CHAPTER 7

Induced representations;
Mackey's criterion

7.1 Induction

Let $H$ be a subgroup of a group $G$ and $R$ a system of left coset representatives for $H$. Let $V$ be a $C[G]$-module and let $W$ be a sub-$C[H]$-module of $V$. Recall (cf. 3.3) that the module $V$ (or the representation $V$) is said to be induced by $W$ if we have $V = \bigoplus_{s \in R} sW$, i.e., if $V$ is a direct sum of the images $sW$, $s \in R$ (a condition which is independent of the choice of $R$). This property can be reformulated in the following way:

Let

$$W' = C[G] \otimes_{C[H]} W$$

be the $C[G]$-module obtained from $W$ by scalar extension from $C[H]$ to $C[G]$. The injection $W \to V$ extends by linearity to a $C[G]$-homomorphism $i: W' \to V$.

**Proposition 18.** In order that $V$ be induced by $W$, it is necessary and sufficient that the homomorphism

$$i: C[G] \otimes_{C[H]} W \to V$$

be an isomorphism.

This is a consequence of the fact that the elements of $R$ form a basis of $C[G]$ considered as a right $C[H]$-module.

**Remarks**

1. This characterization of the representation induced by $W$ makes it obvious that the induced representation exists and is unique (cf. 3.3, th. 11).
In what follows, the representation of $G$ induced by $W$ will be denoted by $\text{Ind}_H^G(W)$, or simply $\text{Ind}(W)$ if there is no danger of confusion.

(2) If $V$ is induced by $W$ and if $E$ is a $C[G]$-module, we have a canonical isomorphism

$$\text{Hom}_H^G(W, E) \cong \text{Hom}_G(V, E),$$

where $\text{Hom}_G(V, E)$ denotes the vector space of $C[G]$-homomorphisms of $V$ into $E$, and $\text{Hom}_H^G(W, E)$ is defined similarly. This follows from an elementary property of tensor products (see also 3.3, lemma 1).

(3) Induction is transitive: if $G$ is a subgroup of a group $K$, we have

$$\text{Ind}_G^K(\text{Ind}_H^G(W)) \cong \text{Ind}_H^K(W).$$

This can be seen directly, or by using the associativity of the tensor product.

**Proposition 19.** Let $V$ be a $C[G]$-module which is a direct sum $V = \bigoplus_{i \in I} W_i$ of vector subspaces permuted transitively by $G$. Let $i_0 \in I$, $W = W_{i_0}$ and let $H$ be the stabilizer of $W$ in $G$ (i.e., the set of all $s \in G$ such that $sW = W$). Then $W$ is stable under the subgroup $H$ and the $C[G]$-module $V$ is induced by the $C[H]$-module $W$.

This is clear.

**Remark.** In order to apply proposition 19 to an irreducible representation $V = \bigoplus W_i$ of $G$, it is enough to check that the $W_i$ are permuted among themselves by $G$; the transitivity condition is automatic, because each orbit of $G$ in the set of $W_i$'s defines a subrepresentation of $V$.

**Example.** When the $W_i$ are of dimension 1, the representation $V$ is said to be monomial.

### 7.2 The character of an induced representation; the reciprocity formula

We keep the preceding notation. If $f$ is a class function on $H$, consider the function $f'$ on $G$ defined by the formula

$$f'(s) = \frac{1}{h} \sum_{t \in G} f(t^{-1}st) \quad \text{where } h = \text{Card}(H).$$

We say that $f'$ is induced by $f$ and denote it by either $\text{Ind}_H^G(f)$ or $\text{Ind}(f)$.
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Proposition 20.

(i) The function \( \text{Ind}(f) \) is a class function on \( G \).
(ii) If \( f \) is the character of a representation \( W \) of \( H \), \( \text{Ind}(f) \) is the character of the induced representation \( \text{Ind}(W) \) of \( G \).

Assertion (ii) has already been proved (3.3, th. 12). Assertion (i) is proved by a direct calculation or can be obtained from (ii) and the observation that each class function is a linear combination of characters.

Recall that, for \( \varphi_1 \) and \( \varphi_2 \) two class functions on \( G \), we set

\[
\langle \varphi_1, \varphi_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \varphi_1(g^{-1}) \varphi_2(g), \quad \text{where} \quad |G| = \text{Card}(G),
\]

cf. 2.2; when we wish to be more explicit about the group \( G \), we write \( \langle \varphi_1, \varphi_2 \rangle_G \) instead of \( \langle \varphi_1, \varphi_2 \rangle \).

