0.1. **Projective representations.** Let G be a group V a vector space. A map $A : G \to GL(V), g \to A_g$ is a *projective representation* if there exist a function $c : G\times G \to \mathbb{C}^\star$ such that $A_{g'}A_{g''} = c(g', g'')A_{g'g''}$ for all $g', g'' \in G$. Why would anybody consider projective representations?

Let H, G be groups, and G acts on H by automorphisms $h \to h^g, h \in H, g \in G$. Let $\pi : H \to GL(V)$ be an irreducible representation of the group H. For any $g \in G$ the map $\pi^g : H \to GL(V), \pi^g(h) := \pi(h^g)$ is also a representation of H. Assume that for any $g \in G$ the representation π^g of H is equivalent to the representation π. Then there exists $A_g \in Gl(V)$ such that $\pi(h)A_g = A_g\pi^g(h)$ for all $h \in H$.

Lemma 0.1. *The map $A : G \to GL(V), g \to A_g$ is a projective representation.*

Proof. Then for any $g', g'' \in G$ we have

$$\pi(h)A_{g'}A_{g''} = A_{g'}\pi^{g'}(h)A_{g''} = A_{g'}\pi(h^{g'})A_{g''} = A_{g'}A_{g''}\pi(h^{g'}g''^{)} = A_{g'}A_{g''}\pi(h^{g'g''})$$

Define $C(g', g'') := A^{-1}_{g'g''}A_{g'}A_{g''} \in GL(V)$. Then

$$C(g', g'')\pi(h) = A^{-1}_{g'g''}A_{g'}A_{g''}\pi(h) = A^{-1}_{g'g''}\pi(h^{g'g''})A_{g'}A_{g''} =$$

$$\pi(h)A^{-1}_{g'g''}A_{g'}A_{g''} = \pi(h)C(g', g'')$$

Since π is an irreducible representation it follows from the Schur lemma that $C(g', g'') = c(g', g'')Id_V, c(g', g'') \in \mathbb{C}^\star$. In other words the map $g \to A(g)$ defines a *projective representation* of the group G. □

Given a projective representation $A : G \to GL(V), g \to A_g$ one can ask whether one can correct a choice of operators $A_g \in Gl(V)$ to obtain an honest representation of the group G. In other way could one find a function $d : G \to \mathbb{C}^\star$ such that the map $g \to d(g)A_g$ is representation of group G. It is clear that this is possible iff one can find $d_g \in \mathbb{C}^\star$ such that $c(g', g'') = d^{-1}_{g'g''}d_{g'}d_{g''}$ for all $g', g'' \in G$.

How to see an obstruction to an existence of $d_g \in \mathbb{C}^\star$ such that $c(g', g'') = d^{-1}_{g'g''}d_{g'}d_{g''}$ for all $g', g'' \in G$? For any pair $g', g'' \in G$ the commutator $q(g', g'') := A_{g'}A_{g''}A^{-1}_{g'}A^{-1}_{g''}$ does not change if you replace A_g by $d(g)A_g$. So if there exist two pairs $(g', g''), ((s', s'')$ of elements of the group G such that $g'g''g'^{-1}g''^{-1} = s's''s'^{-1}s''^{-1}$ but $q(g', g'') \neq q(s', s'')$ the projective representation $g \to A_g$ can not be corrected to obtain an honest representation $g \to d(g)A_g$. In particular if there exists a pair $g', g'' \in G$ of commuting elements such that $q(g', g'') \neq Id$ then there is no way to correct the projective representation $A : g \to A_g$.

Example 0.2. Since the subgroup $Z = \{(0,0;a)\} \subset H$ is in the center of the group H the group $L = H/Z$ acts on H be the conjugation. In this case we can choose $A_l = \pi(l;0)$. It is clear that for any $l', l'' \in L$ we have $q(l', l'') = \psi(< l', l'' >)$. Since the group L is commutative we see that there exists no representation $A : L \to GL(V)$ such that $A(l)\pi(h) = \pi^l(h)A(l)$ for all $h \in H$.

Remark 0.3. R:pr a) Let $A : G \to GL(V), g \to A_g$ be a projective representation which could be corrected to an honest representation $g \to B(g)$. That is there exists a function $d : G \to \mathbb{C}^\star$ such that the map $g \to B(g) := d(g)A_g$ is representation of group G. Then $B(g'g''g'^{-1}g''^{-1}) = q(g', g'')$ for any pair $g', g'' \in G$.

b) If $G = [G, G]$ then a correction $B(g) := d(g)A_g$ is unique if it exists.

