CHAPTER 5

Examples

5.1 The cyclic group C_n

This is the group of order n consisting of the powers $1, r, \ldots, r^{n-1}$ of an element r such that $r^n = 1$. It can be realized as the group of rotations through angles $2k\pi/n$ around an axis. It is an abelian group.

According to th. 9, the irreducible representations of C_n are of degree 1. Such a representation associates with r a complex number $\chi(r) = w$, and with r^k the number $\chi(r^k) = w^k$; since $r^n = 1$, we have $w^n = 1$, that is, $w = e^{2\pi i h/n}$, with $h = 0, 1, \ldots, n-1$. We thus obtain *n irreducible representations of degree* 1 whose characters $\chi_0, \chi_1, \ldots, \chi_{n-1}$ are given by

$$\chi_h(r^k) = e^{2\pi i hk/n}.$$

We have $\chi_h \cdot \chi_{h'} = \chi_{h+h'}$, with the convention that $\chi_{h+h'} = \chi_{h+h'-n}$ if $h + h' \geqslant n$ (in other words, the index h of χ_h is taken *modulo n*).

For $n = 3$, for example, the *character table* is the following:

	1	r	r^2
χ_0	1	1	1
χ_1	1	w	w^2
χ_2	1	w^2	w

where

$$w = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

We have

$$\chi_0 \cdot \chi_i = \chi_i,\ \chi_1 \cdot \chi_1 = \chi_2,\ \chi_2 \cdot \chi_2 = \chi_1 \text{ and } \chi_1 \cdot \chi_2 = \chi_0.$$

5.2 The group C_∞

This is the group of *rotations* of the plane. If we denote by r_α the rotation through an angle α (determined modulo 2π), the *invariant measure* on C_∞ is $(1/2\pi)\,d\alpha$ (cf. 4.2).

The irreducible representations of C_∞ are of degree 1. They are given by:

$$\chi_n(r_\alpha) = e^{in\alpha} \quad (n \text{ an arbitrary integer}).$$

The orthogonality relations give here the well known formulas:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha} \cdot e^{im\alpha}\, d\alpha = \delta_{nm},$$

and th. 6 gives the expansion of a periodic function as a Fourier series.

5.3 The dihedral group D_n

This is the group of rotations and reflections of the plane which preserve a regular polygon with n vertices. It contains n rotations, which form a subgroup isomorphic to C_n, and n reflections. Its order is $2n$. If we denote by r the rotation through an angle $2\pi/n$ and if s is any one of the reflections, we have:

$$r^n = 1, \qquad s^2 = 1, \qquad srs = r^{-1}.$$

Each element of D_n can be written uniquely, either in the form r^k, with $0 \leqslant k \leqslant n-1$ (if it belongs to C_n), or in the form sr^k, with $0 \leqslant k \leqslant n-1$ (if it does not belong to C_n). Observe that the relation $srs = r^{-1}$ implies $sr^ks = r^{-k}$, whence $(sr^k)^2 = 1$.

Realization of D_n as a group of rigid motions of 3-space

There are several such:

(a) The usual realization (the one traditionally denoted D_n cf. Eyring [5]). One takes for rotations the rotations around the axis Oz, and for reflections, the reflections through n lines of the plane Oxy, these lines forming angles which are multiples of π/n.

(b) The realization by means of the group C_{nv} (notation of Eyring [5]): instead of the reflections with respect to the *lines* of Oxy, one takes reflections with respect to *planes* containing the axis Oz.

(c) The group D_{2n} can also be realized as the group D_{nd} (notation of Eyring [5]).

