indeed, $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{g}$ because $\mathfrak{g}$ is an ideal, and $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$ because $\mathfrak{a}$ is stable by $\mathfrak{g}$. It follows that $\mathfrak{a}$ consists *exactly* of those $y \in \mathfrak{h}$ such that $[\mathfrak{g}, y] = (0)$, because, writing $y = x + a$, with $x \in \mathfrak{g}$, $a \in \mathfrak{a}$, we have $[\mathfrak{g}, y] = [\mathfrak{g}, x]$, and $[\mathfrak{g}, x] = (0)$ implies $x = 0$ because the center of $\mathfrak{g}$ is zero. This shows that $\mathfrak{a}$ is unique, even as $\mathfrak{g}$-submodule, and also that $\mathfrak{a}$ is an ideal in $\mathfrak{h}$, because it is the annihilator of the $\mathfrak{h}$-module $\mathfrak{g}$.

Corollary 2. *If $\mathfrak{g}$ is semisimple, then every derivation of $\mathfrak{g}$ is of the form $\operatorname{ad} x$, with $x \in \mathfrak{g}$.*

Apply the preceding corollary with $\mathfrak{h} = \operatorname{Der}(\mathfrak{g})$, the Lie algebra of derivations of $\mathfrak{g}$. It is true that $\mathfrak{g}$ is an ideal in $\operatorname{Der}(\mathfrak{g})$, because for $x \in \mathfrak{g}$ and $D \in \operatorname{Der}(\mathfrak{g})$, we have $[D, \operatorname{ad} x] = \operatorname{ad}(Dx)$. Hence $\operatorname{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{a}$, where $\mathfrak{a}$ consists of the derivations commuting with $\operatorname{ad} \mathfrak{g}$. Let $D \in \mathfrak{a}$. Then $\operatorname{ad}(Dx) = [D, \operatorname{ad} x] = 0$. Hence $Dx = 0$, because the center of $\mathfrak{g}$ is zero. Hence $\mathfrak{a} = 0$. q.e.d.

4. Levi's theorem

Let $\mathfrak{g}$ be a Lie algebra.

Theorem 4.1 (Levi). *Let $\phi : \mathfrak{g} \to \mathfrak{s}$ be a surjective homomorphism of $\mathfrak{g}$ onto a semisimple Lie algebra $\mathfrak{s}$. Then there exists a homomorphism $\varepsilon : \mathfrak{s} \to \mathfrak{g}$ such that $\phi \circ \varepsilon = 1_{\mathfrak{s}}$.*

Let $\mathfrak{a} = \operatorname{Ker} \phi$, and write $\mathfrak{s} = \mathfrak{g}/\mathfrak{a}$. The *crucial case* of the theorem is that in which $\mathfrak{a}$ is abelian, and is a simple $\mathfrak{g}$- (or $\mathfrak{s}$-) module with non-trivial action. The first step of the proof is the *reduction to the crucial case.* Suppose $\mathfrak{a}_1$ is an ideal in $\mathfrak{g}$, and $0 \subset \mathfrak{a}_1 \subset \mathfrak{a}$. If we can find a supplementary subalgebra $\mathfrak{s}_1 = \mathfrak{g}_1/\mathfrak{a}_1$ to $\mathfrak{a}/\mathfrak{a}_1$ in $\mathfrak{g}/\mathfrak{a}_1$, and a supplementary subalgebra $\mathfrak{s}_2$ to $\mathfrak{a}_1$ in $\mathfrak{g}_1$, then $\mathfrak{s}_2$ is supplementary to $\mathfrak{a}$ in $\mathfrak{g}$. Hence, by induction on $\dim \mathfrak{a}$, we may suppose $\mathfrak{a}$ is a simple $\mathfrak{g}$-module. The radical $\mathfrak{r}$ of $\mathfrak{g}$ is in $\mathfrak{a}$. If $\mathfrak{r} = 0$, then $\mathfrak{g}$ is semisimple, and we are done, by Theorem 2.2. If $\mathfrak{r} = \mathfrak{a}$, then $\mathfrak{a}$ is solvable, hence $\mathfrak{a} \neq [\mathfrak{a}, \mathfrak{a}]$. But $[\mathfrak{a}, \mathfrak{a}]$ is an ideal in $\mathfrak{g}$, so $[\mathfrak{a}, \mathfrak{a}] = 0$, i.e., $\mathfrak{a}$ is abelian. If $\mathfrak{g}$ acts trivially on $\mathfrak{a}$, then $\mathfrak{a}$ is in the center of $\mathfrak{g}$, hence $\mathfrak{g}$ operates on $\mathfrak{g}$ through $\mathfrak{g}/\mathfrak{a} \simeq \mathfrak{s}$, and $\mathfrak{g}$ is completely reducible as an $\mathfrak{s}$-module, so there is an ideal supplementary to $\mathfrak{a}$.

