CHAPTER 7

Induced representations; Mackey's criterion

7.1 Induction

Let H be a subgroup of a group G and R a system of left coset representatives for H. Let V be a $\mathbf{C}[G]$-module and let W be a sub-$\mathbf{C}[H]$-module of V. Recall (cf. 3.3) that the module V (or the representation V) is said to be *induced* by W if we have $V = \oplus_{s \in R} sW$, i.e., if V is a direct sum of the images sW, $s \in R$ (a condition which is independent of the choice of R). This property can be reformulated in the following way:

Let

$$W' = \mathbf{C}[G] \otimes_{\mathbf{C}[H]} W$$

be the $\mathbf{C}[G]$-module obtained from W by scalar extension from $\mathbf{C}[H]$ to $\mathbf{C}[G]$. The injection $W \to V$ extends by linearity to a $\mathbf{C}[G]$-homomorphism $i: W' \to V$.

Proposition 18. *In order that* V *be induced by* W, *it is necessary and sufficient that the homomorphism*

$$i: \mathbf{C}[G] \otimes_{\mathbf{C}[H]} W \to V$$

be an isomorphism.

This is a consequence of the fact that the elements of R form a basis of $\mathbf{C}[G]$ considered as a right $\mathbf{C}[H]$-module.

Remarks

(1) This characterization of the representation induced by W makes it obvious that the induced representation *exists* and is *unique* (cf. 3.3, th. 11).

In what follows, the representation of G induced by W will be denoted by $\mathrm{Ind}_H^G(W)$, or simply $\mathrm{Ind}(W)$ if there is no danger of confusion.

(2) If V is induced by W and if E is a $\mathbf{C}[G]$-module, we have a canonical isomorphism

$$\mathrm{Hom}^H(W, E) \cong \mathrm{Hom}^G(V, E),$$

where $\mathrm{Hom}^G(V, E)$ denotes the vector space of $\mathbf{C}[G]$-homomorphisms of V into E, and $\mathrm{Hom}^H(W, E)$ is defined similarly. This follows from an elementary property of tensor products (see also 3.3, lemma 1).

(3) Induction is *transitive*: if G is a subgroup of a group K, we have

$$\mathrm{Ind}_G^K(\mathrm{Ind}_H^G(W)) \cong \mathrm{Ind}_H^K(W).$$

This can be seen directly, or by using the associativity of the tensor product.

Proposition 19. *Let* V *be a* $\mathbf{C}[G]$*-module which is a direct sum* $V = \oplus_{i \in I} W_i$ *of vector subspaces permuted transitively by* G. *Let* $i_0 \in I$, $W = W_{i_0}$ *and let* H *be the stabilizer of* W *in* G (*i.e., the set of all* $s \in G$ *such that* $sW = W$). *Then* W *is stable under the subgroup* H *and the* $\mathbf{C}[G]$*-module* V *is induced by the* $\mathbf{C}[H]$*-module* W.

This is clear.

Remark. In order to apply proposition 19 to an irreducible representation $V = \oplus W_i$ of G, it is enough to check that the W_i are permuted among themselves by G; the transitivity condition is automatic, because each orbit of G in the set of W_i's defines a subrepresentation of V.

EXAMPLE. When the W_i are of dimension 1, the representation V is said to be *monomial*.

7.2 The character of an induced representation; the reciprocity formula

We keep the preceding notation. If f is a class function on H, consider the function f' on G defined by the formula

$$f'(s) = \frac{1}{h} \sum_{\substack{t \in G \\ t^{-1}st \in H}} f(t^{-1}st) \qquad \text{where } h = \mathrm{Card}(H).$$

We say that f' is *induced* by f and denote it by either $\mathrm{Ind}_H^G(f)$ or $\mathrm{Ind}(f)$.

Proposition 20.

(i) *The function* $\mathrm{Ind}(f)$ *is a class function on* G.
(ii) *If* f *is the character of a representation* W *of* H, $\mathrm{Ind}(f)$ *is the character of the induced representation* $\mathrm{Ind}(\mathrm{W})$ *of* G.

Assertion (ii) has already been proved (3.3, th. 12). Assertion (i) is proved by a direct calculation or can be obtained from (ii) and the observation that each class function is a linear combination of characters. □

Recall that, for φ_1 and φ_2 two class functions on G, we set

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{g} \sum_{s \in \mathrm{G}} \varphi_1(s^{-1})\varphi_2(s), \quad \text{where } g = \mathrm{Card}(\mathrm{G}),$$

cf. 2.2; when we wish to be more explicit about the group G, we write $\langle \varphi_1, \varphi_2 \rangle_{\mathrm{G}}$ instead of $\langle \varphi_1, \varphi_2 \rangle$.