Irreducible representations of the group D_n *(n even $\geqslant 2$)*

First, there are 4 representations of degree 1, obtained by letting ± 1 correspond to r and s in all possible ways. Their characters $\psi_1, \psi_2, \psi_3, \psi_4$ are given by the following table:

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

Next we consider representations of degree 2. Put $w = e^{2\pi i/n}$ and let h be an arbitrary integer. We define a representation ρ^h of D_n by setting:

$$\rho^h(r^k) = \begin{pmatrix} w^{hk} & 0 \\ 0 & w^{-hk} \end{pmatrix}, \qquad \rho^h(sr^k) = \begin{pmatrix} 0 & w^{-hk} \\ w^{hk} & 0 \end{pmatrix}.$$

A direct calculation shows that this is indeed a representation. This representation is *induced* (in the sense of 3.3) by the representation of C_n with character χ_h (5.1). It depends only on the residue class of h modulo n; moreover ρ^h and ρ^{n-h} are isomorphic. Hence we may assume $0 \leqslant h \leqslant n/2$. The extreme cases $h = 0$ and $h = n/2$ are uninteresting: the corresponding representations are reducible, with characters $\psi_1 + \psi_2$ and $\psi_3 + \psi_4$ respectively. On the other hand, for $0 < h < n/2$, the representation ρ^h is *irreducible*: since $w^h \neq w^{-h}$, the only lines stable under $\rho^h(r)$ are the coordinate axes, and these are not stable under $\rho^h(s)$. The same argument shows that these representations are pairwise nonisomorphic. The corresponding characters χ^h are given by:

$$\chi_h(r^k) = w^{hk} + w^{-hk} = 2\cos\frac{2\pi hk}{n}$$

$$\chi_h(sr^k) = 0.$$

The irreducible representations of degree 1 and 2 constructed above are the *only irreducible representations* of D_n (up to isomorphism). Indeed, the sum of the squares of their degrees is equal to $4 \times 1 + ((n/2) - 1) \times 4 = 2n$, which is the order of D_n.

EXAMPLE. The group D_6 has 4 representations of degree 1, with characters $\psi_1, \psi_2, \psi_3, \psi_4$ and 2 irreducible representations of degree 2, with characters χ_1 and χ_2.

Irreducible representations of the group D_n *(n odd)*

There are only two representations of degree 1, and their characters ψ_1 and ψ_2 are given by the table:

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1

The representations ρ^h of degree 2 are defined by the same formulas as in the case where n is even. Those corresponding to $0 < h < n/2$ are irreducible and pairwise nonisomorphic (observe that, since n is odd, the condition $h < n/2$ can also be written $h \leqslant (n-1)/2$). The formulas giving their characters are the same.

These representations are the *only* ones. Indeed, the sum of the squares of their degrees is equal to $2 \times 1 + \frac{1}{2}(n-1) \times 4 = 2n$, and this is the order of D_n.

EXERCISES

5.1. Show that in D_n, n even (resp. odd), the reflections form two conjugacy classes (resp. one), and that the elements of C_n form $(n/2) + 1$ classes (resp. $(n+1)/2$ classes). Obtain from this the number of classes of D_n and check that it coincides with the number of irreducible characters.

5.2. Show that $\chi_h \cdot \chi_{h'} = \chi_{h+h'} + \chi_{h-h'}$. In particular, we have

$$\chi_h \cdot \chi_h = \chi_{2h} + \chi_0 = \chi_{2h} + \psi_1 + \psi_2.$$

Show that ψ_2 is the character of the alternating square of ρ^h, and that $\chi_{2h} + \psi_1$ is the character of its symmetric square (cf. 1.5 and prop. 3).

5.3. Show that the usual realization of D_n as a group of rigid motions in $\mathbf{R}^3$ (Eyring [5]) is reducible and has character $\chi_1 + \psi_2$, and that the realization of D_n as C_{nv} (*loc. cit.*) has $\chi_1 + \psi_1$ for its character.

5.4 The group D_{nh}

This group is the product $D_n \times I$, where I is a group of order 2 consisting of elements $\{1, \iota\}$ with $\iota^2 = 1$. Its order is $4n$. If D_n is realized in the usual way as a group of rotations and reflections of 3-space [cf. 5.3, (a)] then D_{nh} can be realized as the group generated by D_n and the reflection ι through the origin.

