CHAPTER 8

Examples of induced representations

8.1 Normal subgroups; applications to the degrees of the irreducible representations

Proposition 24. *Let* A *be a normal subgroup of a group* G, *and let* ρ: G $\rightarrow$ **GL**(V) *be an irreducible representation of* G. *Then*:

(a) *either there exists a subgroup* H *of* G, *unequal to* G *and containing* A, *and an irreducible representation* σ *of* H *such that* ρ *is induced by* σ;

(b) *or else the restriction of* ρ *to* A *is isotypic.*

(A representation is said to be *isotypic* if it is a direct sum of isomorphic irreducible representations.)

Let $V = \oplus V_i$ be the canonical decomposition of the representation ρ (restricted to A) into a direct sum of isotypic representations (cf. 2.6). For $s \in G$ we see by "transport de structure" that $\rho(s)$ permutes the V_i; since V is irreducible, G permutes them transitively. Let V_{i_0} be one of these; if V_{i_0} is equal to V, we have case (b). Otherwise, let H be the subgroup of G consisting of those $s \in G$ such that $\rho(s)V_{i_0} = V_{i_0}$. We have $A \subset H$, $H \neq G$, and ρ is induced by the natural representation σ of H in V_{i_0}, which is case (a). □

Remark. If A is abelian, (b) is equivalent to saying that $\rho(a)$ is a *homothety* for each $a \in A$.

Corollary. *If* A *is an abelian normal subgroup of* G, *the degree of each irreducible representation* ρ *of* G *divides the index* (G: A) *of* A *in* G.

The proof is by induction on the order of G. In case (a) of the preceding proposition the induction hypothesis shows that the degree of σ divides

(H: A), and by multiplying this relation by (G: H) we see that the degree of ρ divides (G: A). In case (b) let $G' = \rho(G)$ and $A' = \rho(A)$; since the canonical map $G/A \to G'/A'$ is surjective, (G': A') divides (G: A). Our previous remark shows now that the elements of A' are homotheties, thus are contained in the center of G'. By prop. 17 of 6.5, it follows that the degree of ρ divides (G': A') and *a fortiori* (G: A). □

Remark. If A is an *abelian subgroup* of G (not necessarily normal) it is no longer true in general that $\deg(\rho)$ divides (G: A), but *nevertheless we have* $\deg(\rho) \leqslant (G: A)$, cf. 3.1, cor. to th. 9.

8.2 Semidirect products by an abelian group

Let A and H be two subgroups of the group G, with A normal. Make the following hypotheses:

(i) A is *abelian.*
(ii) G is the *semidirect product* of H by A.

[Recall that (ii) means that $G = A \cdot H$ and that $A \cap H = \{1\}$, or in other words, that each element of G can be written uniquely as a product ah, with $a \in A$ and $h \in H$.]

We are going to show that the irreducible representations of G can be constructed from those of certain subgroups of H (this is the method of "little groups" of Wigner and Mackey).

Since A is abelian, its irreducible characters are of degree 1 and form a group $X = \mathrm{Hom}(A, \mathbf{C}^*)$. The group G acts on X by

$$(s\chi)(a) = \chi(s^{-1}as) \quad \text{for } s \in G, \chi \in X, a \in A.$$

Let $(\chi_i)_{i \in X/H}$ be a system of representatives for the orbits of H in X. For each $i \in X/H$, let H_i be the subgroup of H consisting of those elements h such that $h\chi_i = \chi_i$, and let $G_i = A \cdot H_i$ be the corresponding subgroup of G. Extend the function χ_i to G_i by setting

$$\chi_i(ah) = \chi_i(a) \quad \text{for } a \in A, h \in H_i.$$

Using the fact that $h\chi_i = \chi_i$ for all $h \in H_i$, we see that χ_i is a *character of degree* 1 of G_i. Now let ρ be an irreducible representation of H_i; by composing ρ with the canonical projection $G_i \to H_i$ we obtain an irreducible representation $\tilde{\rho}$ of G_i. Finally, by taking the tensor product of χ_i and $\tilde{\rho}$ we obtain an irreducible representation $\chi_i \otimes \tilde{\rho}$ of G_i; let $\theta_{i,\rho}$ be the corresponding induced representation of G.

