x_j\in\mathbb{R}^\wp$ be a function on $\mathbb{R}^{\wp\ell}$ such that for some $\iota,K>0,\kappa\in(0,1]$ and all $x_i,y_i\in\mathbb{R}^\wp$, $i=1,...,\ell$, we have

$$|F(x_1,...,x_\ell)-F(y_1,...,y_\ell)|\le K\big(1+\sum_{j=1}^\ell|x_j|^\iota+\sum_{j=1}^\ell|y_j|^\iota\big)\sum_{j=1}^\ell|x_j-y_j|^\kappa \tag{2.5}$$

and

$$|F(x_1,...,x_\ell)|\le K\big(1+\sum_{j=1}^\ell|x_j|^\iota\big). \tag{2.6}$$

Our assumptions on F enable us to include, for instance, products $F(x_1,...,x_\ell)=x_{11}x_{22}\cdots x_{\ell\ell}$, where $x_i=(x_{i1},...,x_{i\ell})\in\mathbb{R}^\ell$, which is sometimes useful. To simplify formulas we assume a centering condition

$$\bar F=\int F(x_1,...,x_\ell)\,d\mu(x_1)\cdots d\mu(x_\ell)=0 \tag{2.7}$$

which is not really a restriction since we always can replace F by $F-\bar F$.

Our goal is to prove an invariance principle (functional central limit theorem) for $\xi_N(t)$, $t\in[0,1]^\nu$ defined by (1.2) where $q_j(n)=jn$ for $j=1,2,...,k$ and $q_j:\mathbb{Z}^\nu\to\mathbb{Z}^\nu$, $j=k+1,...,\ell$ satisfy the following conditions. First, we assume that $|q_1(n)|<|q_2(n)|<\cdots<|q_\ell(n)|$ and $|q_i(n)|<|q_i(\tilde n)|$ for each i if $|n|<|\tilde n|$ whenever $n,\tilde n\in\mathbb{Z}^\nu_+=\{m=(m_1,...,m_\nu)\in\mathbb{Z}^\nu:\,m_i\ge 0\,\forall i\}$. Furthermore, we assume that for $k+1\le i\le\ell$,

$$\lim_{|n|\to\infty}\inf_{\tilde n:\tilde n\ne n}\big(|q_i(\tilde n)-q_i(n)|-|\tilde n-n|\big)=\infty, \tag{2.8}$$

$$q_i(n)\ne q_i(\tilde n)\text{ if }n\ne\tilde n,\ \lim_{|n|\to\infty}\min_{l<i}\big(|q_i(n)|-|q_l(n)|-|n|\big)=\infty \tag{2.9}$$

and for any $\varepsilon>0$,

$$\lim_{|n|\to\infty}\inf_{\tilde n:\,|\tilde n|\ge\varepsilon|n|}\min_{l<i}\big(|q_i(\tilde n)|-|q_l(n)|-|\tilde n-n|\big)=\infty. \tag{2.10}$$

For each $\theta>0$ set

$$\gamma_\theta^\theta=\|X\|_\theta^\theta=E|X(n)|^\theta=\int|x|^\theta d\mu. \tag{2.11}$$

Our main result relies on

2.1. **Assumption.** With $d=(\ell-1)\wp$ there exist $p,q\geq 1$, $m\geq 4$ and $\delta>0$ with $\delta\leq\kappa$, $p\kappa>d$ satisfying

$$\sum_{l=0}^{\infty} l^{3(\nu+1)}\varpi_{q,p}(l)=\theta(p,q)<\infty, \tag{2.12}$$

$$\sum_{r=0}^{\infty} r^{3(\nu+1)}\beta^{\delta}(q,r)<\infty, \tag{2.13}$$

$$\gamma_m<\infty,\ \gamma_{2q\iota}<\infty \text{ with } \frac{1}{2}\geq\frac{1}{p}+\frac{\iota+1}{m}+\frac{\delta}{q}. \tag{2.14}$$

In order to give a detailed statement of our main result as well as for its proof it will be essential to represent the function $F=F(x_1,x_2,\dots,x_\ell)$ in the form

$$F=F_1(x_1)+\cdots+F_\ell(x_1,x_2,\dots,x_\ell) \tag{2.15}$$

where for $i<\ell$,

$$\begin{aligned}F_i(x_1,\dots,x_i)&=\int F(x_1,x_2,\dots,x_\ell)\,d\mu(x_{i+1})\cdots d\mu(x_\ell)\\ &\quad-\int F(x_1,x_2,\dots,x_\ell)\,d\mu(x_i)\cdots d\mu(x_\ell)\end{aligned} \tag{2.16}$$

and

$$F_\ell(x_1,x_2,\dots,x_\ell)=F(x_1,x_2,\dots,x_\ell)-\int F(x_1,x_2,\dots,x_\ell)\,d\mu(x_\ell)$$

which ensures, in particular, that

$$\int F_i(x_1,x_2,\dots,x_{i-1},x_i)\,d\mu(x_i)\equiv 0\quad\forall\quad x_1,x_2,\dots,x_{i-1}. \tag{2.17}$$

We write $t=(t_1,...,t_\nu)\geq s=(s_1,...,s_\nu)$ if $t_i\geq s_i$ for all i and for such $s,t\in[0,1]^\nu$ we set $\Delta_N(s,t)=\{n=(n_1,...,n_\nu)\in\mathbb{Z}^\nu:\ Ns_i\leq n_i\leq Nt_i\,\forall i\}$ and $\Delta_N(t)=\Delta_N(0,t)$. These together with (2.15)–(2.17) enable us to represent $\xi_N(t)$ given by (1.2) in the form

$$\xi_N(t)=\sum_{i=1}^{k}\xi_{i,N}(it)+\sum_{i=k+1}^{\ell}\xi_{i,N}(t) \tag{2.18}$$

where for $1\leq i\leq k$,

$$\xi_{i,N}(t)=N^{-\nu/2}\sum_{n\in\Delta_N(t/i)}F_i(X(n),X(2n),\dots,X(in)) \tag{2.19}$$

and for $i\geq k+1$,

$$\xi_{i,N}(t)=N^{-\nu/2}\sum_{n\in\Delta_N(t)}F_i(X(q_1(n)),\dots,X(q_i(n))). \tag{2.20}$$

2.2. **Theorem.** *Suppose that Assumption 2.1 holds true then each random field $\xi_{i,N}(t)$, $i=1,2,...,\ell$ converges in distribution as $N\to\infty$ to a Gaussian random field $\eta_i(t)$. Moreover, $(\eta_1(t),\eta_2(t),...,\eta_\ell(t))$ is an ℓ-dimensional Gaussian random field such that $\eta_i(t)$, $i\leq k$ have covariances*

$$E\eta_i(s)\eta_j(t)=D_{i,j}\prod_{l=1}^{\nu}\min(s_l,t_l),\ i,j\leq k$$

with matrix $D_{i,j}$ *described in Proposition 4.5 while the random fields* $\eta_i(t)$, $i\geq k+1$ *are independent of each other and of* η_j*'s with* $j\leq k$ *and have variances*

$$E|\eta_i(t)|^2=\int |F_i(x_1,x_2,...,x_i)|^2 d\mu(x_1)d\mu(x_2)\cdots d\mu(x_i),\ i\geq k+1.$$

Finally, $\xi_N(t)$ *converges in distribution to a Gaussian random field* $\xi(t)$ *which can be represented in the form*

$$\xi(t)=\sum_{i=1}^{k}\eta_i(it)+\sum_{i=k+1}^{\ell}\eta_i(t). \tag{2.21}$$

In order to understand our assumptions observe that $\varpi_{q,p}$ is clearly non-increasing in q and non-decreasing in p. Hence, for any pair $p,q\geq 1$,

$$\varpi_{q,p}(n)\leq\psi(n).$$

Furthermore, by the real version of the Riesz–Thorin interpolation theorem or the Riesz convexity theorem (see [13], Section 9.3 and [11], Section VI.10.11) whenever $\theta\in[0,1]$, $1\leq p_0,p_1,q_0,q_1\leq\infty$ and