According to th. 10, the irreducible representations of D_{nh} are the tensor products of those of D_n and those of I. The group I has just two irreducible representations, both of degree 1. Their characters g and u are given by the table:

	1	ι
g	1	1
u	1	-1

Consequently, D_{nh} *has twice as many irreducible representations as* D_n. More precisely, each irreducible character χ of D_n defines two irreducible characters χ_g and χ_u of D_{nh} as follows:

	x	ιx
χ_g	$\chi(x)$	$\chi(x)$
χ_u	$\chi(x)$	$-\chi(x)$

$(x \in D_n)$

For example, the character χ_1 of D_n gives rise to characters χ_{1g} and χ_{1u}:

	r^k	sr^k	ιr^k	ιsr^k
χ_{1g}	$2\cos 2\pi k/n$	0	$2\cos 2\pi k/n$	0
χ_{1u}	$2\cos 2\pi k/n$	0	$-2\cos 2\pi k/n$	0

The same applies to the other characters of D_n.

5.5 The group D_∞

This is the group of rotations and reflections of the plane which preserve the origin. It contains the group C_∞ of rotations r_α; if s is an arbitrary reflection, we have the relations:

$$s^2 = 1, \qquad sr_\alpha s = r_{-\alpha}.$$

Each element of D_∞ can be written uniquely either in the form r_α (if it belongs to C_∞) or in the form sr_α (if it does not belong to C_∞); as a topological space, D_∞ consists of two disjoint circles. The *invariant measure* of D_∞ is the measure $d\alpha/4\pi$. More precisely, the *average* $\int_G f(t)\,dt$ of a function f is given by the formula

$$\int_G f(t)\,dt = \frac{1}{4\pi}\int_0^{2\pi} f(r_\alpha)\,d\alpha + \frac{1}{4\pi}\int_0^{2\pi} f(sr_\alpha)\,d\alpha.$$

In particular, the projections p_i of 2.6 are:

$$p_i x = \frac{n_i}{4\pi}\int_0^{2\pi} \chi_i(r_\alpha)^* \rho_{r_\alpha}(x)\,d\alpha + \frac{n_i}{4\pi}\int_0^{2\pi} \chi_i(sr_\alpha)^* \rho_{sr_\alpha}(x)\,d\alpha.$$

Realizations of D_α *as a group of rigid motions in* 3-*space*

There are two of these:

(a) The usual realization (denoted D_∞ in Eyring [5]). Rotations are taken around Oz and reflections with respect to lines of the plane Oxy passing through O.

(b) The realization by means of the group $C_{\infty v}$ (notations of Eyring [5]): the reflections are taken with respect to planes passing through Oz, instead of lines of Oxy.

Irreducible representations of the group D_∞

They are constructed like those for D_n. There are first two representations of degree 1, with characters ψ_1 and ψ_2 given by the table:

	r_α	sr_α
ψ_1	1	1
ψ_2	1	−1

There is a series of irreducible representations ρ^h of degree 2 ($h = 1, 2, \ldots$) defined by the formulas:

$$\rho^h(r_\alpha) = \begin{pmatrix} e^{ih\alpha} & 0 \\ 0 & e^{-ih\alpha} \end{pmatrix}, \qquad \rho^h(sr_\alpha) = \begin{pmatrix} 0 & e^{-ih\alpha} \\ e^{ih\alpha} & 0 \end{pmatrix}.$$

Their characters $\chi_1, \chi_2, \ldots$ have the following values:

$$\chi_h(r_\alpha) = 2\cos(h\alpha), \qquad \chi_h(sr_\alpha) = 0.$$

It can be shown that these are *all the irreducible representations* of D_∞ (up to isomorphism).

