

Chapter III. Universal Algebra of a Lie Algebra

1. Definition

Let k be a commutative ring and let \mathfrak{g} be a Lie algebra over k .

Definition 1.1. A *universal algebra* of \mathfrak{g} is a map $\varepsilon : \mathfrak{g} \rightarrow U\mathfrak{g}$, where $U\mathfrak{g}$ is an associative algebra, with a unit satisfying the following properties:

- 1). ε is a Lie algebra homomorphism,

$$\text{(i.e., } \varepsilon \text{ is } k\text{-linear and } \varepsilon[x, y] = \varepsilon x \cdot \varepsilon y - \varepsilon y \cdot \varepsilon x \text{)}.$$

- 2). If A is any associative algebra with a unit and $\alpha : \mathfrak{g} \rightarrow A$ is any Lie algebra homomorphism, there is a unique homomorphism of associative algebras $\varphi : U\mathfrak{g} \rightarrow A$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varepsilon} & U\mathfrak{g} \\ \alpha \downarrow & \swarrow \varphi & \\ A & & \end{array}$$

is commutative [i.e., there is an isomorphism

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, LA) \cong \text{Hom}_{\text{Ass}}(U\mathfrak{g}, A)$$

where LA is the Lie algebra associated to A , cf. Chap. I, example (iii).]

It is trivial that $U\mathfrak{g}$, if it exists, is unique (up to a unique isomorphism). To prove its existence, we use the *tensor algebra* $T\mathfrak{g}$ of \mathfrak{g} , i.e., $T\mathfrak{g} = \sum_{n=0}^{\infty} T^n\mathfrak{g}$, where $T^n\mathfrak{g} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} = \bigotimes^n \mathfrak{g}$ for $n \geq 0$. For any associative algebra A with a unit, one has: $\text{Hom}_{\text{Mod}}(\mathfrak{g}, A) = \text{Hom}_{\text{Ass}}(T\mathfrak{g}, A)$.

Now let I be the two-sided ideal of $T\mathfrak{g}$ generated by the elements of the form $[x, y] - x \otimes y + y \otimes x$, $x, y \in \mathfrak{g}$.

Take $U\mathfrak{g} = T\mathfrak{g}/I$, then we have:

Theorem 1.2. Let $\varepsilon : \mathfrak{g} \rightarrow U\mathfrak{g}$ be the composition $\mathfrak{g} \rightarrow T^1\mathfrak{g} \rightarrow T\mathfrak{g} \rightarrow U\mathfrak{g}$. Then the pair $(U\mathfrak{g}, \varepsilon)$ is a universal algebra of \mathfrak{g} .

In fact, let α be a Lie homomorphism of \mathfrak{g} into an associative algebra A . Since α is k -linear, it extends to a unique homomorphism $\psi : T\mathfrak{g} \rightarrow A$. It is clear that $\psi(I) = 0$, hence ψ defines $\varphi : U\mathfrak{g} \rightarrow A$, and we have checked the universal property of $U\mathfrak{g}$.

Remark. Let E be a \mathfrak{g} -module (i.e., a k -module with a bilinear product $\mathfrak{g} \times E \rightarrow E$ such that $[x, y]e = x(ye) - y(x \cdot e)$ for $x, y \in \mathfrak{g}$, $e \in E$). The map $\mathfrak{g} \rightarrow \text{End}(E, E)$ which defines the module structure of E is a Lie homomorphism. Hence it extends to an algebra homomorphism $U\mathfrak{g} \rightarrow \text{End}(E, E)$ and E becomes a $U\mathfrak{g}$ -left-module. It is easy to check that one obtains in this

way an *isomorphism* of the category of \mathfrak{g} -modules onto the category of $U\mathfrak{g}$ -left-modules.

Exercise (Bergman). Prove that $U\mathfrak{g} = k \iff \mathfrak{g} = 0$. (Hint: use the adjoint representation.)