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\ \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$$

then

$$\varpi_{q,p}(n)\leq 2(\varpi_{q_0,p_0}(n))^{1-\theta}(\varpi_{q_1,p_1}(n))^{\theta}. \tag{2.22}$$

In particular, using the obvious bound $\varpi_{q_1,p_1}\leq 2$ valid for any $q_1\geq p_1$ we obtain from (2.22) for pairs $(\infty,1)$, $(2,2)$ and (∞,∞) that for all $q\geq p\geq 1$,

$$\varpi_{q,p}(n)\leq(2\alpha(n))^{\frac{1}{p}-\frac{1}{q}},\ \varpi_{q,p}(n)\leq 2^{1+\frac{1}{p}-\frac{1}{q}}(\rho(n))^{1-\frac{1}{p}+\frac{1}{q}} \text{ and } \varpi_{q,p}(n)\leq 2^{1+\frac{1}{p}}(\phi(n))^{1-\frac{1}{p}}. \tag{2.23}$$

We observe also that by the Hölder inequality for $q\geq p\geq 1$ and $\alpha\in(0,p/q)$,

$$\beta(q,r)\leq 2^{1-\alpha}[\beta(p,r)]^{\alpha}\gamma^{1-\alpha}_{\frac{pq(1-\alpha)}{p-q\alpha}} \tag{2.24}$$

with γ_θ defined in (2.11). Thus, we can formulate Assumption 2.1 in terms of more familiar α, ρ, ϕ, and ψ–mixing coefficients and with various moment conditions. It follows also from (2.22) that if $\varpi_{q,p}(n)\to 0$ as $n\to\infty$ for some $q\geq p\geq 1$ then

$$\varpi_{q,p}(n)\to 0 \text{ as } n\to\infty \text{ for all } q\geq p\geq 1, \tag{2.25}$$

and so (2.25) holds true under Assumption 2.1.

In order to prove Theorem 2.2 we will represent $\xi_{i,N}(t)$ in the form $\sum_{1\leq l\leq N}Z_{t,N}(l)$ where now l is one dimensional which together with estimates of the next section will enable us to apply central limit theorems for mixingale arrays (see [18] and [19]). This will lead to Gaussian one dimensional distributions in the limit but combining this with a kind of the Cramér-Wold argument, covariances computation in Section 4 and tightness estimates of Section 5 will yield appropriate Gaussian random fields as asserted in the theorem. Recall (see [16]), that already in the one parameter case $\nu=1$ the process $\xi(t)$, in general, does not have independent increments so also in the random field case $\xi(t)$, in general, is not a multiparameter Brownian motion.

2.3**. Remark.** As a part of tightness estimates of Section 5 we will see that $\sup_{N\geq 1,\, t\in[0,1]^\nu} E|\xi_{i,N}(t)|^4 \leq C < \infty$. Hence, applying the Borel–Cantelli lemma we obtain as a byproduct that if $S_{i,N} = N^{\nu/2}\xi_{i,N}(t)$ and $S_N = N^{\nu/2}\sum_{i=1}^{\ell}\xi_{i,N}(t)$ then with probability one

$$\lim_{N\to\infty}\frac{1}{N^\nu}S_{i,N}(t) = 0 \text{ for each } i, \text{ and so } \lim_{N\to\infty}\frac{1}{N^\nu}S_N(t) = 0.$$

Still, we observe that this strong law of large numbers can be obtained under more general circumstances here since, in particular, we do not need for it convergence of covariances derived in Section 4 which requires, for instance, more specific assumptions on q_j's.

3. Blocks and mixingale type estimates

We rely on the following result which appears as Corollary 3.6 in [16].

3.1**. Proposition.** *Let $\mathcal{G}$ and $\mathcal{H}$ be σ-subalgebras on a probability space $(\Omega,\mathcal{F},P)$, X and Y be d-dimensional random vectors and $f = f(x,\omega)$, $x\in\mathbb{R}^d$ be a collection of random variables measurable with respect to $\mathcal{H}$ and satisfying*

$$\|f(x,\omega)-f(y,\omega)\|_q \leq C(1+|x|^\iota+|y|^\iota)|x-y|^\kappa \text{ and } \|f(x,\omega)\|_q \leq C(1+|x|^\iota) \tag{3.1}$$

where $g\geq 1$. Set $g(x) = Ef(x,\omega)$. Then

$$\|E(f(X,\cdot)|\mathcal{G}) - g(X)\|_\upsilon \leq c(1+\|X\|^{\iota+2}_{b(\iota+2)})(\varpi_{q,p}(\mathcal{G},\mathcal{H}) + \|X-E(X|\mathcal{G})\|_q^\delta) \tag{3.2}$$

provided $\kappa - \frac{d}{p} > \theta > 0$, $\frac{1}{\upsilon}\geq\frac{1}{p}+\frac{1}{b}+\frac{\delta}{q}$ with $c = c(C,\iota,\kappa,\delta,p,q,\upsilon,d) > 0$ depending only on parameters in brackets. Moreover, let $x=(v,z)$ and $X=(V,Z)$, where V and Z are d_1 and $d-d_1$-dimensional random vectors, respectively, and let $f(x,\omega) = f(v,z,\omega)$ satisfy (3.1) in $x=(v,z)$. Set $\tilde g(v) = Ef(v,Z(\omega),\omega)$. Then

$$\begin{aligned}\|E(f(V,Z,\cdot)|\mathcal{G}) - \tilde g(V)\|_\upsilon &\leq c(1+\|X\|^{\iota+2}_{b(\iota+2)})\\ &\times\big(\varpi_{q,p}(\mathcal{G},\mathcal{H}) + \|V-E(V|\mathcal{G})\|_q^\delta + \|Z-E(Z|\mathcal{H})\|_q^\delta\big).\end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned}&\|\tilde f(X(\omega),\omega) - \tilde f(Y(\omega),\omega) - g(X) + g(Y)\|_\upsilon\\ &\leq c\,\varpi_{q,p}(\mathcal{G},\mathcal{H})\big(1+\|X\|_m^{\iota+2}+\|Y\|_m^{\iota+2}\big)\|X-Y\|_q^\delta.\end{aligned} \tag{3.4}$$

We will use the following notations

$$\begin{aligned}&F_{i,n,r} = F_{i,n,r}(x_1,x_2,\dots,x_{i-1},\omega) = E\big(F(x_1,x_2,\dots,x_{i-1},X(n))|\mathcal{F}_{U_r(n)}\big),\\ &Y_i(q_i(n)) = F_i(X(q_1(n)),\dots,X(q_i(n))) \text{ and } Y_i(m)=0 \text{ if } m\neq q_i(n) \text{ for any } n,\\ &X_r(n) = E(X(n)|\mathcal{F}_{U_r(n)}),\ Y_{i,r}(q_i(n)) = F_{i,q_i(n),r}(X_r(q_1(n)),\\ &\dots,X_r(q_{i-1}(n)),\omega) \text{ and } Y_{i,r}(m)=0 \text{ if } m\neq q_i(n) \text{ for any } n;\\ &\bar Y_i(n) = Y_i(n) - EY_i(n),\ \bar Y_{i,r}(n) = Y_{i,r}(n) - EY_{i,r}(n).\end{aligned} \tag{3.5}$$