Proposition 25

(a) *$\theta_{i,\rho}$ is irreducible.*
(b) *If $\theta_{i,\rho}$ and $\theta_{i',\rho'}$ are isomorphic, then $i = i'$ and ρ is isomorphic to ρ'.*
(c) *Every irreducible representation of G is isomorphic to one of the $\theta_{i,\rho}$.*

(Thus we have all the irreducible representations of G.)

We prove (a) using *Mackey's criterion* (7.4, prop. 23) as follows: Let $s \notin G_i = A \cdot H_i$, and let $K_s = G_i \cap sG_is^{-1}$. We have to show that, if we compose the representation $\chi_i \otimes \tilde{\rho}$ of G_i with the two injections $K_s \to G_i$ defined by $x \mapsto x$ and $x \mapsto s^{-1}xs$, we obtain two disjoint representations of K_s. To do this, it is enough to check that the restrictions of these representations to the subgroup A of K_s are disjoint. But the first restricts to a multiple of χ_i and the second to a multiple of $s\chi_i$; since $s \notin A \cdot H_i$ we have $s\chi_i \neq \chi_i$ and so the two representations in question are indeed disjoint.

Now we prove (b). First of all, the restriction of $\theta_{i,\rho}$ to A only involves characters χ belonging to the orbit $H\chi_i$ of χ_i. This shows that $\theta_{i,\rho}$ determines i. Next, let W be the representation space for $\theta_{i,\rho}$, and let W_i be the subspace of W corresponding to χ_i [i.e., the set of $x \in W$ such that $\theta_{i,\rho}(a)x = \chi_i(a)x$ for all $a \in A$]. The subspace W_i is stable under H_i, and one checks immediately that the representation of H_i in W_i is isomorphic to ρ; whence $\theta_{i,\rho}$ determines ρ.

Finally, let σ: $G \to \mathbf{GL}(W)$ be an irreducible representation of G. Let $W = \bigoplus_{\chi \in X} W$ be the canonical decomposition of $\mathrm{Res}_A W$. At least one of the W_χ is nonzero; if $s \in G$, $\sigma(s)$ transforms W_χ into $W_{s(\chi)}$. The group H_i maps W_{χ_i} into itself; let W_i be an irreducible sub - $\mathbf{C}[H_i]$-module of W_{χ_i} and let ρ be the corresponding representation of H_i. It is clear that the representation of $G_i = A \cdot H_i$ is isomorphic to $\chi_i \otimes \tilde{\rho}$. Thus the restriction of σ to G_i contains $\chi_i \otimes \tilde{\rho}$ at least once. By prop. 21, it follows that σ occurs at least once in the induced representation $\theta_{i,\rho}$; since $\theta_{i,\rho}$ is irreducible, this implies that σ and $\theta_{i,\rho}$ are isomorphic, which proves (c). □

EXERCISES

8.1. Let a, h, h_i be the orders of A, H, H_i respectively. Show that $a = \sum (h/h_i)$. Show that, for fixed i, the sum of the squares of the degrees of the representations $\theta_{i,\rho}$ is h^2/h_i. Deduce from this another proof of (c).

8.2. Use prop. 25 to recompute the irreducible representations of the groups D_n, $\mathfrak{A}_4$, and $\mathfrak{S}_4$ (cf. Ch. 5).

8.3 A review of some classes of finite groups

For more details on the results of this section and the following, see Bourbaki, Alg. I, §7.

Solvable groups. One says that G is solvable if there exists a sequence

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

of subgroups of G, with G_{i-1} normal in G_i and G_i/G_{i-1} abelian for

$1 \leqslant i \leqslant n$. (Equivalent definition: G is obtained from the group $\{1\}$ by a finite number of *extensions with abelian kernels*.)

Supersolvable groups. Same as above, except that one requires that all the G_i be normal in G and that G_i/G_{i-1} be cyclic.

Nilpotent groups. As above, except that G_i/G_{i-1} is required to be in the center of G/G_{i-1} for $1 \leqslant i \leqslant n$. (Equivalent definition: G is obtained from the group $\{1\}$ by a finite number of *central extensions*.)

It is clear that supersolvable $\Rightarrow$ solvable. On the other hand, one checks immediately that each central extension of a supersolvable group is supersolvable; thus nilpotent $\Rightarrow$ supersolvable.

p-groups. If p is a prime, a group whose order is a power of p is called a p-group.

Theorem 14. *Every p-group is nilpotent* (*thus supersolvable*).