Also, if V_1 and V_2 are two $\mathbf{C}[\mathrm{G}]$-modules, we set

$$\langle \mathrm{V}_1, \mathrm{V}_2 \rangle_{\mathrm{G}} = \dim . \mathrm{Hom}^{\mathrm{G}}(\mathrm{V}_1, \mathrm{V}_2).$$

Lemma 2. *If* φ_1 *and* φ_2 *are the characters of* V_1 *and* V_2, *we have*

$$\langle \varphi_1, \varphi_2 \rangle_{\mathrm{G}} = \langle \mathrm{V}_1, \mathrm{V}_2 \rangle_{\mathrm{G}}.$$

Decomposing V_1 and V_2 into direct sums, we can assume that they are irreducible, in which case the lemma follows from the orthogonality formulas for characters (2.3, th. 3). □

If φ (resp. V) is a function on G (resp. a representation of G), we denote by Res φ (resp. Res V) its *restriction* to the subgroup H.

Theorem 13 (Frobenius reciprocity). *If* ψ *is a class function on* H *and* φ *a class function on* G, *we have*

$$\langle \psi, \mathrm{Res}\ \varphi \rangle_{\mathrm{H}} = \langle \mathrm{Ind}\ \psi, \varphi \rangle_{\mathrm{G}}.$$

Since each class function is a linear combination of characters, we can assume that ψ is the character of a $\mathbf{C}[\mathrm{H}]$-module W and φ is the character of a $\mathbf{C}[\mathrm{G}]$-module E. In view of lemma 2, it is enough to show that

$$(*) \qquad \langle \mathrm{W}, \mathrm{Res}\ \mathrm{E} \rangle_{\mathrm{H}} = \langle \mathrm{Ind}\ \mathrm{W}, \mathrm{E} \rangle_{\mathrm{G}},$$

that is,

$$\dim.\mathrm{Hom}^{\mathrm{H}}(\mathrm{W}, \mathrm{Res}\ \mathrm{E}) = \dim.\mathrm{Hom}^{\mathrm{G}}(\mathrm{Ind}\ \mathrm{W}, \mathrm{E}),$$

which follows from remark 2 in 7.1 (or from lemma 1 of 3.3, which amounts to the same thing). Of course it is also possible to prove theorem 13 by direct calculation. □

Remarks

(1) Theorem 13 expresses the fact that the maps Res and Ind are *adjoints* of each other.

(2) Instead of the bilinear form $\langle \alpha, \beta \rangle$, we can use the scalar product $(\alpha|\beta)$ defined in 2.3. We have the same formula:

$$(\psi|\mathrm{Res}\ \varphi)_{\mathrm{H}} = (\mathrm{Ind}\ \psi|\varphi)_{\mathrm{G}}.$$

(3) We mention also the following useful formula

$$\mathrm{Ind}(\psi \cdot \mathrm{Res}\ \varphi) = (\mathrm{Ind}\ \psi) \cdot \varphi.$$

It can be checked by a simple calculation, or deduced from the formula $\mathrm{Ind}(\mathrm{W}) \otimes \mathrm{E} \cong \mathrm{Ind}(\mathrm{W} \otimes \mathrm{Res}\ \mathrm{E})$, cf. 3.3, example 5.

Proposition 21. *Let* W *be an irreducible representation of* H *and* E *an irreducible representation of* G. *Then the number of times that* W *occurs in* Res E *is equal to the number of times that E occurs in* Ind W

This follows from th. 13, applied to the character ψ of W and to the character φ of E (one may also apply formula (*)). □

EXERCISES

7.1. (Generalization of the concept of induced representation.) Let α: $\mathrm{H} \to \mathrm{G}$ be a homomorphism of groups (not necessarily injective), and let $\tilde{\alpha}$: $\mathbf{C}[\mathrm{H}] \to \mathbf{C}[\mathrm{G}]$ be the corresponding algebra homomorphism. If E is a $\mathbf{C}[\mathrm{G}]$-module we denote by $\mathrm{Res}_\alpha \mathrm{E}$ the $\mathbf{C}[\mathrm{H}]$-module obtained from E by means of $\tilde{\alpha}$; if φ is the character of E, that of $\mathrm{Res}_\alpha \mathrm{E}$ is $\mathrm{Res}_\alpha \varphi = \varphi \circ \alpha$. If W is a $\mathbf{C}[\mathrm{H}]$-module, we denote by $\mathrm{Ind}_\alpha \mathrm{W}$ the $\mathbf{C}[\mathrm{G}]$-module $\mathbf{C}[\mathrm{G}] \otimes_{\mathbf{C}[\mathrm{H}]} \mathrm{W}$, and if ψ is the character of W, we denote by $\mathrm{Ind}_\alpha \psi$ the character of $\mathrm{Ind}_\alpha \mathrm{W}$.