For each $l\in\mathbb{Z}_+$ introduce cubes $\square(l) = \{n=(n_1,...,n_\nu)\in\mathbb{Z}^\nu : 0\leq n_i\leq l \text{ for } i=1,...,\nu\}$ and for $l>\tilde l$ we set also $\Upsilon(l,\tilde l) = \square(l)\setminus\square(\tilde l)$. Fix some positive numbers $4\eta < 2\theta < \tau < 1/2$ and set $a(1)=0, b(1)=1$ and for $j>1$,

$$a(j) = b(j-1) + [(j-1)^\theta],\ b(j) = a(j) + [j^\tau] \text{ and } r(j) = [j^\eta]. \tag{3.6}$$

We define also

$$V_{i,t,N}(l)=\sum_{n\in\Delta_N(t)\cap\Upsilon(a(l),b(l))}Y_{i,r(l)}(\hat q_i(n)) \quad \text{and } W_{i,t,N}(l)=\sum_{n\in\Delta_N(t)\cap\Upsilon(b(l),a(l+1))}Y_{i,r(l)}(\hat q(n)) \tag{3.7}$$

where $\hat q_i(n)=n$ for $i=1,2,...,k$ and $\hat q_i(n)=q_i(n)$ for $i=k+1,...,\ell$. The sets $\Upsilon(b(l),a(l+1))$ will play the role of gaps between $\Upsilon(a(l),b(l))$ and $\Upsilon(a(l+1),b(l+1))$ and we will see that the random variables $W_{i,t,N}(l)$ can be disregarded for our purposes while dealing with the random variables $V_{i,t,N}(l)$ we will take advantage of our mixing conditions in order to show that their centered versions $\bar V_{i,t,N}(l)=V_{i,t,N}(l)-EV_{i,t,N}(l)$ satisfy mixingale estimates (see [18] and [19]) with respect to the nested family of σ-algebras $\mathcal{G}_l^{(i)}=\mathcal{F}_{\Gamma_i(l)}$, $l=0,1,2,...$ where $\Gamma_i(l)=\Box(b(l))$ for $i\le k$,

$$\Gamma_i(l)=\{n\in\mathbb{Z}_+^\nu:\ \mathrm{dist}\big(n,\cup_{j\le i}q_j(\Box(b(l)))\big)\le r(l)\}$$

for $i\ge k+1$ and we take $\mathcal{G}_l^{(i)}$ to be the trivial σ-algebra $\{\emptyset,\Omega\}$ for $l<0$.

Namely, for any $u\in\mathbb{N}$ we have

$$\begin{aligned}\|E\big(\bar V_{i,t,N}(l)|\mathcal{G}_{l-u}^{(i)}\big)\|^2 &\le \sum_{n\in\Delta_N(t)\cap\Upsilon(a(l),b(l))}\|E(\bar Y_i(\hat q_i(n))|\mathcal{G}_{l-u}^{(i)})\| \\ &\le |\Upsilon(a(l),b(l))|\max_{n\in\Upsilon(a(l),b(l))}\|E(\bar Y_i(\hat q_i(n))|\mathcal{G}_{l-u}^{(i)})\|_2\end{aligned} \tag{3.8}$$

where $|A|$ for a set A denotes its cardinality. Next, for $u>l$,

$$E(\bar Y_i(\hat q_i(n))|\mathcal{G}_{l-u}^{(i)})=0 \tag{3.9}$$

while for all $u\ge 1$ we can write by the triangle inequality and the contraction property of conditional expectations that

$$\|E(\bar Y_i(\hat q_i(n))|\mathcal{G}_{l-u}^{(i)})\|_2\le 2\|E(Y_{i,r(l)}(\hat q_i(n))|\mathcal{G}_{l-1}^{(i)})\|_2+2\|Y_i(\hat q_i(n))-Y_{i,r(l)}(\hat q_i(n))\|_2. \tag{3.10}$$

It follows from Proposition 3.1 together with (2.8), (2.9) and (2.14) that for all $n\in\Upsilon(a(l),b(l))$ and $\tilde r,l\ge 1$ (see Lemma 3.12 in [16]),

$$\|Y_i(\hat q_i(n))-Y_{i,\tilde r}(\hat q_i(n))\|_2\le C\beta_q^\delta(\tilde r) \tag{3.11}$$

and

$$\|E(Y_{i,r(l)}(\hat q_i(n))|\mathcal{G}_{l-1}^{(i)})\|_2\le C\big(l^{\nu+\eta}\varpi_{p,q}(a(l)-b(l-1)-2r(l))+\beta_q^\delta(r(l))\big) \tag{3.12}$$

for some $C>0$ independent of n,l provided p,q and δ satisfy the conditions of Assumption 2.1. Collecting (3.7)–(3.10) we obtain that for any $u\ge 1$,

$$\|E(\bar V_{i,t,N}(l)|\mathcal{G}_{l-u}^{(i)})\|_2\le\tilde C a^\nu(l)[l^\tau]\big(l^{\nu+\eta}\varpi_{p,q}([(l-1)^\theta]-2[l^\eta])+\beta_q^\delta([l^\eta])\big) \tag{3.13}$$

for some $\tilde C>0$ independent of t,N,l and u. On the other hand, by (3.11),

$$\|E(\bar V_{i,t,N}(l)|\mathcal{G}_{l+u}^{(i)})-\bar V_{i,t,N}(l)\|_2\le\tilde{\tilde C}\beta_q^\delta(r(l+u)) \tag{3.14}$$

where $\tilde{\tilde C}>0$ is independent of t,N,l and $u\ge 1$ which together with (3.13) yields the mixingale type estimates we will rely on.

Next, we estimate contribution of small blocks (gaps) $W_{i,t,N}(j)$, $j\ge 1$. Let $l>j$, $n\in\Upsilon(b(l),a(l+1))$ and set $\hat\Gamma_i(j)=\Box(a(j+1))$ for $i\le k$ while

$$\hat\Gamma_i(j)=\{n\in\mathbb{Z}_+^\nu:\ \mathrm{dist}\big(n,\cup_{a\le i}q_a\big(\Box(a(j+1))\big)\big)\le r(j)\}.$$

Then employing Proposition 3.1 with $\mathcal{G}^{(i)}=\mathcal{F}_{\hat\Gamma_i(j)}$ and taking into account (2.8)–(2.10) we obtain that

$$|EW_{i,t,N}(j)Y_{i,\hat q_i(n),r(l)}|=\big|E\big(W_{i,t,N}(j)E(Y_{i,\hat q(n),r(l)}|\mathcal{G})\big)\big| \le C_1\|W_{i,t,N}(j)\|_2\big(\varpi_{q,p}(b(l)-r(l)-a(j+1)-r(j)) +\beta_q^\delta(b(l)-r(l)-a(j+1)-r(j))\big) \tag{3.15}$$

for some $C_1>0$ independent of t,N,n,l and j. Hence,

$$|EW_{i,y,N}(j)W_{i,t,N}(l)|\le C_1\|W_{i,t,N}(j)\|_2|\Upsilon(b(l),a(l+1))| \times\big(\varpi_{q,p}(b(l)-r(l)-a(j+1)-r(j))+\beta_q^\delta(b(l)-r(l)-a(j+1)-r(j))\big). \tag{3.16}$$

To specify the above estimates we observe that

$$|\Upsilon(l,\tilde l)|\le C_2l^{\nu-1}(l-\tilde l),\text{ in particular, }|\Upsilon(b(l),a(l+1))|\le C_2b^{\nu-1}(l)[l^\theta] \tag{3.17}$$

for some $C_2>0$ independent of $l>\tilde l$ while

$$b(l)\le\frac{2}{1+\tau}(l+1)^{1+\tau}\text{ and }b(l)-r(l)-a(j+1)-r(j)\ge[l^\tau]-2[l^\eta].$$

Set $L(N)=\min\{j:a(j+1)\ge N\}$ and since

$$N\ge\sum_{1\le j\le L(N)-1}[j^\tau]\ge(1+\tau)^{-1}(L(N)-1)^{1+\tau}$$

we have that

$$L(N)\le(N(1+\tau))^{1/(1+\tau)}+1\le 2N^{1/(1+\tau)}+1. \tag{3.18}$$

It follows from (2.12)–(2.14) and (3.15)–(3.17) that

$$E\big(\sum_{0\le j\le L(N)}W_{i,t,N}(j)\big)^2\le\sum_{0\le j\le L(N)}\big(EW^2_{i,t,N}(j) +2\sum_{l:L(N)\ge l>j}|EW_{i,t,N}(j)W_{i,t,N}(l)|\big)\le C_3\sum_{0\le j\le L(N)}(\|W_{i,t,N}(j)\|_2^2+1) \tag{3.19}$$

for some $C_3>0$ independent of N.