Also, if \( V_1 \) and \( V_2 \) are two \( \mathbb{C}[G] \)-modules, we set

\[
\langle V_1, V_2 \rangle_G = \dim \text{Hom}^G(V_1, V_2).
\]

Lemma 2. If \( \varphi_1 \) and \( \varphi_2 \) are the characters of \( V_1 \) and \( V_2 \), we have

\[
\langle \varphi_1, \varphi_2 \rangle_G = \langle V_1, V_2 \rangle_G.
\]

Decomposing \( V_1 \) and \( V_2 \) into direct sums, we can assume that they are irreducible, in which case the lemma follows from the orthogonality formulas for characters (2.3, th. 3).

If \( \varphi \) (resp. \( V \)) is a function on \( G \) (resp. a representation of \( G \)), we denote by \( \text{Res} \varphi \) (resp. \( \text{Res} V \)) its restriction to the subgroup \( H \).

Theorem 13 (Frobenius reciprocity). If \( \psi \) is a class function on \( H \) and \( \varphi \) a class function on \( G \), we have

\[
\langle \psi, \text{Res} \varphi \rangle_H = \langle \text{Ind} \psi, \varphi \rangle_G.
\]

Since each class function is a linear combination of characters, we can assume that \( \psi \) is the character of a \( \mathbb{C}[H] \)-module \( W \) and \( \varphi \) is the character of a \( \mathbb{C}[G] \)-module \( E \). In view of lemma 2, it is enough to show that

\[
\langle W, \text{Res} E \rangle_H = \langle \text{Ind} W, E \rangle_G,
\]

that is,

\[
\dim \text{Hom}^H(W, \text{Res} E) = \dim \text{Hom}^G(\text{Ind} W, E),
\]

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which follows from remark 2 in 7.1 (or from lemma 1 of 3.3, which amounts to the same thing). Of course it is also possible to prove theorem 13 by direct calculation. □

Remarks

(1) Theorem 13 expresses the fact that the maps Res and Ind are adjoints of each other.

(2) Instead of the bilinear form $\langle \alpha, \beta \rangle$, we can use the scalar product $(\alpha | \beta)$ defined in 2.3. We have the same formula:

$$
(\psi | \text{Res } \varphi)_H = (\text{Ind } \psi | \varphi)_G.
$$

(3) We mention also the following useful formula

$$
\text{Ind}(\psi \cdot \text{Res } \varphi) = (\text{Ind } \psi) \cdot \varphi.
$$

It can be checked by a simple calculation, or deduced from the formula

$$
\text{Ind}(W) \otimes E \cong \text{Ind}(W \otimes \text{Res } E),
$$

cf. 3.3, example 5.

Proposition 21. Let $W$ be an irreducible representation of $H$ and $E$ an irreducible representation of $G$. Then the number of times that $W$ occurs in Res $E$ is equal to the number of times that $E$ occurs in Ind $W$.

This follows from th. 13, applied to the character $\psi$ of $W$ and to the character $\varphi$ of $E$ (one may also apply formula (*)). □

Exercises

7.1. (Generalization of the concept of induced representation.) Let $\alpha: H \to G$ be a homomorphism of groups (not necessarily injective), and let $\tilde{\alpha}: C[H] \to C[G]$ be the corresponding algebra homomorphism. If $E$ is a $C[G]$-module we denote by Res$_\alpha$ $E$ the $C[H]$-module obtained from $E$ by means of $\tilde{\alpha}$; if $\varphi$ is the character of $E$, that of Res$_\alpha$ $E$ is Res$_\alpha \varphi = \varphi \circ \alpha$. If $W$ is a $C[H]$-module, we denote by Ind$_\alpha$ $W$ the $C[G]$-module $C[G] \otimes_{C[H]} W$, and if $\psi$ is the character of $W$, we denote by Ind$_\alpha \psi$ the character of Ind$_\alpha W$.