In view of the preceding it suffices to show that the center of every nontrivial p-group G is nontrivial. This is a consequence of the following lemma:

Lemma 3. *Let* G *be a p-group acting on a finite set* X, *and let* X^G *be the set of elements of* X *fixed by* G. *We have*

$$\mathrm{Card}(X) \equiv \mathrm{Card}(X^G) \qquad (\mathrm{mod.}\, p).$$

Indeed $X - X^G$ is a union of nontrivial orbits of G, and the cardinality of each of these orbits is a power p^α of p, with $\alpha \geqslant 1$; hence $\mathrm{Card}(X - X^G)$ is divisible by p. □

Let us now apply this lemma to the case $X = G$ with G acting by inner automorphisms. The set X^G is just the *center* C of G. Thus

$$\mathrm{Card}(C) \equiv \mathrm{Card}(G) \equiv 0 \ (\mathrm{mod.}\, p),$$

whence $C \neq \{1\}$, which proves the theorem.

We record another application of lemma 3 which will be used in Part III:

Proposition 26. *Let* V *be a vector space* $\neq 0$ *over a field k of characteristic p and let* $\rho\colon G \to \mathbf{GL}(V)$ *be a linear representation of a p-group* G *in* V. *Then there exists a nonzero element of* V *which is fixed by all* $\rho(s)$, $s \in G$.

Let x be a nonzero element of V, and let X be the subgroup of V generated by the $\rho(s)x$, $s \in G$. We apply lemma 3 to X, observing that X is finite and of order a power of p. Therefore $X^G \neq \{0\}$, which proves the proposition. □

Corollary. *The only irreducible representation of a p-group in characteristic p is the trival representation.*

EXERCISES

8.3. Show that the dihedral group D_n is supersolvable, and that it is nilpotent if and only if n is a power of 2.

8.4. Show that the alternating group $\mathfrak{A}_4$ is solvable, but not supersolvable. Same question for the group $\mathfrak{S}_4$.

8.5. Show that each subgroup and each quotient of a solvable group (resp. supersolvable, nilpotent) is solvable (resp. supersolvable, nilpotent).

8.6. Let p and q be distinct prime numbers and let G be a group of order $p^a q^b$ where a and b are integers > 0.

(i) Assume that the center of G is $\{1\}$. For $s \in G$ denote by $c(s)$ the number of elements in the conjugacy class of s. Show that there exists $s \neq 1$ such that $c(s) \not\equiv 0$ (mod.q). (Otherwise the number of elements of $G - \{1\}$ would be divisible by q.) For such an s, $c(s)$ is a power of p; derive from this the existence of a normal subgroup of G unequal to $\{1\}$ or G [apply ex. 6.10].

(ii) Show that G is solvable (*Burnside's theorem*). [Use induction on the order of G and distinguish two cases, depending on whether the center of G is equal or unequal to $\{1\}$.]

(iii) Show by example that G is not necessarily supersolvable (cf. ex. 8.4).

(iv) Give an example of a nonsolvable group whose order is divisible by just three prime numbers [$\mathfrak{S}_5$, $\mathfrak{S}_6$, $\mathbf{GL}_2(\mathbf{F}_7)$ will do].

8.4 Sylow's theorem

Let p be a prime number, and let G be a group of order $g = p^n m$, where m is prime to p. A subgroup of G of order p^n is called a *Sylow p-subgroup* of G.

Theorem 15

(a) *There exist Sylow p-subgroups.*
(b) *They are conjugate by inner automorphisms.*
(c) *Each p-subgroup of* G *is contained in a Sylow p-subgroup.*

To prove (a) we use induction on the order of G. We may assume $n \geqslant 1$, i.e. Card (G) $\equiv 0$ (mod. p). Let C be the center of G. If Card (C) is divisible of order p, an elementary argument shows that C contains a subgroup D cyclic of order p. By the induction hypothesis, G/D has a Sylow p-subgroup, and the inverse image of this subgroup in G is a Sylow p-subgroup of G. If Card (C) $\not\equiv 0$ (mod. p) let G act on $G - C$ by inner automorphisms; this gives a partition of G − C into orbits (conjugacy classes). As Card(G − C) $\not\equiv$ 0 (mod. p), one of these orbits has a cardinality prime to p. It follows that there is a subgroup H unequal to G such that (G: H) $\not\equiv$ 0 (mod. p). The order of H is thus divisible by p^n, and the induction hypothesis shows that H contains a subgroup of order p^n.