2. Functorial properties

- 1). If $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$, then $U\mathfrak{g} = \varinjlim U\mathfrak{g}_\alpha$.
- 2). If $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, where \mathfrak{g}_1 and \mathfrak{g}_2 commute, then $U\mathfrak{g} = U\mathfrak{g}_1 \otimes U\mathfrak{g}_2$.
- 3). Let k' be an extension of k and let $\mathfrak{g}' = \mathfrak{g} \otimes_k k'$, then $U\mathfrak{g}' = U\mathfrak{g} \otimes_k k'$.

Proof of 2). Consider the homomorphisms $\varepsilon_i : \mathfrak{g}_i \rightarrow U\mathfrak{g}_i$, $i = 1, 2$, $f : \mathfrak{g} \rightarrow U\mathfrak{g}_1 \otimes U\mathfrak{g}_2$ given by $f(x) = \varepsilon(x_1) \otimes 1 + 1 \otimes \varepsilon(x_2)$ where $x = x_1 + x_2$ with $x_1 \in \mathfrak{g}_1$, $x_2 \in \mathfrak{g}_2$. The map f is a Lie algebra homomorphism since \mathfrak{g}_1 commutes with \mathfrak{g}_2 . Hence f induces an associative algebra homomorphism $\psi : U\mathfrak{g} \rightarrow U\mathfrak{g}_1 \otimes U\mathfrak{g}_2$.

On the other hand we have the homomorphisms $\mathfrak{g}_i \rightarrow \mathfrak{g} \rightarrow U\mathfrak{g}$, $i = 1, 2$, which induce homomorphisms $\varphi_i : U\mathfrak{g}_i \rightarrow U\mathfrak{g}$ and since \mathfrak{g}_1 commutes with \mathfrak{g}_2 we have that $\varphi_1(x_1)\varphi_2(x_2) = \varphi_2(x_2)\varphi_1(x_1)$ for all $x_1 \in \mathfrak{g}_1$, $x_2 \in \mathfrak{g}_2$.

Finally take $\varphi : U\mathfrak{g}_1 \otimes U\mathfrak{g}_2 \rightarrow U\mathfrak{g}$ given by $\varphi(x_1 \otimes x_2) = \varphi_1(x_1)\varphi_2(x_2)$, then we have $\psi \circ \varphi = \text{id}$ and $\varphi \circ \psi = \text{id}$.

The proof of 1) and 3) are similar.

3. Symmetric algebra of a module

Let \mathfrak{g} be a k -module and define $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. In this case, the universal algebra $U\mathfrak{g}$ of \mathfrak{g} is called the *symmetric algebra* of the k -module \mathfrak{g} and it is denoted by $S\mathfrak{g}$.

We can define $S\mathfrak{g}$ as the largest commutative quotient of $T\mathfrak{g}$, i.e., $S\mathfrak{g} = \sum_{n=0}^{\infty} S^n\mathfrak{g}$ where $S^n\mathfrak{g} = (\otimes^n \mathfrak{g})/I$ where I is generated by the elements of the form $a - \sigma a$ where σ is a permutation of $[1, n]$, and $a \in \otimes^n \mathfrak{g}$.

We will consider the case where \mathfrak{g} is a free k -module with basis $(e_i)_{i \in I}$.

Let $\varepsilon : \mathfrak{g} \rightarrow k[(X_i)_{i \in I}]$ be the homomorphism given by $\varepsilon(e_i) = X_i$ where $k[(X_i)_{i \in I}]$ is the polynomial ring in the indeterminates X_i , $i \in I$. Then $(\varepsilon, k[(X_i)_{i \in I}])$ has the universal property of 1.1, i.e., ε is a k -linear map such that $\varepsilon(x)\varepsilon(y) = \varepsilon(y)\varepsilon(x)$ and if $f : \mathfrak{g} \rightarrow A$ is a k -linear map with $f(x)f(y) = f(y)f(x)$ for all $x, y \in \mathfrak{g}$ where A is an associative algebra, then there exists an associative algebra homomorphism $f^* : k[(X_i)] \rightarrow A$ such that $f^* \circ \varepsilon = f$. In fact if $P(x_i) \in k[(X_i)]$ then $f^*(P) = P(f(e_i))$. This shows that we can identify $S\mathfrak{g}$ with the polynomial algebra $k[(X_i)_{i \in I}]$.

