CHAPTER 3

Subgroups, products, induced representations

All the groups considered below are assumed to be finite.

3.1 Abelian subgroups

Let G be a group. One says that G is *abelian* (or *commutative*) if $st = ts$ for all $s, t \in G$. This amounts to saying that each conjugacy class of G consists of a single element, also that each function on G is a class function. The linear representations of such a group are particularly simple:

Theorem 9. *The following properties are equivalent:*

(i) G *is abelian.*

(ii) *All the irreducible representations of* G *have degree* 1.

Let g be the order of G, and let $(n_1, \ldots, n_h)$ be the degrees of the distinct irreducible representations of G; we know, cf. Ch. 2, that h is the number of classes of G, and that $g = n_1^2 + \cdots + n_h^2$. Hence g is equal to h if and only if all the n_i are equal to 1, which proves the theorem. □

Corollary. *Let* A *be an abelian subgroup of* G, *let a be its order and let g be that of* G. *Each irreducible representation of* G *has degree* $\leqslant g/a$.

(The quotient g/a is the *index* of A in G.)

Let $\rho\colon G \to \mathbf{GL}(V)$ be an irreducible representation of G. Through *restriction* to the subgroup A, it defines a representation $\rho_A : A \to \mathbf{GL}(V)$ of A. Let $W \subset V$ be an irreducible subrepresentation of ρ_A; by th. 9, we have $\dim(W) = 1$. Let V′ be the vector subspace of V generated by the images $\rho_s W$ of W, s ranging over G. It is clear that V′ is stable under G; since ρ is irreducible, we thus have $V' = V$. But, for $s \in G$ and $t \in A$ we have

$$\rho_{st} W = \rho_s \rho_t W = \rho_s W.$$

It follows that the number of distinct $\rho_s W$ is at most equal to g/a, hence the desired inequality $\dim(V) \leqslant g/a$, since V is the sum of the $\rho_s W$. □

EXAMPLE. A dihedral group contains a cyclic subgroup of index 2. Its irreducible representations thus have degree 1 or 2; we will determine them later (5.3).

EXERCISES

3.1. Show directly, using Schur's lemma, that each irreducible representation of an abelian group, finite or not, has degree 1.

3.2. Let ρ be an irreducible representation of G of degree n and character χ; let C be the *center* of G (i.e., the set of $s \in G$ such that $st = ts$ for all $t \in G$), and let c be its order.

(a) Show that ρ_s is a homothety for each $s \in C$. [Use Schur's lemma.] Deduce from this that $|\chi(s)| = n$ for all $s \in C$.

(b) Prove the inequality $n^2 \leqslant g/c$. [Use the formula $\sum_{s \in G} |\chi(s)|^2 = g$, combined with (a).]

(c) Show that, if ρ is *faithful* (i.e., $\rho_s \neq 1$ for $s \neq 1$), the group C is cyclic.

3.3. Let G be an abelian group of order g, and let $\hat{G}$ be the set of irreducible characters of G. If χ_1, χ_2 belong to $\hat{G}$, the same is true of their product $\chi_1 \chi_2$. Show that this makes $\hat{G}$ an abelian group of order g; the group $\hat{G}$ is called the dual of the group G. For $x \in G$ the mapping $\chi \mapsto \chi(x)$ is an irreducible character of $\hat{G}$ and so an element of the dual $\hat{\hat{G}}$ of $\hat{G}$. Show that the map of G into $\hat{\hat{G}}$ thus obtained is an injective homomorphism; conclude (by comparing the orders of the two groups) that it is an *isomorphism*.

3.2 Product of two groups

Let G_1 and G_2 be two groups, and let $G_1 \times G_2$ be their *product*, that is, the set of pairs (s_1, s_2), with $s_1 \in G_1$ and $s_2 \in G_2$.

Putting

$$(s_1, s_2) \cdot (t_1, t_2) = (s_1 t_1, s_2 t_2),$$

we define a group structure on $G_1 \times G_2$; endowed with this structure, $G_1 \times G_2$ is called the *group product* of G_1 and G_2. If G_1 has order g_1 and G_2 has order g_2, $G_1 \times G_2$ has order $g = g_1 g_2$. The group G_1 can be identified with the subgroup of $G_1 \times G_2$ consisting of elements $(s_1, 1)$, where s_1 ranges over G_1; similarly, G_2 can be identified with a subgroup of $G_1 \times G_2$. With these identifications, each element of G_1 *commutes* with each element of G_2.