Assume now we are in the *crucial case*: $\mathfrak{a}$ abelian, and a simple $\mathfrak{s}$-module with non-trivial action. If we had cohomology at our disposal, and knew that the extensions of $\mathfrak{s}$ by $\mathfrak{a}$ are classified by $H^2(\mathfrak{s}, \mathfrak{a}) = \operatorname{Ext}^2_{U\mathfrak{s}}(k, \mathfrak{a})$, we would be finished, because we could use a Casimir element to show that the Ext group is zero. But not having cohomology, we resort to the following argument of Bourbaki:

Lemma. *Let W be a $\mathfrak{g}$-module. Suppose an element $w \in W$ satisfies the conditions*

a) *the map* $a \mapsto aw$ *is a bijection* $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}w$;

b) $\mathfrak{g}w = \mathfrak{a}w$.

Let $\mathfrak{i}_w = \{ x \in \mathfrak{g} \mid xw = 0 \}$ *be the stabilizer of* w. *Then* $\mathfrak{i}_w$ *is a Lie subalgebra of* $\mathfrak{g}$, *and* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{i}_w$ (direct sum as vector spaces).

The lemma is completely trivial. Our problem now is to construct a suitable w. We let $W = \mathrm{End}(\mathfrak{g})$, viewed as $\mathfrak{g}$-module in the usual way, the representation $\sigma : \mathfrak{g} \to \mathrm{End}\, W = \mathrm{End}\,\mathrm{End}\,\mathfrak{g}$ being defined by $\sigma(x)\phi = \mathrm{ad}\, x \circ \phi - \phi \circ \mathrm{ad}\, x = [\mathrm{ad}\, x, \phi]$. We define three subspaces $P \subset Q \subset R \subset W$ as follows:

$$P = \{ \mathrm{ad}_{\mathfrak{g}} a \mid a \in \mathfrak{a} \}$$
$$Q = \{ \phi \in W \mid \phi\mathfrak{g} \subset \mathfrak{a} \text{ and } \phi\mathfrak{a} = 0 \}$$
$$R = \{ \phi \in W \mid \phi\mathfrak{g} \subset \mathfrak{a} \text{ and } \phi \mid \mathfrak{a} \text{ is a homothety} \} .$$

We leave to the reader the task of showing that these are $\mathfrak{g}$-submodules of W. We have an exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow Q \xrightarrow{i} R \xrightarrow{\varrho} k \to 0$$

where i is the inclusion, and ϱ the map which associates with each $r \in R$ the scalar by which r multiplies elements of $\mathfrak{a}$. If $x \in \mathfrak{a}$ and $\phi \in R$, then $\sigma(x)\phi = \mathrm{ad}\, x \circ \phi - \phi \circ \mathrm{ad}\, x = -\lambda\, \mathrm{ad}\, x$, where $\lambda = \varrho(\phi) \in k$. Thus, $\sigma(x)R \subset P$, for $x \in \mathfrak{a}$, and the exact sequence

$$0 \longrightarrow Q/P \longrightarrow R/P \xrightarrow{\bar{\varrho}} k \to 0$$

may be viewed as a sequence of $\mathfrak{s}$-modules. By the principle of lifting invariants, there exists $\bar{w} \in R/P$ such that $\bar{\varrho}(\bar{w}) = 1$, and such that $\bar{w}$ is invariant by $\mathfrak{s}$. Let w be an inverse of image of $\bar{w}$ in R. We contend that w satisfies the conditions of the lemma above.

a) Let $a \in \mathfrak{a}$. Then $\sigma(a)w = -\mathrm{ad}\, a$. If $\sigma(a)w = 0$, then $\mathrm{ad}_{\mathfrak{g}}\, a = 0$, that is, $[a, x] = 0$ for all $x \in \mathfrak{g}$. This implies $a = 0$, because $\mathfrak{a}$ is simple, and $\mathfrak{g}$ acts non-trivially.

b) Let $x \in \mathfrak{g}$. We must show that $\sigma(x)w$ is of the form $\sigma(a)w$ for some $a \in \mathfrak{a}$. Since $\sigma(a)w = -\mathrm{ad}_{\mathfrak{g}}\, a$, this amount to showing $\sigma(x)w \in P$. But that is just the invariance of $\bar{w}$. q.e.d.

Corollary 1. *An arbitrary Lie algebra* $\mathfrak{g}$ *is the semi-direct product of its radical* $\mathfrak{r}$ *and a semisimple subalgebra.*

One applies Theorem 4.1 to $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$.

Remark. This corollary has a complement, due to Malcev, which says that, if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two subalgebras of $\mathfrak{g}$ such that $\mathfrak{r} \oplus \mathfrak{s}_i = \mathfrak{g}$, there exists an automorphism σ of $\mathfrak{g}$ such that $\sigma(\mathfrak{s}_1) = \mathfrak{s}_2$ [one can even choose σ of the special form $e^{\mathrm{ad}(a)}$, where $a \in \mathfrak{r}$, and $\mathrm{ad}(a)$ is nilpotent]. When $\mathfrak{r}$ is