If we assume that I is totally ordered, then $S\mathfrak{g}$ has for basis the set of monomials $e_{i_1} \cdots e_{i_n}$, $i_1 \leq i_2 \leq \cdots \leq i_n$, $n \geq 0$.

4. Filtration of $U\mathfrak{g}$

Let \mathfrak{g} be a Lie algebra over k , and let $U\mathfrak{g}$ be the universal algebra of \mathfrak{g} . We define a filtration of $U\mathfrak{g}$ as follows: Let $U_n\mathfrak{g}$ be the submodule of $U\mathfrak{g}$ generated by the products $\varepsilon(x_1)\cdots\varepsilon(x_m)$, $m \leq n$, where $x_i \in \mathfrak{g}$. We have

$$\begin{aligned} U_0\mathfrak{g} &= k \\ U_1\mathfrak{g} &= k \oplus \varepsilon(\mathfrak{g}) \end{aligned}$$

and $U_0\mathfrak{g} \subset U_1\mathfrak{g} \subset \cdots \subset U_n\mathfrak{g} \subset U_{n+1}\mathfrak{g} \subset \cdots$.

Now we define $\text{gr } U\mathfrak{g} = \sum_{n=0}^{\infty} \text{gr}_n U\mathfrak{g}$, where $\text{gr}_n U\mathfrak{g} = U_n\mathfrak{g}/U_{n-1}\mathfrak{g}$.

The map $U_p\mathfrak{g} \times U_q\mathfrak{g} \rightarrow U_{p+q}\mathfrak{g}$ given by $(a, b) \mapsto ab$ defines, by passage to quotient, a bilinear map

$$\text{gr}_p U\mathfrak{g} \times \text{gr}_q U\mathfrak{g} \rightarrow \text{gr}_{p+q} U\mathfrak{g}.$$

We then obtain a structure of *graded algebra* on $\text{gr } U\mathfrak{g}$; with this structure $\text{gr } U\mathfrak{g}$ is called the *graded algebra* associated to $U\mathfrak{g}$. It is associative and has a unit.

Proposition 4.1. *The algebra $\text{gr } U\mathfrak{g}$ is generated by the image of \mathfrak{g} under the map induced by $\varepsilon : \mathfrak{g} \rightarrow U\mathfrak{g}$.*

Proof. Let $\alpha \in \text{gr}_n U\mathfrak{g}$ and let $a \in U_n\mathfrak{g}$ be a representative of α , i.e., $\bar{a} = \alpha$. Now, we have $a = \sum_{m_\mu \leq n} \lambda_\mu \varepsilon(x_1^{(\mu)}) \cdots \varepsilon(x_{m_\mu}^{(\mu)})$. Thus we have

$$\alpha = \sum_{m_\mu = n} \lambda_\mu \overline{\varepsilon(x_1^{(\mu)}) \cdots \varepsilon(x_{m_\mu}^{(\mu)})} \quad \text{q.e.d.}$$

Theorem 4.2. *The algebra $\text{gr } U\mathfrak{g}$ is commutative.*

Proof. Using 4.1 it is enough to prove that $\overline{\varepsilon(x)}, \overline{\varepsilon(y)}$ commute in $\text{gr}_2 U\mathfrak{g}$ for all $x, y \in \mathfrak{g}$.