(a) Show that we still have the reciprocity formula

$$\langle \psi, \mathrm{Res}_\alpha \varphi \rangle_{\mathrm{H}} = \langle \mathrm{Ind}_\alpha \psi, \varphi \rangle_{\mathrm{G}}.$$

(b) Assume that α is surjective and identify G with the quotient of H by the kernel N of α. Show that $\mathrm{Ind}_\alpha \mathrm{W}$ is isomorphic to the module obtained by having $\mathrm{G} = \mathrm{H}/\mathrm{N}$ act on the subspace of W consisting of the elements invariant under N. Deduce the formula

$$(\mathrm{Ind}_\alpha \psi)(s) = \frac{1}{n} \sum_{\alpha(t)=s} \psi(t) \quad \text{where } n = \mathrm{Card}(\mathrm{N}).$$

7.2. Let H be a subgroup of G and let χ be the character of the permutation representation associated with G/H (cf. 1.2). Show that $\chi = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(1)$, and

that $\psi = \chi - 1$ is the character of a representation of G; determine under what condition the latter representation is irreducible [use ex. 2.6, or apply the reciprocity formula].

7.3. Let H be a subgroup of G. Assume that for each $t \notin H$ we have $H \cap tHt^{-1} = \{1\}$, in which case H is said to be a Frobenius subgroup of G. Denote by N the set of elements of G which are not conjugate to any element of H.

(a) Let $g = \mathrm{Card}(G)$ and let $h = \mathrm{Card}(H)$. Show that the number of elements of N is $(g/h) - 1$.

(b) Let f be a class function on H. Show that there exists a unique class function $\tilde{f}$ on G which extends f and takes the value $f(1)$ on N.

(c) Show that $\tilde{f} = \mathrm{Ind}_H^G f - f(1)\psi$, where ψ is the character $\mathrm{Ind}_H^G(1) - 1$ of G, cf. ex. 7.2.

(d) Show that $\langle f_1, f_2 \rangle_H = \langle \tilde{f}_1, \tilde{f}_2 \rangle_G$.

(e) Take f to be an irreducible character of H. Show, using (c) and (d), that $\langle \tilde{f}, \tilde{f} \rangle_G = 1$, $\tilde{f}(1) \geq 0$, and that $\tilde{f}$ is a linear combination with integer coefficients of irreducible characters of G. Conclude that $\tilde{f}$ is an irreducible character of G. If ρ is a corresponding representation of G, show that $\rho(s) = 1$ for each $s \in N$ [use ex. 6.7].

(f) Show that each linear representation of H extends to a linear representation of G whose kernel contains N. Conclude that $N \cup \{1\}$ is a normal subgroup of G and that G is the semidirect product of H and $N \cup \{1\}$ (*Frobenius' theorem*).

(g) Conversely, suppose G is the semidirect product of H and a normal subgroup A. Show that H is a Frobenius subgroup of G if and only if for each $s \in H - \{1\}$ and each $t \in A - \{1\}$, we have $sts^{-1} \neq t$ (i.e., H acts freely on $A - \{1\}$). (If $H \neq \{1\}$, this property implies that A is nilpotent, by a theorem of Thompson.)

7.3 Restriction to subgroups

Let H and K be two subgroups of G, and let ρ: $H \to \mathbf{GL}(W)$ be a linear representation of H, and let $V = \mathrm{Ind}_H^G(W)$ be the corresponding induced representation of G. We shall determine the restriction $\mathrm{Res}_K V$ of V to K.

First choose a set of representatives S for the (H, K) double cosets of G; this means that G is the disjoint union of the KsH for $s \in S$ (we could also write $s \in K\backslash G/H$). For $s \in S$, let $H_s = sHs^{-1} \cap K$, which is a subgroup of K. If we set

$$\rho^s(x) = \rho(s^{-1}xs), \quad \text{for } x \in H_s,$$

we obtain a homomorphism ρ^s: $H_s \to \mathbf{GL}(W)$, and hence a linear representation of H_s, denoted W_s. Since H_s is a subgroup of K, the induced representation $\mathrm{Ind}_{H_s}^K(W_s)$ is defined.

Proposition 22. *The representation* $\mathrm{Res}_K \mathrm{Ind}_H^G(W)$ *is isomorphic to the direct sum of the representations* $\mathrm{Ind}_{H_s}^K(W_s)$, *for* $s \in S \simeq K\backslash G/H$.