Conversely, let G be a group containing G_1 and G_2 as subgroups, and suppose the following two conditions are satisfied:

(i) *Each $s \in G$ can be written uniquely in the form $s = s_1 s_2$ with $s_1 \in G_1$ and $s_2 \in G_2$.*

(ii) *For $s_1 \in G_1$ and $s_2 \in G_2$, we have $s_1 s_2 = s_2 s_1$.*

The product of two elements $s = s_1 s_2$, $t = t_1 t_2$ can then be written

$$st = s_1 s_2 t_1 t_2 = (s_1 t_1)(s_2 t_2).$$

It follows that, if we let $(s_1, s_2) \in G_1 \times G_2$ correspond to the element $s_1 s_2$ of G, we obtain an *isomorphism of* $G_1 \times G_2$ *onto* G. In this case, we also say that G is the *product* (or the *direct product*) of its subgroups G_1 and G_2, and we identify it with $G_1 \times G_2$.

Now let $\rho^1 : G_1 \to \mathbf{GL}(V_1)$ and $\rho^2 : G_2 \to \mathbf{GL}(V_2)$ be linear representations of G_1 and G_2 respectively. We define a linear representation $\rho^1 \otimes \rho^2$ of $G_1 \times G_2$ into $V_1 \otimes V_2$ by a procedure analogous to 1.5 by setting

$$(\rho^1 \otimes \rho^2)(s_1, s_2) = \rho^1(s_1) \otimes \rho^2(s_2).$$

This representation is called the *tensor product* of the representations ρ^1 and ρ^2. If χ_i is the character of ρ_i $(i = 1, 2)$, the character χ of $\rho^1 \otimes \rho^2$ is given by:

$$\chi(s_1, s_2) = \chi_1(s_1) \cdot \chi_2(s_2).$$

When G_1 and G_2 are equal to the same group G, the representation $\rho^1 \otimes \rho^2$ defined above is a representation of $G \times G$. When restricted to the *diagonal* subgroup of $G \times G$ (consisting of (s, s), where s ranges over G), it gives the representation of G denoted $\rho^1 \otimes \rho^2$ in 1.5; in spite of the identity of notations, it is important to distinguish these two representations.

Theorem 10

(i) *If* ρ^1 *and* ρ^2 *are irreducible,* $\rho^1 \otimes \rho^2$ *is an irreducible representation of* $G_1 \times G_2$.

(ii) *Each irreducible representation of* $G_1 \times G_2$ *is isomorphic to a representation* $\rho^1 \otimes \rho^2$, *where* ρ^i *is an irreducible representation of* G_i $(i = 1, 2)$.

If ρ^1 and ρ^2 are irreducible, we have (cf. 2.3):

$$\frac{1}{g_1} \sum_{s_1} |\chi_1(s_1)|^2 = 1, \qquad \frac{1}{g_2} \sum_{s_2} |\chi_2(s_2)|^2 = 1.$$

By multiplication, this gives:

$$\frac{1}{g} \sum_{s_1, s_2} |\chi(s_1, s_2)|^2 = 1$$

which shows that $\rho^1 \otimes \rho^2$ is irreducible (th. 5). In order to prove (ii), it suffices to show that each class function f on $G_1 \times G_2$, which is orthogonal to the characters of the form $\chi_1(s_1)\chi_2(s_2)$, is zero. Suppose then that we have:

$$\sum_{s_1, s_2} f(s_1, s_2)\chi_1(s_1)^* \chi_2(s_2)^* = 0.$$

Fixing χ_2 and putting $g(s_1) = \sum_{s_2} f(s_1, s_2)\chi_2(s_2)^*$ we have:

$$\sum_{s_1} g(s_1)\chi_1(s_1)^* = 0 \quad \text{for all } \chi_1 .$$

Since g is a class function, this implies $g = 0$, and, since the same is true for each χ_2, we conclude by the same argument that $f(s_1, s_2) = 0$. □

[It is also possible to prove (ii) by computing the sum of the squares of the degrees of the representations $\rho^1 \otimes \rho^2$, and applying 2.4.]