Since ε is a Lie algebra homomorphism we have

$$\varepsilon(x)\varepsilon(y) - \varepsilon(y)\varepsilon(x) = \varepsilon([x, y]),$$

but $\varepsilon([x, y]) \in U_1\mathfrak{g}$ so $\varepsilon(x)\varepsilon(y) \equiv \varepsilon(y)\varepsilon(x) \pmod{U_1\mathfrak{g}}$. Therefore

$$\overline{\varepsilon(x)\varepsilon(y)} = \overline{\varepsilon(y)\varepsilon(x)}.$$

It follows from Theorem 4.2 that the canonical map $\mathfrak{g} \rightarrow \text{gr } U\mathfrak{g}$ extends to a homomorphism

$$\iota : S\mathfrak{g} \rightarrow \text{gr } U\mathfrak{g}$$

where $S\mathfrak{g}$ is the symmetric algebra of \mathfrak{g} (cf. III.3).

Since $\text{gr } U\mathfrak{g}$ is generated by the image of \mathfrak{g} , ι is *surjective*.

Theorem 4.3 (Poincaré-Birkhoff-Witt). *If \mathfrak{g} is a k -free module, then ι is an isomorphism.*

In order to prove the theorem we will prove first two lemmas.
 Let $(x_i)_{i \in I}$ be a basis of \mathfrak{g} and choose a total order in I .

Lemma 4.4. *The family of monomials $\varepsilon(x_{i_1}) \cdots \varepsilon(x_{i_m})$, $i_1 \leq \cdots \leq i_m$, $m \leq n$, generate $U^n \mathfrak{g}$ (as a k -module).*

Proof. We proceed by induction with respect to n .

For $n = 0$ the statement is trivial.

Suppose now $n > 0$ and take $a \in U^n \mathfrak{g}$. Then its image $\bar{a} \in \text{gr}^n U \mathfrak{g}$ is a polynomial of degree n in the $\varepsilon(x_i)$, but this implies a is a linear combination of products $\varepsilon(x_{i_1}) \cdots \varepsilon(x_{i_n})$, $i_1 \leq \cdots \leq i_n$ plus an element $a_1 \in U^{n-1} \mathfrak{g}$.

By the hypothesis of induction a_1 is a linear combination of products $\varepsilon(x_{i_1}) \cdots \varepsilon(x_{i_m})$, $i_1 \leq \cdots \leq i_m$, $m < n$. q.e.d.

Lemma 4.5. *The following statement is equivalent to 4.3:*

The family of monomials $\varepsilon(x_{i_1}) \cdots \varepsilon(x_{i_n})$, $i_1 \leq \cdots \leq i_n$, $n \geq 0$ is a basis of $U \mathfrak{g}$.

For $M = (i_1, \dots, i_m)$ with $i_1 \leq i_2 \leq \cdots \leq i_m$, write

$$x_M = \varepsilon(x_{i_1}) \cdots \varepsilon(x_{i_m}),$$

and denote the *length* of M by $\ell(M) = m$. For each $n \geq 0$ the elements x_M with $\ell(M) = n$ lie in $U_n \mathfrak{g}$, and their images \bar{x}_M in $\text{gr}_n U \mathfrak{g} = U_n \mathfrak{g} / U_{n-1} \mathfrak{g}$ are the images, under the map $\iota : S^n \mathfrak{g} \rightarrow \text{gr}_n U \mathfrak{g}$, of the monomial basis elements of $S^n \mathfrak{g}$. Thus, the injectivity of ι is equivalent to the non-existence of a relation

$$\sum_{\ell(M)=n} c_M x_M \equiv 0 \pmod{U_{n-1} \mathfrak{g}}$$

with some $c_M \neq 0$. By Lemma 4.4 this is the same as the non-existence of a relation

$$\sum_{\ell(M)=n} c_M x_M = \sum_{\ell(M)<n} c_M x_M,$$

with some c_M on the left not zero. But any non-trivial k -linear dependence relation among the x_M can be put in the latter form. Hence Lemma 4.5 is true, and we can now proceed to prove Theorem 4.3 in the new form.