(a) Show that we still have the reciprocity formula

$$
(\psi | \text{Res } \varphi)_H = (\text{Ind } \psi | \varphi)_G.
$$

(b) Assume that $\alpha$ is surjective and identify $G$ with the quotient of $H$ by the kernel $N$ of $\alpha$. Show that Ind$_\alpha W$ is isomorphic to the module obtained by having $G = H/N$ act on the subspace of $W$ consisting of the elements invariant under $N$. Deduce the formula

$$
(\text{Ind } \psi)(s) = \frac{1}{n} \sum_{\alpha(s) = t} \psi(t) \quad \text{where } n = \text{Card}(N).
$$

7.2. Let $H$ be a subgroup of $G$ and let $\chi$ be the character of the permutation representation associated with $G/H$ (cf. 1.2). Show that $\chi = \text{Ind}_H^G(1)$, and

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that \( \psi = \chi - 1 \) is the character of a representation of \( G \); determine under what condition the latter representation is irreducible [use ex. 2.6, or apply the reciprocity formula].

7.3. Let \( H \) be a subgroup of \( G \). Assume that for each \( t \not\in H \) we have \( H \cap H t^{-1} = \{1\} \), in which case \( H \) is said to be a Frobenius subgroup of \( G \). Denote by \( N \) the set of elements of \( G \) which are not conjugate to any element of \( H \).

(a) Let \( g = \text{Card}(G) \) and let \( h = \text{Card}(H) \). Show that the number of elements of \( N \) is \((g/h) - 1\).

(b) Let \( f \) be a class function on \( H \). Show that there exists a unique class function \( \tilde{f} \) on \( G \) which extends \( f \) and takes the value \( f(1) \) on \( N \).

(c) Show that \( \tilde{f} = \text{Ind}_{H}^{G}(f - f(1)\psi) \), where \( \psi \) is the character \( \text{Ind}_{H}^{G}(1) - 1 \) of \( G \), cf. ex. 7.2.

(d) Show that \( \langle f_1, f_2 \rangle_{H} = \langle f_1, f_2 \rangle_{G} \).

(e) Take \( f \) to be an irreducible character of \( H \). Show, using (c) and (d), that \( \langle f, f \rangle_{G} = 1 \), \( f(1) \geq 0 \), and that \( f \) is a linear combination with integer coefficients of irreducible characters of \( G \). Conclude that \( f \) is an irreducible character of \( G \). If \( \rho \) is a corresponding representation of \( G \), show that \( \rho(s) = 1 \) for each \( s \in N \) [use ex. 6.7].

(f) Show that each linear representation of \( H \) extends to a linear representation of \( G \) whose kernel contains \( N \). Conclude that \( N \cup \{1\} \) is a normal subgroup of \( G \) and that \( G \) is the semidirect product of \( H \) and \( N \cup \{1\} \) (Frobenius’ theorem).

(g) Conversely, suppose \( G \) is the semidirect product of \( H \) and a normal subgroup \( A \). Show that \( H \) is a Frobenius subgroup of \( G \) if and only if for each \( s \in H - \{1\} \) and each \( t \in A - \{1\} \), we have \( sts^{-1} \neq t \) (i.e., \( H \) acts freely on \( A - \{1\} \)). (If \( H \neq \{1\} \), this property implies that \( A \) is nilpotent, by a theorem of Thompson.)

7.3 Restriction to subgroups

Let \( H \) and \( K \) be two subgroups of \( G \), and let \( \rho: H \to \text{GL}(W) \) be a linear representation of \( H \), and let \( \psi = \text{Ind}_{H}^{G}(W) \) be the corresponding induced representation of \( G \). We shall determine the restriction \( \text{Res}_{K} \psi \) of \( \psi \) to \( K \).

First choose a set of representatives \( S \) for the \((H, K)\) double cosets of \( G \); this means that \( G \) is the disjoint union of the \( K s H \) for \( s \in S \) (we could also write \( s \in K \backslash G / H \)). For \( s \in S \), let \( H_{s} = s H s^{-1} \cap K \), which is a subgroup of \( K \). If we set

\[
\rho'(x) = \rho(s^{-1}xs), \quad \text{for} \ x \in H_{s},
\]

we obtain a homomorphism \( \rho' : H_{s} \to \text{GL}(W) \), and hence a linear representation of \( H_{s} \), denoted \( W_{s} \). Since \( H_{s} \) is a subgroup of \( K \), the induced representation \( \text{Ind}_{H_{s}}^{K}(W_{s}) \) is defined.