Relying on (2.3), (2.5), (2.14) and the Hölder inequality we can estimate the error of replacement of $Y_i(q_i(n))$ by its $r(l)$-approximation $Y_{i,r(j)}(q_i(n))$ (see Lemma 3.12 in [16]),

$$\|Y_i(q_i(n))-Y_{i,r(j)}(q_i(n))\|_2\le C_4\beta_q^\delta(r(j)) \tag{3.20}$$

for some $C_4>0$ independent of i,j and n. Now, set

$$\zeta_{i,N}(t)=N^{-\nu/2}\sum_{1\le l\le L(N)}V_{i,t,N}(l).$$

Then by (3.17), (3.19) and (3.20),

$$\|\xi_{i,N}(t)-\zeta_{i,N}(t)\|_2\le C_5N^{-\nu/2}\big(\sum_{1\le l\le L(N)}l^{\nu-1+\tau}\beta_q^\delta(r(l)) +\big(\sum_{0\le l\le L(N)}(\|W_{i,t,N}(l)\|_2^2+1)\big)^{1/2}\big). \tag{3.21}$$

It follows from (2.14) and Lemma 4.2 of the next section that

$$\|W_{i,t,N}(l)\|_2^2=\mathrm{O}(|\Upsilon(b(l),a(l+1))|) \tag{3.22}$$

which together with (2.13), (3.17) and (3.18) yields that

$$\xi_{i,N}(t)-\zeta_{i,N}(t)\to 0\text{ in probability as }N\to\infty, \tag{3.23}$$

and so for each t the limits in distribution as $N \to \infty$ of $\xi_{i,N}(t)$ and of $\zeta_{i,N}(t)$ coincide (if they exist).

4. Limiting covariances

The first step in our limiting covariances computations is the following estimate of

$$b_{i,j}(m,n) = EY_{i,q_i(m)}Y_{j,q_j(n)},\ m,n \in \mathbb{Z}_+^\nu$$

where $Y_{i,q_i(n)}$ was defined in (3.4).

4.1**. Lemma.** *(i) For $i,j = 1,...,\ell$ and any $m,n \in \mathbb{Z}_+^\nu$ set*

$$\hat{s}_{i,j}(m,n) = \min\big(\min_{1\le l\le j}|q_i(m) - q_l(n)|,\ |m|\big) \tag{4.1}$$
and $s_{i,j}(m,n) = \max(\hat{s}_{i,j}(m,n),\ \hat{s}_{j,i}(n,m))$.

Then for all $i \le k$,

$$s_{i,i}(m,n) \ge \frac{1}{8k^2}|m-n|. \tag{4.2}$$

Furthermore, if $i \ge k+1$ then for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that if $\max(|m|,|n|) \ge M_\varepsilon$ and $m \ne n$ then

$$s_{i,i}(m,n) \ge \min\big(|m-n| + \varepsilon^{-1}, \max(|m|,|n|)\big) \ge \frac{1}{2}|m-n|. \tag{4.3}$$

(ii) There exists a function $h(l) \ge 0$ defined on integers such that $\sum_{l=1}^\infty l^{2\nu}h(l) < \infty$ and for any $i,j = 1,2,...,\ell$ and $l = 0,1,2,...,$

$$\sup_{m,n\in\mathbb{Z}_+^\nu:\, s_{i,j}(m,n)\ge l}|b_{i,j}(m,n)| \le h(l). \tag{4.4}$$

Proof. (i) First, suppose that $i \le k$. If $|im - ln| \ge \frac{1}{2}|m-n|$ for all $l \le i$ and $|m| \ge \frac{1}{4k}|m-n|$ then $s_{i,i}(m,n) \ge 14|m-n|$ and there is nothing to prove. If $|im - ln| \ge 12|m-n|$ for all $l \le i$ and $|m| \le \frac{1}{4k}|m-n|$ then $l|n| \ge |im-ln| - i|m| \ge \frac{1}{4}|m-n|$, and so $|n| \ge \frac{1}{4k}|m-n|$. If $|in - \tilde{l}m| \ge \frac{1}{8k}|m-n|$ for all $\tilde{l} \le i$ then $s_{i,i}(m,n) \ge \frac{1}{8k}|m-n|$ and we are done. On the other hand, if $|n| \ge \frac{1}{4k}|m-n|$ and $|in - \tilde{l}m| \le \frac{1}{8k}|m-n|$ for some $\tilde{l} \le i$ then $\frac{1}{4k}|m-n| \ge i|n| - \tilde{l}|m|$, i.e. $|m| \ge \frac{1}{k}(|n| - \frac{1}{8k}|m-n|) \ge \frac{1}{8k^2}|m-n|$ implying that $s_{i,i}(m,n) \ge \frac{1}{8k^2}|m-n|$. Next, consider the case when $|im - ln| \le \frac{1}{2}|m-n|$ for some $l \le i$ which can only happen when $l < i$. Since for any $\tilde{l} < i$,

$$|im - ln| \ge i|m| - l|n| \ge \tilde{l}|m| - i|n| + (i-\tilde{l})|m| + (i-l)|n| \ge -|in - \tilde{l}m| + |m-n|$$

we obtain that $|in - \tilde{l}m| \ge 12|m-n|$. Now, we are in the situation as above with m and n exchanged, and so (4.2) follows. In order to obtain (4.3) we rely on the definition (4.1) together with the assumptions (2.8) and (2.10).

(ii) By (2.9) there exists M such that $|q_i(m) - q_l(m)| \ge |m|$ for all $l < i$ provided $|m| \ge M$. Hence, for such m,

$$\min\big(\min_{1\le l\le j}|q_i(m) - q_l(n)|,\ \min_{1\le l<i}|q_i(m) - q_l(m)| \ge \hat{s}_{i,j}(m,n) = s_{i,j}(m,n).$$

Assume that $s_{i,j}(m,n) = \hat{s}_{i,j}(m,n) \ge 2r$ and set

$$b_{i,j}^{(r)}(m,n) = EY_{i,q_i(m),r}Y_{j,q_j(n),r}$$

where $Y_{i,l,r}$ is defined in (3.4). It follows from (2.3), (2.5), (2.6) together with the Hölder inequality (cf. Lemma 3.12 in [16]) that

$$|b^{(r)}_{i,j}(m,n)-b_{i,j}(m,n)|\leq C(\beta_q(r))^\delta \tag{4.5}$$

where a constant $C>0$ does not depend on i,j,m,n and r. Set

$$\Gamma_r(m,n)=\cup_{u=1}^{i-1}U_r(q_u(m))\cup_{v=1}^{j}U_r(q_v(n))$$

where $U_\rho(x)=\{y:\,|x-y|\leq\rho\}$. Applying Proposition 3.1 we conclude that

$$\begin{aligned}|b^{(r)}_{i,j}(m,n)|&=\big|E\big(E(Y_{i,q_i(m),r}|\mathcal{F}_{\Gamma_r(m,n)})Y_{j,q_j(n),r}\big)\big| \\ &\leq\|Y_{j,q_j(n),r}\|_2\|E(Y_{j,q_i(m),r}|\mathcal{F}_{\Gamma_r(m,n)}\|_2\leq Cr^\nu\varpi_{q,p}(s_{i,j}(m,n)-2r).\end{aligned} \tag{4.6}$$

The estimate in the case $s_{i,j}(m,n)=\hat s_{j,i}(n,m)\geq 2r$ is similar. Now we choose $r=\frac14 s_{i,j}(m,n)$ and the result follows from (4.5) and (4.6) taking into account Assumption 2.1. □

For $s,t\in[0,1]^\nu$ with $s\leq t$ set

$$\xi_{i,N}(s,t)=N^{-\nu/2}\sum_{n\in\Delta_N(s,t)}F_i(X(q_1(n)),...,X(q_i(n))).$$

Now we can obtain an appropriate estimate of the second moment of $\xi_{i,N}$.