We know that V is the direct sum of the images $x\mathrm{W}$, for $x \in \mathrm{G}/\mathrm{H}$. Let $s \in \mathrm{S}$ and let $\mathrm{V}(s)$ be the subspace of V generated by the images $x\mathrm{W}$, for $x \in \mathrm{K}s\mathrm{H}$; the space V is a direct sum of the $\mathrm{V}(s)$, and it is clear that $\mathrm{V}(s)$ is stable under K. It remains to prove that $\mathrm{V}(s)$ is K-isomorphic to $\mathrm{Ind}_{\mathrm{H}_s}^{\mathrm{K}}(\mathrm{W}_s)$. But the subgroup of K consisting of the elements x such that $x(s\mathrm{W}) = s\mathrm{W}$ is evidently equal to H_s, and $\mathrm{V}(s)$ is a direct sum of the images $x(s\mathrm{W})$, $x \in \mathrm{K}/\mathrm{H}_s$. Therefore $\mathrm{V}(s) = \mathrm{Ind}_{\mathrm{H}_s}^{\mathrm{K}}(s\mathrm{W})$. Now it remains to check that $s\mathrm{W}$ is H_s-isomorphic to W_s, and this is immediate: the isomorphism is given by s: $\mathrm{W}_s \to s\mathrm{W}$. □

Remark. Since $\mathrm{V}(s)$ depends only on the image of s in $\mathrm{K}\backslash\mathrm{G}/\mathrm{H}$, we also see that the representation $\mathrm{Ind}_{\mathrm{H}_s}^{\mathrm{K}}(\mathrm{W}_s)$ depends (up to isomorphism) only on the double coset of s.

7.4 Mackey's irreducibility criterion

We apply the preceding results to the case $\mathrm{K} = \mathrm{H}$. For $s \in \mathrm{G}$, we still denote by H_s the subgroup $s\mathrm{H}s^{-1} \cap \mathrm{H}$ of H; the representation ρ of H defines a representation $\mathrm{Res}_s(\rho)$ by restriction to H_s, which should not be confused with the representation ρ^s defined in 7.3.

Proposition 23. *In order that the induced representation* $\mathrm{V} = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{W}$ *be irreducible, it is necessary and sufficient that the following two conditions be satisfied*:

(a) W *is irreducible.*
(b) *For each* $s \in \mathrm{G} - \mathrm{H}$ *the two representations* ρ^s *and* $\mathrm{Res}_s(\rho)$ *of* H_s *are disjoint.*

(Two representations V_1 and V_2 of a group K are said to be *disjoint* if they have no irreducible component in common, i.e., if $\langle \mathrm{V}_1, \mathrm{V}_2 \rangle_{\mathrm{K}} = 0$.)

In order that V be irreducible, it is necessary and sufficient that $\langle \mathrm{V}, \mathrm{V} \rangle_{\mathrm{G}} = 1$. But, according to Frobenius reciprocity, we have:

$$\langle \mathrm{V}, \mathrm{V} \rangle_{\mathrm{G}} = \langle \mathrm{W}, \mathrm{Res}_{\mathrm{H}}\,\mathrm{V} \rangle_{\mathrm{H}}.$$

However, from 7.3 we have:

$$\mathrm{Res}_{\mathrm{H}}\,\mathrm{V} = \bigoplus_{s \in \mathrm{H}\backslash\mathrm{G}/\mathrm{H}} \mathrm{Ind}_{\mathrm{H}_s}^{\mathrm{H}}(\rho^s).$$

Once more applying the Frobenius formula, we obtain:

$$\langle \mathrm{V}, \mathrm{V} \rangle_{\mathrm{G}} = \sum_{s \in \mathrm{H}\backslash\mathrm{G}/\mathrm{H}} d_s, \quad \text{with } d_s = \langle \mathrm{Res}_s(\rho), \rho^s \rangle_{\mathrm{H}_s}.$$

For $s = 1$ we have $d_s = \langle \rho, \rho \rangle \geqslant 1$. In order that $\langle V, V \rangle_G = 1$, it is thus necessary and sufficient that $d_1 = 1$ and $d_s = 0$ for $s \neq 1$; these are exactly the conditions (a) and (b). □

Corollary. *Suppose* H *is normal in* G. *In order that* $\mathrm{Ind}_H^G(\rho)$ *be irreducible, it is necessary and sufficient that* ρ *be irreducible and not isomorphic to any of its conjugates* ρ^s *for* $s \notin H$.

Indeed, we have then $H_s = H$ and $\mathrm{Res}_s(\rho) = \rho$.

EXERCISE

7.4. Let k be a finite field, let $G = \mathbf{SL}_2(k)$ and let H be the subgroup of G consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c = 0$. Let ω be a homomorphism of k^* into $\mathbf{C}^*$ and let χ_ω be the character of degree 1 of H defined by

$$\chi_\omega\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \omega(a).$$

Show that the representation of G induced by χ_ω is irreducible if $\omega^2 \neq 1$. Compute χ_ω.