Now let P be a Sylow p-subgroup of G and Q a p-subgroup of G. The p-group Q acts on X = G/P by left translations. By lemma 3 of 8.3 we have

$$\mathrm{Card}(X^Q) \equiv \mathrm{Card}(X) \not\equiv 0 \ (\mathrm{mod}.\, p),$$

whence $X^Q \neq \varnothing$. Thus there exists an element $x \in G$ such that $QxP = xP$, hence $Q \subset xPx^{-1}$, which proves (c). If in addition Card(Q) = p^n, the groups Q and xPx^{-1} have the same order, and $Q = xPx^{-1}$, which proves (b). □

EXERCISES

8.7. Let H be a normal subgroup of a group G and let P_H be a Sylow p-subgroup of G/H.
 (a) Show that there exists a Sylow p-subgroup P of G whose image in G/H is P_H [use the conjugacy of Sylow subgroups].
 (b) Show that P is unique if H is a p-group or if H is in the center of G [reduce to the case where H has order prime to p, and use the fact that each homomorphism from P_H into H is trivial].

8.8. Let G be a nilpotent group. Show that, for each prime number p, G contains a unique Sylow p-subgroup, which is normal [use induction on the order of G, and apply the induction hypothesis to the quotient of G by its center, cf. ex. 8.7(b)]. Conclude that G is a direct product of p-groups.

8.9. Let $G = \mathbf{GL}_n(k)$, where k is a finite field of characteristic p. Show that the subgroup of G which consists of all upper triangular matrices having only 1's on the diagonal is a Sylow p-subgroup of G.

8.5 Linear representations of supersolvable groups

Lemma 4. *Let* G *be a nonabelian supersolvable group. Then there exists a normal abelian subgroup of* G *which is not contained in the center of* G.

Let C be the center of G. The quotient H = G/C is supersolvable, thus has a composition series in which the first nontrivial term H_1 is a cyclic normal subgroup of H. The inverse image of H_1 in G has the required properties. □

Theorem 16. *Let* G *be a supersolvable group. Then each irreducible representation of* G *is induced by a representation of degree* 1 *of a subgroup of* G (*i.e., is monomial*).

We prove the theorem by induction on the order of G. Consequently we may consider only those irreducible representations ρ which are *faithful*, i.e., such that $\mathrm{Ker}(\rho) = \{1\}$. If G is abelian, such a ρ is of degree 1 and there is nothing to prove. Suppose G is not abelian, and let A be a normal abelian subgroup of G which is not contained in the center of G (cf. lemma 4). Since ρ is faithful, this implies that $\rho(A)$ is not contained in the center of $\rho(G)$; thus there exists $a \in A$ such that $\rho(a)$ is not a homothety. The restriction of ρ to A is thus not isotypic. By prop. 24, this implies that ρ is induced by an irreducible representation of a subgroup H of G which is unequal to G. The theorem now follows by applying induction to H. □

Exercises

8.10. Extend Theorem 16 to groups which are semidirect products of a supersolvable group by an abelian normal subgroup [use prop. 25 to reduce to the supersolvable case].

8.11. Let **H** be the field of quaternions over **R**, with basis $\{1, i, j, k\}$ satisfying

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i,$$
$$ki = -ik = j.$$

Let E be the subgroup of $\mathbf{H}^*$ consisting of the eight elements ± 1, $\pm i$, $\pm j$, $\pm k$ (quaternion group), and let G be the union of E and the sixteen elements $(\pm 1 \pm i \pm j \pm k)/2$. Show that G is a solvable subgroup of $\mathbf{H}^*$ which is a semidirect product of a cyclic group of order 3 by the normal subgroup E. Use the isomorphism $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{M}_2(\mathbf{C})$ to define an irreducible representation of degree 2 of G. Show that this representation is not monomial (observe that G has no subgroup of index 2). [The group G is the group of invertible elements of the ring of Hurwitz "integral quaternions"; it is also the group of automorphisms of the elliptic curve $y^2 - y = x^3$ in characteristic 2. It is isomorphic to $\mathbf{SL}_2(\mathbf{F}_3)$.]

8.12. Let G be a p-group. Show that, for each irreducible character χ of G, we have $\sum \chi'(1)^2 \equiv 0 \,(\mathrm{mod.}\,\chi(1)^2)$, the sum being over all irreducible characters χ' such that $\chi'(1) < \chi(1)$. [Use the fact that $\chi(1)$ is a power of p, and apply cor. 2(a) to prop. 5.]