0.2. **Weil representations.** We choose a basis e_1, e_2 in L such that $< e_1, e_2 >= 1$ and identify L with $\mathbb{F}_q^2$. The group $SL_2(\mathbb{F}_q)$ acts on L and preserves the form $<,>$. Therefore the map $h \to h^g, (l, a) \to (gl, a), (l, a) \in H, g \in SL_2(\mathbb{F}_q)$ is an automorphism of a group H which acts trivially on Z. Let $\pi : H \to GL(V)$ be an irreducible representation such that $\pi(0, a) = \psi(a)Id$ for any $a \in \mathbb{F}_q$. As follows from Proposition 21 c) in *examples* the representation π^g is equivalent to the representation π for any $g \in SL_2(\mathbb{F}_q)$.

Let $\pi : H \to GL(V)$ be the realization of π as in Example 1.22 in *examples*. How to find a map $A : G \to GL(V), g \to A_g$ such that $A_g\pi(h) = \pi^g(h)A_g$ for all $h \in H$? Consider elements $u_a, t_\lambda, w \in SL_2(\mathbb{F}_q)$ where

$$u_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad , t_\lambda = \begin{pmatrix} \lambda & \\ 0 & \lambda^{-1} \end{pmatrix} \quad , w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $a \in \mathbb{F}_q, \lambda \in \mathbb{F}_q^\star$.

How to find A_{u_a}? Let $h = (x, y; a)$. By the definition $h^{u_a} = (x + ay, y; a)$. We want to find an operator A_{u_a} such that $\pi(x, y; 0)A_{u_a} = A_{u_a}\pi(x, y+ax; 0)$ for all $x, y \in \mathbb{F}_q$. In particular we have $\pi(0, y; 0)A_{u_a} = A_{u_a}\pi(0, y; 0)$ for all $a \in \mathbb{F}_q$. So we see that the operator A_{u_a} commutes with operators $\pi(0, y; 0)$ for all $y \in \mathbb{F}_q$. In other word the operator $A_{u_a} : \mathbb{C}[\mathbb{F}_q] \to \mathbb{C}[\mathbb{F}_q]$ commutes with the multiplication by functions $\psi(az/2), z \in \mathbb{F}_q$ for all $a \in \mathbb{F}_q$. Since the span of functions $\psi(az/2), \in \mathbb{F}_q$ is equal to the set of all functions on $\mathbb{F}_q$ we see that an operator A_{u_a} commutes with the multiplication by any function on $\mathbb{F}_q$. Therefore A_{u_a} has a form $f(z) \to r_a(z)$ where $r_a(z)$ is a function on $\mathbb{F}_q$.

To find the function $r_a(z)$ we use the equation $\pi(x,0;0)A_{u_a}(\phi) = A_{u_a}\pi(x,ax;0)(\phi), \phi \in \mathbb{C}[\mathbb{F}_q]$. In other words

$$r_a(x+z)\phi(x+z) = \psi(ax^2/2 + axz)r_a(z)\phi(z)\forall\phi \in \mathbb{C}[\mathbb{F}_q]$$

In other words we have $r_a(x+z) = \psi(ax^2/2+axz)r_a(z)$. If we put $z = 0$ we see that $r_a(x) = c_a\psi(ax^2/2), c_a \in \mathbb{C}^\star$. In other words $A_{u_a}(\phi)(x) = c_a\psi(ax^2/2)\phi(x), \phi \in \mathbb{C}[\mathbb{F}_q]$.

Problem 0.4. *Show that* $A_{t_\lambda}(\phi)(x) = c_\lambda\phi(\lambda x)$ *and* $A_{t_\lambda}(\phi)(x) = c\mathcal{F}(\phi)(x)$.

Theorem 0.5. *T:W There exists a representation* $\omega : SL_2(\mathbb{F}_q) \to GL(V)$ *such that* $\omega(g)\pi(h) = \pi^g(h)\omega(g)$ *for all* $h \in H$ *and*

a) $\omega(u_a)(\phi)(x) = \psi(ax^2/2)\phi(x)$.

b) $\omega(t_\lambda)(\phi)(x) = \epsilon(\lambda)\phi(\lambda x)$ *where* $\epsilon : \mathbb{F}_q^\star \to \mathbb{C}^\star$ *is the unique non-trivial character of* $\mathbb{F}_q^\star$ *of order* 2.

c) $\omega(w) = 1/sqrtq\mathcal{F}$