5.6 The group $D_{\infty h}$

This group is the product $D_\infty \times I$; it can be realized as the group generated by D_∞ and the reflection ι through the origin. Its elements can be written uniquely in one of the four forms:

$$r_\alpha, \qquad sr_\alpha, \qquad \iota r_\alpha, \qquad \iota sr_\alpha.$$

As a topological space, it is the union of four disjoint circles. The *invariant measure* of $D_{\infty h}$ is $(1/8\pi)\,d\alpha$. As above, this means that the average $\int_G f(t)\,dt$ of a function f on $D_{\infty h}$ is given by:

$$\int_G f(t)\,dt = \frac{1}{8\pi}\int_0^{2\pi} f(r_\alpha)\,d\alpha + \frac{1}{8\pi}\int_0^{2\pi} f(sr_\alpha)\,d\alpha + \frac{1}{8\pi}\int_0^{2\pi} f(\iota r_\alpha)\,d\alpha + \frac{1}{8\pi}\int_0^{2\pi} f(\iota sr_\alpha)\,d\alpha.$$

We leave it to the reader to derive the explicit expressions for the projections p_i of 2.6.

As in the case of D_{nh}, the irreducible representations of $D_{\infty h}$ come in pairs from D_∞: each character χ of D_∞ gives rise to two characters χ_g and χ_u of $D_{\infty h}$.

So, for example, the character χ_3 of D_∞ gives:

	r_α	sr_α	ιr_α	ιsr_α
χ_{3g}	$2\cos 3\alpha$	0	$2\cos 3\alpha$	0
χ_{3u}	$2\cos 3\alpha$	0	$-2\cos 3\alpha$	0

5.7 The alternating group $\mathfrak{A}_4$

This is the group of even permutations of a set $\{a,b,c,d\}$ having 4 elements; it is isomorphic to the group of rotations in $\mathbf{R}^3$ which stabilize a regular tetrahedron with barycenter the origin. It has 12 elements:

the identity element 1;
3 elements of order 2, $x = (ab)(cd)$, $y = (ac)(bd)$, $z = (ad)(bc)$, which correspond to reflections of the tetrahedron through lines joining the midpoints of two opposite edges;
8 elements of order 3: (abc), (acb), $\ldots$, (bcd), which correspond to rotations of $\pm 120°$ with respect to lines joining a vertex to the barycenter of the opposite face.

We denote by (abc) the cyclic permutation $a \mapsto b$, $b \mapsto c$, $c \mapsto a$, $d \mapsto d$; likewise, $(ab)(cd)$ denotes the permutation $a \mapsto b$, $b \mapsto a$, $c \mapsto d$, $d \mapsto c$, product of the transpositions (ab) and (cd).

Set $t = (abc)$, $K = \{1, t, t^2\}$ and $H = \{1, x, y, z\}$. We have

$$txt^{-1} = z, \qquad tzt^{-1} = y, \qquad tyt^{-1} = x;$$

moreover H and K are subgroups of $\mathfrak{A}_4$, H is normal, and $H \cap K = \{1\}$. It is easy to see that each element of $\mathfrak{A}_4$ *can be written uniquely as a product* $h \cdot k$, with $h \in H$ and $k \in K$.

One also says that $\mathfrak{A}_4$ is the *semidirect product* of K by the normal subgroup H; note that this is not a direct product, because the elements of K do not commute with those of H.

There are 4 *conjugacy classes in* $\mathfrak{A}_4$: $\{1\}$, $\{x,y,z\}$, $\{t, tx, ty, tz\}$, and $\{t^2, t^2x, t^2y, t^2z\}$, hence 4 irreducible characters. There are three characters of degree 1, corresponding to the three characters χ_0, χ_1, and χ_2 of the group K (cf. 5.1) extended to $\mathfrak{A}_4$ by setting $\chi_i(h \cdot k) = \chi_i(k)$ for $h \in H$ and $k \in K$. The last character ψ is determined, for example, by means of cor. 2