The above theorem completely reduces the study of representations of $G_1 \times G_2$ to that of representations of G_1 and of representations of G_2.

3.3 Induced representations

Left cosets of a subgroup

Recall the following definition: Let H be a subgroup of a group G. For $s \in G$, we denote by sH the set of products st with $t \in H$, and say that sH is the *left coset* of H containing s. Two elements s, s' of G are said to be *congruent modulo* H if they belong to the same left coset, i.e., if $s^{-1}s'$ belongs to H; we write then $s' \equiv s \pmod{H}$. The set of left cosets of H is denoted by G/H; it is a partition of G. If G has g elements and H has h elements, G/H has g/h elements; the integer g/h is the *index* of H in G and is denoted by (G:H).

If we choose an element from each left coset of H, we obtain a subset R of G called a system of representatives of G/H; each s in G can be written uniquely $s = rt$, with $r \in R$ and $t \in H$.

Definition of induced representations

Let $\rho\colon G \to \mathbf{GL}(V)$ be a linear representation of G, and let ρ_H be its restriction to H. Let W be a subrepresentation of ρ_H, that is, a vector subspace of V stable under the ρ_t, $t \in H$. Denote by $\theta\colon H \to \mathbf{GL}(W)$ the representation of H in W thus defined. Let $s \in G$; the vector space $\rho_s W$ depends only on the left coset sH of s; indeed, if we replace s by st, with $t \in H$, we have $\rho_{st} W = \rho_s \rho_t W = \rho_s W$ since $\rho_t W = W$. If σ is a left coset of H, we can thus define a subspace W_σ of V to be $\rho_s W$ for any $s \in \sigma$. It is clear that the W_σ are permuted among themselves by the ρ_s, $s \in G$. Their sum $\sum_{\sigma \in G/H} W_\sigma$ is thus a subrepresentation of V.

Definition. We say that the representation ρ of G in V is *induced* by the representation θ of H in W if V is equal to the sum of the W_σ ($\sigma \in G/H$) and if this sum is direct (that is, if $V = \bigoplus_{\sigma \in G/H} W_\sigma$).

We can reformulate this condition in several ways:

(i) Each $x \in V$ can be written uniquely as $\sum_{\sigma \in G/H} x_\sigma$, with $x_\sigma \in W_\sigma$ for each σ.

(ii) If R is a system of representatives of G/H, the vector space V is the *direct sum* of the $\rho_r W$, with $r \in R$.

In particular, we have $\dim(V) = \sum_{r \in R} \dim(\rho_r W) = (G: H) \cdot \dim(W)$.

EXAMPLES 1. Take for V the regular representation of G; the space V has a basis $(e_t)_{t \in G}$ such that $\rho_s e_t = e_{st}$ for $s \in G$, $t \in G$. Let W be the subspace of V with basis $(e_t)_{t \in H}$. The representation θ of H in W is the *regular representation* of H, and it is clear that ρ is induced by θ.

2. Take for V a vector space having a basis (e_σ) indexed by the elements σ of G/H and define a representation ρ of G in V by $\rho_s e_\sigma = e_{s\sigma}$ for $s \in G$ and $\sigma \in G/H$ (this formula makes sense, because, if σ is a left coset of H, so is $s\sigma$). We thus obtain a representation of G which is the *permutation representation* of G associated with G/H [cf. 1.2, example (*c*)]. The vector e_H corresponding to the coset H is invariant under H; the representation of H in the subspace Ce_H is thus the *unit representation* of H, and it is clear that this representation induces the representation ρ of G in V.

3. If ρ_1 is induced by θ_1 and if ρ_2 is induced by θ_2, then $\rho_1 \oplus \rho_2$ is induced by $\theta_1 \oplus \theta_2$.

4. If (V, ρ) is induced by (W, θ), and if W_1 is a stable subspace of W, the subspace $V_1 = \sum_{r \in R} \rho_r W_1$ of V is stable under G, and the representation of G in V_1 is induced by the representation of H in W_1.

5. If ρ is induced by θ, if ρ' is a representation of G, and if ρ'_H is the restriction of ρ' to H, then $\rho \otimes \rho'$ is induced by $\theta \otimes \rho'_H$.