4.2. **Lemma.** *There exists $C>0$ such that for all $t=(t_1,...,t_\nu)\geq s=(s_1,...,s_\nu)\geq 0$ and $i=1,...,\ell$,*

$$E|\xi_{i,N}(s,t)|^2\leq CN^{-\nu}|\Delta_N(s,t)|\leq C\prod_{l=1}^{\nu}(t_l-s_l+N^{-1}). \tag{4.7}$$

Proof. Set $G_i(m,l)=\{n\in\mathbb{Z}^\nu_+:\,l\leq|m-n|<l+1\}$. Then by (4.2) and (4.4) for $i\leq k$,

$$\begin{aligned}E|\xi_{i,N}(s,t)|^2&=N^{-\nu}\textstyle\sum_{m,n\in\Delta_N(s,t)}b_{i,i}(m,n)\\ &\leq 2N^{-\nu}\textstyle\sum_{l=0}^\infty\sum_{m\in\Delta_N(s,t)}\sum_{n\in G_i(m,l)}b_{i,i}(m,n)\\ &\leq 2N^{-\nu}\textstyle\sum_{l=0}^\infty h(l)\sum_{m\in\Delta_N(s,t)}|G_i(m,8k^2l)|.\end{aligned} \tag{4.8}$$

If $i>k$ then by (4.3) and (4.4),

$$\begin{aligned}E|\xi_{i,N}(s,t)|^2&\leq N^{-\nu}\textstyle\sum_{m,n:|m|,|n|\leq M_1}b_{i,i}(m,n)\\ &+2N^{-\nu}\textstyle\sum_{l=0}^\infty\sum_{m\in\Delta_N(s,t)}\sum_{n\in G_i(m,2l)}b_{i,i}(m,n)\\ &\leq N^{-\nu}(C_1+\textstyle\sum_{l=0}^\infty h(l)\sum_{m\in\Delta_N(s,t)}|G_i(m,2l)|)\end{aligned} \tag{4.9}$$

for some $C_1>0$ independent of s,t and N. Clearly, for any $\tilde l\geq 0$,

$$|G_i(m,l)|\leq C_2(\tilde l+1)^{\nu-1} \tag{4.10}$$

for some $C_2>0$ independent of m and l since it is bounded by the number of integer points between spheres of radius l and $l+1$. Finally, by (4.8)–(4.10) and the summability of $l^{\nu-1}h(l)$ we derive (4.7). □

4.3. **Lemma.** *For each $t>0$ and $i,j\leq k$,*

$$\lim_{u\to\infty}\limsup_{N\to\infty}N^{-\nu}\sum_{\substack{0\leq n,n'\leq Nt\\ |in-jn'|\geq u}}|b_{i,j}(n,n')|=0.$$

Proof. The result is a strightforward corollary of Lemma 4.1. □

4.4. **Proposition.** *For any $i,j\le k$ and $s=(s_1,...,s_\nu),t=(t_1,...,t_\nu)\ge 0$ the limit*

$$\lim_{N\to\infty} N^{-\nu}\sum_{\substack{0\le in\le Ns\\ 0\le jn'\le Nt,\, in-jn'=u}} b_{i,j}(n,n')=\frac{\upsilon\prod_{l=1}^{\nu}\min(s_l,t_l)}{ij}c_{i,j}(u) \tag{4.11}$$

exists for any $u\in\mathbb{Z}^\nu$ where υ is the greatest common divisor of i and j and for $i'=i/\upsilon$, $j'=j/\upsilon$,

$$c_{i,j}(u)=\int F_i(x_1,...,x_i)F_j(y_1,...,y_j)\prod_{\alpha=1}^{\upsilon}d\mu_{\alpha u/\upsilon}(x_{\alpha i'},y_{\alpha j'}) \prod_{\sigma\notin\{i',2i',...,\upsilon i'\}}d\mu(x_\sigma)\prod_{\sigma'\notin\{j',2j',...,\upsilon j'\}}d\mu(y_{\sigma'}) \tag{4.12}$$

with μ_0 being the diagonal measure, i.e. $\int f(x,y)d\mu_0(x,y)=\int f(x,x)d\mu(x)$. If υ does not divide all components of $u\in\mathbb{Z}^\nu$ then $c_{i,j}(u)=0$. Finally,

$$\begin{aligned}\lim_{N\to\infty}E\xi_{i,N}(s)\xi_{j,N}(t)&=\lim_{N\to\infty}N^{-\nu}\sum_{\substack{0\le in\le Ns\\ 0\le jn'\le Nt}}b_{i,j}(n,n')\\ &=D_{i,j}\prod_{l=1}^{\nu}\min(s_l,t_l)\end{aligned} \tag{4.13}$$

where

$$D_{i,j}=\frac{\upsilon}{ij}\sum_{u\in\mathbb{Z}^\nu}c_{i,j}(u). \tag{4.14}$$

Proof. Let υ be the greatest common divisor of i and j. If $u\in\mathbb{Z}^\nu$ has components which are not divisible by υ then the sum in (4.11) is empty, and so in this case $c_{i,j}(u)=0$. Thus it remains to deal with this sum when $in-jn'=\upsilon u$ for $u\in\mathbb{Z}^\nu$. We will show first that the limit

$$c_{i,j}(u)=\lim_{|n|,|n'|\to\infty,\, in-jn'=\upsilon u} b_{i,j}(n,n') \tag{4.15}$$

exists. Observe that if we consider two strings $(n,2n\dots,in)$ and $(n',2n',\dots,jn')$ with $in-jn'=\upsilon u$ then there will also be pairs (i_α,j_α), $\alpha=1,2,\dots,\upsilon-1$ such that $i_\alpha n-j_\alpha n'=\alpha u$ where $i_\alpha=\alpha i_1$ and $j_\alpha=\alpha j_1$ with i_1 and j_1 being coprime. On the other hand, if $\tilde i/\tilde j\ne i_1/j_1$ then

$$\begin{aligned}|\tilde in-\tilde jn'|&=\tilde j\big|\tfrac{\tilde i}{\tilde j}n-n'\big|=\tilde j\big|(\tfrac{\tilde i}{\tilde j}-\tfrac{i_1}{j_1})n+\tfrac{1}{j_1}(i_1n-j_1n')\big|\\ &\ge |n|\tilde j\big|\tfrac{\tilde i}{\tilde j}-\tfrac{i_1}{j_1}\big|-\tfrac{\tilde j}{j_1}|u|\to\infty \text{ as } |n|\to\infty.\end{aligned} \tag{4.16}$$

We split the collection of numbers $(n,2n,...,in;n',2n',...,jn')$ into disjoint sets $\Gamma_1,...,\Gamma_\upsilon,...,\Gamma_{i+j-\upsilon}$ where $\Gamma_\alpha=\{\alpha i_1,\alpha j_1\}$, $\alpha=1,...,\upsilon$ are pairs and $\Gamma_{\upsilon+\beta}$, $\beta=1,2,...,i+j-\upsilon$ are singeltons. We order the latter so that $\Gamma_\beta=\{l_\beta n\}$, $1\le l_\beta<i$, $\beta=\upsilon+1,...,i$ with $l_\beta\ne\alpha i_1$ for $\alpha=1,...,\upsilon$ and $\Gamma_{\upsilon+\beta}=\{l'_\beta n'\}$ for $\beta=i+1,...,i+j-\upsilon$, $1\le l'_\beta<i$, $l'_\beta\ne\alpha j_1$ for $\alpha=1,...,\upsilon$. By (4.16) there is $\delta>0$ depending on u but not on n and n' such that

$$\min_{1\le l\ne l'\le i+j-\upsilon}\operatorname{dist}(\Gamma_l,\Gamma_{l'})\ge\delta n.$$

Set

$$U_r(\Gamma_l)=\{n\in\mathbb{Z}_+^\nu:\ \operatorname{dist}(n,\Gamma_l)\le r\},\ l=1,2,...,i+j-\upsilon$$

and choose $r=r(n)\to\infty$ as $|n|\to\infty$ so that all $U_{r(n)}(\Gamma_l)$, $l=1,2,...,i+j-\upsilon$ were disjoint.