To do so we can (and will) assume that I is *well-ordered*. Let V be the free k -module with basis $\{z_M\}$ where M runs through the set of all sequences (i_1, \dots, i_n) with $n \geq 0$ and $i_1 \leq i_2 \leq \cdots \leq i_n$ as above. If $i \in I$ and $M = (i_1, \dots, i_n)$, we define $i \leq M \iff i \leq i_1$, in which case we introduce the notation $iM = (i, i_1, \dots, i_n)$.

Main lemma. *We can make V into a \mathfrak{g} -module in such a way that $x_i z_M = z_{iM}$ whenever $i \leq M$.*

We shall first define a k -bilinear map $(x, v) \mapsto xv$ of $\mathfrak{g} \times V$ into V , and will then prove that it makes V a \mathfrak{g} -module, that is, satisfies

$$(1) \quad xyv - yxv = [x, y]v, \quad \text{for } x, y \in \mathfrak{g}, \text{ and } v \in V.$$

To define xv it suffices to define $x_i Z_M$ for all i and M , and to define $x_i Z_M$ we may assume by induction that $x_j Z_N$ has been defined for all $j \in I$ when $\ell(N) < \ell(M)$ and for $j < i$ when $\ell(N) = \ell(M)$. Moreover we assume that this has been done in such a way that the following holds:

$$(*) \quad x_j Z_N \text{ is a } k\text{-linear combination of } Z_L\text{'s with } \ell(L) \leq \ell(N) + 1.$$

We then put

$$(2) \quad x_i Z_M = \begin{cases} Z_{iM} & , \text{ if } i \leq M \\ x_j(x_i Z_N) + [x_i, x_j]Z_N & , \text{ if } M = jN \text{ with } i > j. \end{cases}$$

This makes sense because, in the second case, $x_i Z_N$ is already defined as a linear combination of Z_L 's with $\ell(L) \leq \ell(N) + 1 = \ell(M)$, and $[x_i, x_j]$ is a linear combination of x_k . Moreover the condition $(*)$ holds with j and N replaced by i and M .

To check (1) it suffices, by linearity, to show

$$(1') \quad x_i x_j Z_N - x_j x_i Z_N = [x_i, x_j]Z_N$$

for all i, j and N . Since both sides are skew symmetric and vanish when $i = j$, we may assume $i > j$. If $j \leq N$, then $x_j Z_N = Z_{jN}$ and (1') follows from the second case of our inductive definition (2) above. There remains the case $N = kL$, with $i > j > k$, when (1') becomes

$$(ijk) \quad x_i x_j x_k Z_L - x_j x_i x_k Z_L = [x_i, x_j]x_k Z_L.$$

By induction on $\inf(i, j)$, we know this equation does hold if we permute ijk cyclically, that is the equations (jki) and (kij) are correct. On the other hand, by induction on $\ell(N)$ we can assume $xyZ_L = yxZ_L + [x, y]Z_L$ for all $x, y \in \mathfrak{g}$. Thus the right hand side of (ijk) can be rewritten:

$$\begin{aligned} [x_i, x_j]x_k Z_L &= x_k[x_i, x_j]Z_L + [[x_i, x_j], x_k]Z_L \\ &= x_k x_i x_j Z_L - x_k x_j x_i Z_L + [[x_i, x_j], x_k]Z_L. \end{aligned}$$

If we substitute this on the right side of (ijk) and then add the three equations $(ijk) + (jki) + (kij)$ we get an equation of the form

$$\sum = \sum + \text{Jac}(x_i, x_j, x_k)Z_L.$$

Hence, (ijk) is true, and our main lemma is proved.

Since V is a \mathfrak{g} -module, it is also a $U\mathfrak{g}$ -left module, cf. Remark at the end of III.1.