**Proposition 22.** The representation \( \text{Res}_{K} \psi = \text{Ind}_{H}^{G}(W) \) is isomorphic to the direct sum of the representations \( \text{Ind}_{H_{s}}^{K}(W_{s}) \), for \( s \in S \simeq K \backslash G / H \).
We know that $V$ is the direct sum of the images $xW$, for $x \in G/H$. Let $s \in S$ and let $V(s)$ be the subspace of $V$ generated by the images $xW$, for $x \in K\cdot sH$; the space $V$ is a direct sum of the $V(s)$, and it is clear that $V(s)$ is stable under $K$. It remains to prove that $V(s)$ is $K$-isomorphic to $\text{Ind}_H^G(W_s)$. But the subgroup of $K$ consisting of the elements $x$ such that $x(sW) = sW$ is evidently equal to $H_s$, and $V(s)$ is a direct sum of the images $x(sW)$, $x \in K/H_s$. Therefore $V(s) = \text{Ind}_H^G(sW)$. Now it remains to check that $sW$ is $H_s$-isomorphic to $W_s$, and this is immediate: the isomorphism is given by $s: W_s \rightarrow sW$.

**Remark.** Since $V(s)$ depends only on the image of $s$ in $K\setminus G/H$, we also see that the representation $\text{Ind}_H^G(W_s)$ depends (up to isomorphism) only on the double coset of $s$.

### 7.4 Mackey’s irreducibility criterion

We apply the preceding results to the case $K = H$. For $s \in G$, we still denote by $H_s$ the subgroup $sHs^{-1} \cap H$ of $H$; the representation $\rho$ of $H$ defines a representation $\text{Res}_s(\rho)$ by restriction to $H_s$, which should not be confused with the representation $\rho^s$ defined in 7.3.

**Proposition 23.** In order that the induced representation $V = \text{Ind}_H^G(W)$ be irreducible, it is necessary and sufficient that the following two conditions be satisfied:

(a) $W$ is irreducible.

(b) For each $s \in G - H$ the two representations $\rho^s$ and $\text{Res}_s(\rho)$ of $H_s$ are disjoint.

(Two representations $V_1$ and $V_2$ of a group $K$ are said to be disjoint if they have no irreducible component in common, i.e., if $\langle V_1, V_2 \rangle_K = 0$.)

In order that $V$ be irreducible, it is necessary and sufficient that $\langle V, V \rangle_G = 1$. But, according to Frobenius reciprocity, we have:

$$\langle V, V \rangle_G = \langle W, \text{Res}_H V \rangle_H.$$

However, from 7.3 we have:

$$\text{Res}_H V = \bigoplus_{s \in H\setminus G/H} \text{Ind}_H^G(\rho^s).$$

Once more applying the Frobenius formula, we obtain:

$$\langle V, V \rangle_G = \sum_{s \in H\setminus G/H} d_s, \quad \text{with } d_s = \langle \text{Res}_s(\rho), \rho^s \rangle_{H_s}. 59$$
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For $s = 1$ we have $d_s = \langle \rho, \rho \rangle \geq 1$. In order that $\langle V, V \rangle_Q = 1$, it is thus necessary and sufficient that $d_1 = 1$ and $d_s = 0$ for $s \neq 1$; these are exactly the conditions (a) and (b).

\[ \square \]

**Corollary.** Suppose $H$ is normal in $G$. In order that $\text{Ind}^G_H(\rho)$ be irreducible, it is necessary and sufficient that $\rho$ be irreducible and not isomorphic to any of its conjugates $\rho^s$ for $s \notin H$.

Indeed, we have then $H_s = H$ and $\text{Res}_s(\rho) = \rho$.

**Exercise**

7.4. Let $k$ be a finite field, let $G = \text{SL}_2(k)$ and let $H$ be the subgroup of $G$ consisting of matrices $\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}$ such that $c = 0$. Let $\omega$ be a homomorphism of $k^*$ into $C^*$ and let $\chi_\omega$ be the character of degree 1 of $H$ defined by

$$
\chi_\omega \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \omega(a).
$$

Show that the representation of $G$ induced by $\chi_\omega$ is irreducible if $\omega^2 \neq 1$. Compute $\chi_\omega$. 

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