Now, observe that $b_{i,j}(n,n')$ has the form $EG(Y_1(n,n'),Y_2(n,n'),...,Y_{i+j-\upsilon}(n,n'))$ where $Y_\alpha(n,n')=(X(\alpha i,n),X(\alpha j,n'))$

for $\alpha=1,...,\upsilon$, $Y_\beta(n,n')=X(l_\beta n)$ for $\beta=\upsilon+1,...,i$ and $Y_\beta(n,n')=X(l'_\beta n')$ for $\beta=i+1,...,i+j-\upsilon$. Define $G_1=G$ and successively

$$G_{l+1}(y_{l+1},y_{l+2},...,y_{i+j-\upsilon})=EG(Y_l(n,n'),y_{l+1},y_{l+2},...,y_{i+j-\upsilon}).$$

Relying on the assumptions (2.12) and (2.13) we can apply (3.3) of Proposition 3.1 with $V=(Y_{l+1}(n,n'),Y_{l+2}(n,n'),...,Y_{i+j-\upsilon}(n,n'))$, $Z=Y_l(n,n')$, $\mathcal{G}=\mathcal{F}_{\tilde U}$ with $\tilde U=\cup_{l+1\leq\beta\leq i+j-\upsilon}U_{r(n)}(\Gamma_\beta)$ and $\mathcal{H}=\mathcal{F}_{U_l(r(n))}$ obtaining that

$$\begin{aligned}&EG_l(Y_l(n,n'),Y_{l+1}(n,n'),...,Y_{i+j-\upsilon}(n,n'))\\ &-EG_{l+1}(Y_{l+1}(Y_{l+1}(n,n'),...,Y_{i+j-\upsilon}(n,n'))\to 0 \text{ as } n\to\infty.\end{aligned}$$

This argument repeated for $l=1,2,...,i+j-\upsilon-1$ yields (4.15) with $c_{i,j}(u)$ given by (4.12).

Finally, in order to obtain (4.11) and (4.13) we have to count the number of solutions n,n' of the vector Diophantine equation $in-jn'=\upsilon u$ where $0\leq in\leq Ns$ and $0\leq jn'\leq Nt$. Since we have to satisfy this equation coordinate wise, the number of solutions is the product of the number of solutions in each coordinate. Let, as before, $i=\upsilon i_1$ and $j=\upsilon j_1$ with i_1 and j_1 being coprime then all solutions of the equation $i_1n_l-j_1n'_l=u_l$ are given by $n=n_0+j_1m$ and $n'=n'_0+i_1m$ where n_0,n'_0 is its particular solution and m is any integer. The number of such solutions with $0\leq in_l\leq Ns_l$ and $0\leq jn'_l\leq Nt_l$ for large N is equal approximately to

$$N\min(\frac{t_l}{ij_1},\frac{s_l}{ji_1})+O(1)=\frac{N\upsilon\min(s_l,t_l)}{ij}+O(1)$$

and taking the product in l we obtain (4.11) while (4.13) follows from (4.11) and Lemmas 4.1 and 4.3. □

4.5. **Proposition.** *For $i\geq k+1$,*

$$\lim_{N\to\infty}E|\xi_{i,N}(t)|^2=(\prod_{l=1}^{\nu}t_l)\int\big(F_i(x_1,x_2,\dots,x_i)\big)^2d\mu(x_1)d\mu(x_2)\cdots d\mu(x_i). \tag{4.17}$$

Moreover, for any $t,s\in\mathbb{R}^\nu_+$ and $j<i$,

$$\lim_{N\to\infty}E\big(\xi_{i,N}(t)\xi_{j,N}(s)\big)=0. \tag{4.18}$$

Proof. By (4.3) and (4.4),

$$b_{i,i}(m,n)\to 0 \text{ as } \max(|m|,|n|)\to\infty \text{ with } |m-n|\geq 1.$$

Therefore, for any fixed L,

$$\begin{aligned}&\limsup_{N\to\infty}N^{-\nu}\textstyle\sum_{m,n\in\Delta_N(t),\,m\neq n}|b_{i,i}(m,n)|\\ &\leq\limsup_{N\to\infty}N^{-\nu}\textstyle\sum_{1\leq|m-n|\leq L}|b_{i,i}(m,n)|\\ &+C|\Delta_N(t)|\textstyle\sum_{l\geq L}l^{\nu-1}h(l)=C|\Delta_N(t)|\sum_{l\geq L}l^{\nu-1}h(l)\end{aligned}$$

for some $C>0$ independent of N and L. We now let $L\to\infty$ and since $l^{\nu-1}h(l)$ is summable it follows that lim sup in the left hand side above equals zero, i.e. the off-diagonal terms do not contribute in (4.17). It remains to deal with the diagonal terms $b_{i,i}(n,n)$. Since $|q_j(n)-q_{j-1}(n)|\to\infty$ for $j=2,3,...,\ell$ as $|n|\to\infty$ it follows by the argument similar to one applied in the proof of Proposition 4.4 (see a more general Lemma 4.3 in [16]) that

$$\lim_{|n|\to\infty}b_{i,i}(n,n)=\int\big(F_i(x_1,x_2,...,x_i)\big)^2d\mu(x_1)\cdots d\mu(x_i). \tag{4.19}$$

Namely, set $G_i(x_1,...,x_i) = \big(F_i(x_1,x_2,...,x_i)\big)^2$ and recursively for $l = i-1,...,2,1,0$,

$$G_l(x_1,...,x_l)=\int G_{l+1}(x_1,...,x_{l+1})d\mu(x_{l+1}).$$

Taking into account that $|q_l(n)-q_{\tilde l}(n)|\to\infty$ as $|n|\to\infty$ when $l\neq\tilde l$ we apply Lemma 3.1 to obtain successively for $l=i,i-1,...,1,0$ that

$$\big|EG_{l+1}\big(X(q_1(n)),...,X(q_l(n))\big)-EG_{l+1}\big(X(q_1(n)),...,X(q_l(n))\big)\big|\to 0 \text{ as } |n|\to\infty.$$

Since $b_{i,i}(n,n)=G_0$ we arrive at (4.19).