In particular we have in V the element Z_\emptyset where \emptyset is the empty set. For all M we have $x_M Z_\emptyset = Z_M$. We will prove this by induction on $\ell(M)$. If

$\ell(M) = 0$ then it is clear because $x_M = 1$. If $\ell(M) > 0$ we write $M = iN$, $i \leq N$. Then $x_M = x_i x_N$ and $x_M Z_\emptyset = x_i x_N Z_\emptyset = x_i Z_N = Z_{iN} = Z_M$.

Finally, suppose we have $\sum c_M x_M = 0$, then

$$0 = \sum c_M x_M Z_\emptyset = \sum c_M Z_M,$$

but this implies $c_M = 0$ for all M . q.e.d.

Corollary 1. *If \mathfrak{g} is a free k -module then $\varepsilon : \mathfrak{g} \rightarrow U\mathfrak{g}$ is injective.*

In fact, in this case $\mathfrak{g} \cong \text{gr}_1 U\mathfrak{g}$.

Corollary 2. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where \mathfrak{g}_1 and \mathfrak{g}_2 are subalgebras of \mathfrak{g} and are free k -modules. Then the map $U\mathfrak{g}_1 \otimes U\mathfrak{g}_2 \rightarrow U\mathfrak{g}$ given by $u_1 \otimes u_2 \mapsto u_1 u_2$ is a k -linear isomorphism.*

Proof. Let $(x_i)_{i \in I}, (y_j)_{j \in J}$ be a basis of \mathfrak{g}_1 and \mathfrak{g}_2 respectively, then $\{(x_i), (x_j)\}$ is a basis of \mathfrak{g} . Take a total order in $I \cup J$ such that every element of I is less than every element of J . Applying 4.5 we have that the families of monomials $\{\varepsilon(x_{i_1}) \cdots \varepsilon(x_{i_n})\}$, $\{\varepsilon(y_{j_1}) \cdots \varepsilon(y_{j_m})\}$ and $\{\varepsilon(x_{i_1}) \cdots \varepsilon(x_{i_n}) \varepsilon(y_{j_1}) \cdots \varepsilon(y_{j_m})\}$ for $i_1 \leq \cdots \leq i_n$ and $j_1 \leq \cdots \leq j_m$ are basis of $U\mathfrak{g}_1$, $U\mathfrak{g}_2$ and $U\mathfrak{g}$ respectively. Thus the map $U\mathfrak{g}_1 \otimes U\mathfrak{g}_2 \rightarrow U\mathfrak{g}$ given by $u_1 \otimes u_2 \mapsto u_1 u_2$ is a bijection on the basis of $U\mathfrak{g}_1 \otimes U\mathfrak{g}_2$ and $U\mathfrak{g}$. q.e.d.

Notice that in this case we have also induced an isomorphism

$$\text{gr } U\mathfrak{g}_1 \otimes \text{gr } U\mathfrak{g}_2 \xrightarrow{\cong} \text{gr } U\mathfrak{g}$$

because $\text{gr } U\mathfrak{g}_i = S\mathfrak{g}_i$ and $\text{gr } U\mathfrak{g} = S\mathfrak{g} \simeq S\mathfrak{g}_1 \otimes S\mathfrak{g}_2$.

5. Diagonal map

Let \mathfrak{g} be a Lie algebra over k and suppose \mathfrak{g} is free as a k -module.

Definition 5.1. The Lie algebra homomorphism $\Delta : \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$ given by $x \mapsto (x, x)$ induces a homomorphism of associative algebras

$$\Delta : U\mathfrak{g} \rightarrow U\mathfrak{g} \otimes U\mathfrak{g},$$

which is called the *diagonal map*.

Proposition 5.2. *The diagonal map Δ is characterized by the following two conditions:*

- 1) Δ is an algebra homomorphism.
- 2) $\Delta x = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$.

Notice that we identify $x \in \mathfrak{g}$ with its image in $U\mathfrak{g}$.