Existence and uniqueness of induced representations

Lemma 1. *Suppose that* (V, ρ) *is induced by* (W, θ). *Let* $\rho' : G \to \mathbf{GL}(V')$ *be a linear representation of* G, *and let* $f: W \to V'$ *be a linear map such that* $f(\theta_t w) = \rho'_t f(w)$ *for all* $t \in H$ *and* $w \in W$. *Then there exists a unique linear map* $F: V \to V'$ *which extends f and satisfies* $F \circ \rho_s = \rho'_s \circ F$ *for all* $s \in G$.

If F satisfies these conditions, and if $x \in \rho_s W$, we have $\rho_s^{-1} x \in W$; hence

$$F(x) = F(\rho_s \rho_s^{-1} x) = \rho'_s F(\rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x).$$

This formula determines F on $\rho_s W$, and so on V, since V is the sum of the $\rho_s W$. This proves the uniqueness of F.

Now let $x \in W_\sigma$, and choose $s \in \sigma$; we *define* $F(x)$ by the formula $F(x) = \rho'_s f(\rho_s^{-1} x)$ as above. This definition does not depend on the choice of s in σ; indeed, if we replace s by st, with $t \in H$, we have

$$\rho'_{st} f(\rho_{st}^{-1} x) = \rho'_s \rho'_t f(\theta_t^{-1} \rho_s^{-1} x) = \rho'_s(\theta_t \theta_t^{-1} \rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x).$$

Since V is the direct sum of the W_σ, there exists a unique linear map

F: $V \to V'$ which extends the partial mappings thus defined on the W_σ. It is easily checked that $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$. □

Theorem 11. *Let* (W, θ) *be a linear representation of* H. *There exists a linear representation* (V, ρ) *of* G *which is induced by* (W, θ), *and it is unique up to isomorphism.*

Let us first prove the existence of the induced representation ρ. In view of example 3, above, we may assume that θ is irreducible. In this case, θ is isomorphic to a subrepresentation of the regular representation of H, which can be induced to the regular representation of G (cf. example 1). Applying example 4, we conclude that θ itself can be induced.

It remains to prove the uniqueness of ρ up to isomorphism. Let (V, ρ) and (V', ρ') be two representations induced by (W, θ). Applying Lemma 1 to the injection of W into V′, we see that there exists a linear map F: $V \to V'$ which is the identity on W and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$. Consequently the image of F contains all the $\rho'_s W$, and thus is equal to V′. Since V′ and V have the same dimension $(G: H) \cdot \dim(W)$, we see that F is an *isomorphism*, which proves the theorem. (For a more natural proof of Theorem 11, see 7.1.) □

Character of an induced representation

Suppose (V, ρ) is induced by (W, θ) and let χ_ρ and χ_θ be the corresponding characters of G and of H. Since (W, θ) determines (V, ρ) up to isomorphism, we ought to be able to compute χ_ρ from χ_θ. The following theorem tells how:

Theorem 12. *Let h be the order of* H *and let* R *be a system of representatives of* G/H. *For each* $u \in G$, *we have*

$$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(r^{-1}ur) = \frac{1}{h} \sum_{\substack{s \in G \\ s^{-1}us \in H}} \chi_\theta(s^{-1}us).$$

(In particular, $\chi_\rho(u)$ is a linear combination of the values of χ_θ on the intersection of H with the conjugacy class of u in G.)

The space V is the direct sum of the $\rho_r W$, $r \in R$. Moreover ρ_u permutes the $\rho_r W$ among themselves. More precisely, if we write ur in the form $r_u t$ with $r_u \in R$ and $t \in H$, we see that ρ_u sends $\rho_r W$ into $\rho_{r_u} W$. To determine $\chi_\rho(u) = \mathrm{Tr}_V(\rho_u)$, we can use a basis of V which is a union of bases of the $\rho_r W$. The indices r such that $r_u \neq r$ give *zero* diagonal terms; the others give the trace of ρ_u on the $\rho_r W$. We thus obtain:

$$\chi_\rho(u) = \sum_{r \in R_u} \mathrm{Tr}_{\rho_r W}(\rho_{u,r}),$$