Next, we deal with (4.18). Since $i>j$ and $i>k$ then by (2.8)–(2.10) for any $\varepsilon>0$ there exists $N(\varepsilon)$ such that whenever $|m|>\varepsilon N$ and $|n|\le N\sqrt{\nu}$ we have $s_{i,j}(m,n)\ge\min(|m-n|+\varepsilon^{-1},|m|)$ provided $N\ge N(\varepsilon)$ where $s_{i,j}(m,n)$ is the same as in Lemma 4.1. It follows from Lemmas 4.1 and 4.2 that

$$\begin{aligned}(4.20)\qquad |E\xi_{i,N}(t)\xi_{j,N}(s)| &\le \big|\textstyle\sum_{|m|\le\varepsilon N,\,n\in\Delta_N(s)} b_{i,j}(m,n)\big|\\ &+N^{-\nu}\textstyle\sum_{|m|>\varepsilon N,\,m\in\Delta_N(t),n\in\Delta_N(s)}|b_{i,j}(m,n)|\\ &\le N^{-\nu}\big(\textstyle\sum_{|m|\le\varepsilon N}EY^2_{i,m}\big)^{1/2}\big(\textstyle\sum_{n\in\Delta_N(s)}EY^2_{j,n}\big)^{1/2}\\ &+C\textstyle\sum_{l\ge\min(\varepsilon^{-1},\varepsilon N)}l^{\nu-1}h(l)\le C\sqrt{\varepsilon}+C\textstyle\sum_{l\ge\min(\varepsilon^{-1},\varepsilon N)}l^{\nu-1}h(l)\end{aligned}$$

for some $C>0$ independent of N and ε. Letting in (4.20), first $N\to\infty$ and then $\varepsilon\to 0$ we arrive at (4.18). □

5. Tightness estimates

First, we will extend the estimate of Lemma 4.2 to the corresponding estimate of the 4th moment.

5.1. Lemma. *There exists $C>0$ such that for all $t=(t_1,...,t_\nu)\ge s=(s_1,...,s_\nu)\ge 0$ and $i=1,...,\ell$,*

$$(5.1)\qquad E|\xi_{i,N}(s,t)|^4\le CN^{-2\nu}|\Delta_N(s,t)|^2\le C\prod_{l=1}^{\nu}(t_l-s_l+N^{-1})^2.$$

Proof. For $n^{(1)},n^{(2)},n^{(3)},n^{(4)}\in\mathbb{Z}^\nu_+$ set

$$d_i(n^{(1)},n^{(2)},n^{(3)},n^{(4)})=EY_{i,q_i(n^{(1)})}Y_{i,q_i(n^{(2)})}Y_{i,q_i(n^{(3)})}Y_{i,q_i(n^{(4)})}$$

and for $r>0$,

$$d_i^{(r)}(n^{(1)},n^{(2)},n^{(3)},n^{(4)})=EY_{i,q_i(n^{(1)}),r}Y_{i,q_i(n^{(2)}),r}Y_{i,q_i(n^{(3)}),r}Y_{i,q_i(n^{(4)}),r}.$$

Then similarly to (4.5),

$$(5.2)\qquad |d_i^{(r)}(n^{(1)},n^{(2)},n^{(3)},n^{(4)})-d_i(n^{(1)},n^{(2)},n^{(3)},n^{(4)})|\le C_1(\beta_q(r))^\delta$$

where $C_1>0$ does not depend on $n^{(1)},n^{(2)},n^{(3)},n^{(4)}$ and r. Define

$$\begin{aligned}v_i(n^{(1)},n^{(2)},n^{(3)},n^{(4)})=\max\nolimits_{1\le j\le4}\big(\min\big(&\min\nolimits_{l<i}|q_i(n^{(j)})-q_l(n^{(j)})|,\\ &\min\nolimits_{\tilde j\ne j,l\le i}|q_i(n^{(j)})-q_l(n^{(\tilde j)})|\big)\big).\end{aligned}$$

Without loss of generality assume that

$$(5.3)\qquad v_i(n^{(1)},n^{(2)},n^{(3)},n^{(4)})=\min\big(\min_{l<i}|q_i(n^{(j)})-q_l(n^{(j)})|,\min_{\tilde j\neq j,l\le i}|q_i(n^{(j)})-q_l(n^{(\tilde j)})|\big).$$

For each $a\geq 0$ introduce the sets

$$\Gamma_a=\{n^{(1)},n^{(2)},n^{(3)},n^{(4)}\in\mathbb{Z}_+:\,a\leq v_i(n^{(1)},n^{(2)},n^{(3)},n^{(4)})<a+1\}$$

and

$$\Gamma_a(N,s,t)=\{(n^{(1)},n^{(2)},n^{(3)},n^{(4)})\in\Gamma_a:\,n^{(1)},n^{(2)},n^{(3)},n^{(4)}\in\Delta_N(s,t)\}.$$

If $i\leq k$ and $(n^{(1)},n^{(2)},n^{(3)},n^{(4)})\in\Gamma_a$ then for $j=2,3,4$,

either $|n^{(j)})<a+1$ or $|in^{(j)}-ln^{(\tilde j)}|<a+1$ for some $l=1,...,i$ and $\tilde j\neq j$.

It follows that

$$|\Gamma_a(N,s,t)|\leq C_2a^2(1+a^2+|\Delta_N(s,t)|^2) \tag{5.4}$$

for some $C_2>0$ independent of a,N,s and t. If $i\geq k+1$ then by (2.8)–(2.10) there exists $M>0$ such that whenever $|n|\geq M$,

$$\min_{l<i}|q_i(n)-q_l(n)|\geq|n|\text{ and }\min_{\tilde i\leq i,\,\tilde n\neq n}|q_i(n)-q_{\tilde i}(\tilde n)|\geq|n-\tilde n|. \tag{5.5}$$

Then similarly to the case $i\leq k$ we conclude from (5.5) that (5.4) holds true also when $i\geq k+1$.

Next, let $r=a/3$ and $(n^{(1)},n^{(2)},n^{(3)},n^{(4)})\in\Gamma_a$ satisfy (5.3). Set

$$\Psi_r(n^{(1)},n^{(2)},n^{(3)},n^{(4)})=\cup_{u=1}^{i-1}U_r(q_u(n^{(1)}))\cup\big(\cup_{j=2}^4\cup_{v=1}^iU_r(q_v(n^{(j)})\big).$$

Then by Proposition 3.1 similarly to (4.6) we derive that

$$\begin{aligned}|d_i^{(r)}(n^{(1)},n^{(2)},n^{(3)},n^{(4)})|\leq\big|E\big(E(Y_{i,q_i(n^{(1)}),r}|\mathcal{F}_{\Psi_r(n^{(1)},n^{(2)},n^{(3)},n^{(4)})})\\ \times Y_{i,q_i(n^{(2)}),r}Y_{i,q_i(n^{(3)}),r}Y_{i,q_i(n^{(4)}),r}\big)\big|\leq C_3a^\nu\varpi_{q,p}(a/3)\end{aligned} \tag{5.6}$$

for some $C_3>0$ independent of $a,n^{(1)},n^{(2)},n^{(3)}$ and $n^{(4)}$. Set $q(a)=C_1(\beta_q(a/3))^\delta+C_3a^\nu\varpi_{q,p}(a/3)$. Then

$$\begin{aligned}E|\xi_{i,N}(s,t)|^4\leq\sum_{a=0}^\infty\sum_{(n^{(1)},n^{(2)},n^{(3)},n^{(4)})\in\Gamma_a(N,s,t)}|d_i^{(r)}(n^{(1)},n^{(2)},n^{(3)},n^{(4)})|\\ \leq C_2\sum_{a=0}^\infty q(a)a^2(1+a^2+|\Delta_N(s,t)|^2)\end{aligned} \tag{5.7}$$

and (5.1) follows from (5.7) and Assumption 2.1. □

Now tightness of each sequence of random fields $\{\xi_{i,N}(t),\,t\in[0,1]^\nu\}$ follows by a slight modification of [9] (see also Ch.5 in [8] and Theorem 1.4.7 in [15]), and so the sequence of random fields $\{\xi_N(t),\,t\in[0,1]^\nu\}$ is tight, as well.

6. Gaussian limits

For each fixed $t\in[0,1]^\nu$ the convergence in distribution as $N\to\infty$ of each $\zeta_{i,N}(t)$, $i=1,...,\ell$ to corresponding Gaussian random variables follows from [18] and [19] in view of the mixingale estimates (3.13) and (3.14) of Section 3. Then $\xi_{i,N}(t)$, $i=1,...,\ell$ also converge in distribution to the same Gaussian random variables in view of (3.23). Furthermore, for any $\mathbf{t}=(t^{(1)},...,t^{(j)})$, $t^{(a)}\in[0,1]^\nu$, $a=1,...,j$ and $\mathbf{d}=(d_1,...,d_j)$ set

$$V_{i,\mathbf{t},\mathbf{d},N}(l)=\sum_{a=1}^jd_aV_{i,t^{(a)},N}(l) \tag{6.1}$$

and

$$\zeta_{i,\mathbf{d},N}(\mathbf{t})=\sum_{a=1}^{j}d_a\zeta_{i,N}(t^{(a)})=N^{-\nu/2}\sum_{1\le l\le L(N)}V_{i,\mathbf{t},\mathbf{d},N}(l). \tag{6.2}$$

Then, we obtain from (3.13) and (3.14) similar mixingale estimates also for $V_{i,\mathbf{t},\mathbf{d},N}(l)$ which via [18] and [19] yields convergence in distribution as $N\to\infty$ to Gaussian random variables of each $\zeta_{i,\mathbf{d},N}(\mathbf{t})$. This together with (3.23) imply that each

$$\xi_{i,\mathbf{d},N}(\mathbf{t})=\sum_{a=1}^{j}d_a\xi_{i,N}(t^{(a)}) \tag{6.3}$$

converges in distribution as $N\to\infty$ to Gaussian random variables. Hence, finite dimensional distributions of each $\xi_{i,N}$ have Gaussian limits which together with tightness results of Section 5 yields that each $\xi_{i,N}$ converges in distribution as $N\to\infty$ to a Gaussian random field η_i.

In fact, we can show that $(\xi_{1,N},...,\xi_{1,k})$ converges in distribution as $N\to\infty$ to a k-dimensional Gaussian random field $(\eta_1,...,\eta_k)$. Indeed, for any $\mathbf{e}=(e_1,...,e_k)\in\mathbb{R}^k$ set

$$V_{\mathbf{t},\mathbf{d},\mathbf{e},N}(l)=\sum_{i=1}^{k}V_{i,\mathbf{t},\mathbf{d},N}(l)\text{ and }\zeta_{\mathbf{d},\mathbf{e},N}(\mathbf{t})=\sum_{i=1}^{k}\zeta_{i,\mathbf{d},N}(\mathbf{t}). \tag{6.4}$$

Then it is easy to see again by (3.13) and (3.14) that similar mixingale estimates hold true also for $V_{\mathbf{t},\mathbf{d},\mathbf{e},N}(l)$ which via [18] and [19] provides convergence in distribution as $N\to\infty$ of $\zeta_{\mathbf{d},\mathbf{e},N}$ to a Gaussian random variable which must have the same distribution as $\sum_{i=1}^k\sum_{a=1}^j e_i\eta_i(t^{(a)})$. As above we conclude from (3.23) and tightness arguments of Section 5 that, in fact, $\sum_{i=1}^k e_i\xi_{i,N}$ converges in distribution as $N\to\infty$ to a Gaussian random field which must have the same distribution as $\sum_{i=1}^k e_i\eta_i$. Thus, $(\eta_1,...,\eta_k)$ is a Gaussian random field and $(\xi_{1,N},...,\xi_{k,N})$ converges in distribution to it as $N\to\infty$. Finally, $\sum_{i=1}^k\xi_{i,N}(it)$ converges in distribution as $N\to\infty$ to the random field $\eta_{i=1}^k\eta_i(it)$ which must be Gaussian as a result of the linear transformation (in the path space) of a Gaussian random field (see, for instance, [4], Section 2.2).

Next, clearly, ξ_N converges in distribution to ξ given by (2.21) and it remains to show that η_i with $i\ge k+1$ are independent of each other and of η_i with $i\le k$ which will imply that ξ is a Gaussian random field. This can be done either via a modified version of Theorem 5.6 from [16] or by the following more direct approach. First, observe that (2.10) implies that there exists $\varepsilon_N\to 0$ as $N\to\infty$ such that

$$\lim_{N\to\infty}\min_{n,\tilde n\in\Delta_N(\mathbf{1})\setminus\Delta_N(\varepsilon_N\mathbf{1})}(|q_i(\tilde n)|-\max_{l<i}|q_l(n)|-|\tilde n|)=\infty \tag{6.5}$$

where $\mathbf{1}=(1,...,1)\in[0,1]^\nu$. Next, it follows from Lemma 4.2 that for each i,

$$\xi_{i,N}(\varepsilon_N\mathbf{1})\to 0\text{ in probability as }N\to\infty, \tag{6.6}$$

and so in all our arguments the sum over $\Delta_N(\varepsilon_N\mathbf{1})$ can be disregarded.

Set $j_N=\min\{j:\,a(j)\ge\varepsilon_N\}$ with $a(j)$ defined in (3.6). Now, for $i\ge k+1$ and $L(N)\ge l\ge j_N$ we set

$$\tilde V_{t,N}(l+(i-k)L(N))=V_{i,t,N}(l). \tag{6.7}$$

If $\tilde{V}_{t,N}(\tilde{l})$ is not defined for some $L(N) < \tilde{l} \leq (\ell - k)L(N)$ by (6.7) we set it to be zero. For $\mathbf{t} = (t^{(1)}, ..., t^{(j)})$, $t^{(a)} \in [0,1]^\nu$, $a = 1, ..., j$, $\mathbf{d} = (d_1, ..., d_j)$ and $\tilde{\mathbf{e}} = (e_{k+1}, ..., e_\ell)$ we define also as in (6.1),

$$\tilde{V}_{\mathbf{t},\mathbf{d},\tilde{\mathbf{e}},N}(l) = \sum_{a=1}^{j} \sum_{i=k+1}^{\ell} d_a e_i V_{i,t^{(a)},N}(l) \tag{6.8}$$

assuming that $t^{(a)} > \varepsilon_N$ for all a. Now, define $V_{\mathbf{t},\mathbf{d},\mathbf{e},N}(l)$ by (6.4) for $l \leq L(N)$ and by $V_{\mathbf{t},\mathbf{d},\mathbf{e},N}(l) = \tilde{V}_{\mathbf{t},\mathbf{d},\tilde{\mathbf{e}},N}(l)$ for $l > L(N)$ setting $V_{\mathbf{t},\mathbf{d},\mathbf{e},N}(l)$ to be zero if it is not defined by one of the above. It is easy to see from the mixingale estimates (3.13) and (3.14) that the sequence $V_{\mathbf{t},\mathbf{d},\mathbf{e},N}(l)$, $l = 0, 1, ..., \ell L(N)$ forms a mixingale array with the corresponding σ-algebras $\mathcal{G}_l = \mathcal{G}_l^{(i)}$ for $i \leq k$ and $\mathcal{G}_{l+(i-k)L(N)} = \mathcal{G}_l^{(i)}$ for $i \geq k+1$. Taking into account (3.23) and (6.6) we conclude by [18] and [19] similarly to above that each linear combination $\sum_{a=1}^{j} \sum_{i=1}^{\ell} d_a e_i \xi_{i,N}(t^{(a)})$ converges in distribution as $N \to \infty$ to some Gaussian random variable which then must be $\sum_{a=1}^{j} \sum_{i=1}^{\ell} d_a e_i \eta_i(t^{(a)})$. Thus $(\eta_1, \eta_2, ..., \eta_\ell)$ is an ℓ-dimensional Gaussian random field. Invoking again the linear transformation argument from Section 2.2 of [4] we conclude both that the field $\xi(t)$ given by (2.21) is a Gaussian one and taking into account the vanishing limiting covariances result (4.18) we obtain also independency of η_i, $i \geq k+1$ of each other and of η_i's with $i \leq k$. $\square$

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