CHAPTER 3

Subgroups, products, induced representations

All the groups considered below are assumed to be finite.

3.1 Abelian subgroups

Let $G$ be a group. One says that $G$ is abelian (or commutative) if $st = ts$ for all $s, t \in G$. This amounts to saying that each conjugacy class of $G$ consists of a single element, also that each function on $G$ is a class function. The linear representations of such a group are particularly simple:

**Theorem 9.** The following properties are equivalent:

(i) $G$ is abelian.

(ii) All the irreducible representations of $G$ have degree 1.

Let $g$ be the order of $G$, and let $(n_1, \ldots, n_h)$ be the degrees of the distinct irreducible representations of $G$; we know, cf. Ch. 2, that $h$ is the number of classes of $G$, and that $g = n_1^2 + \cdots + n_h^2$. Hence $g$ is equal to $h$ if and only if all the $n_i$ are equal to 1, which proves the theorem.

**Corollary.** Let $A$ be an abelian subgroup of $G$, let $a$ be its order and let $g$ be that of $G$. Each irreducible representation of $G$ has degree $\leq g/a$.

(The quotient $g/a$ is the index of $A$ in $G$.)

Let $\rho: G \to \text{GL}(V)$ be an irreducible representation of $G$. Through restriction to the subgroup $A$, it defines a representation $\rho_A: A \to \text{GL}(V)$ of $A$. Let $W \subset V$ be an irreducible subrepresentation of $\rho_A$; by th. 9, we have $\dim(W) = 1$. Let $V'$ be the vector subspace of $V$ generated by the images $\rho_s W$ of $W$, $s$ ranging over $G$. It is clear that $V'$ is stable under $G$; since $\rho$ is irreducible, we thus have $V' = V$. But, for $s \in G$ and $t \in A$ we have

$$\rho_{st} W = \rho_s \rho_t W = \rho_s W.$$
Chapter 3: Subgroups, products, induced representations

It follows that the number of distinct $\rho_s W$ is at most equal to $g/a$, hence the desired inequality $\dim(V) \leq g/a$, since $V$ is the sum of the $\rho_s W$. □

EXAMPLE. A dihedral group contains a cyclic subgroup of index 2. Its irreducible representations thus have degree 1 or 2; we will determine them later (5.3).

EXERCISES

3.1. Show directly, using Schur's lemma, that each irreducible representation of an abelian group, finite or not, has degree 1.

3.2. Let $\rho$ be an irreducible representation of $G$ of degree $n$ and character $\chi$; let $C$ be the center of $G$ (i.e., the set of $s \in G$ such that $st = ts$ for all $t \in G$), and let $c$ be its order.

(a) Show that $\rho_s$ is a homothety for each $s \in C$. [Use Schur's lemma.]
Deduce from this that $|\chi(s)| = n$ for all $s \in C$.

(b) Prove the inequality $n^2 \leq g/c$. [Use the formula $\sum_{s \in G} |\chi(s)|^2 = g$, combined with (a).]

(c) Show that, if $\rho$ is faithful (i.e., $\rho_s \neq 1$ for $s \neq 1$), the group $C$ is cyclic.

3.3. Let $G$ be an abelian group of order $g$, and let $\hat{G}$ be the set of irreducible characters of $G$. If $\chi_1, \chi_2$ belong to $\hat{G}$, the same is true of their product $\chi_1 \chi_2$.
Show that this makes $\hat{G}$ an abelian group of order $g$; the group $\hat{G}$ is called the dual of the group $G$. For $x \in G$ the mapping $\chi \mapsto \chi(x)$ is an irreducible character of $\hat{G}$ and so an element of the dual $\hat{G}$ of $G$. Show that the map of $G$ into $\hat{G}$ thus obtained is an injective homomorphism; conclude (by comparing the orders of the two groups) that it is an isomorphism.

3.2 Product of two groups

Let $G_1$ and $G_2$ be two groups, and let $G_1 \times G_2$ be their product, that is, the set of pairs $(s_1, s_2)$, with $s_1 \in G_1$ and $s_2 \in G_2$.

Putting

$$(s_1, s_2) \cdot (t_1, t_2) = (s_1 t_1, s_2 t_2),$$

we define a group structure on $G_1 \times G_2$: endowed with this structure, $G_1 \times G_2$ is called the group product of $G_1$ and $G_2$. If $G_1$ has order $g_1$, and $G_2$ has order $g_2$, $G_1 \times G_2$ has order $g = g_1 g_2$. The group $G_1$ can be identified with the subgroup of $G_1 \times G_2$ consisting of elements $(s_1, 1)$, where $s_1$ ranges over $G_1$; similarly, $G_2$ can be identified with a subgroup of $G_1 \times G_2$. With these identifications, each element of $G_1$ commutes with each element of $G_2$.

Conversely, let $G$ be a group containing $G_1$ and $G_2$ as subgroups, and suppose the following two conditions are satisfied:

(i) Each $s \in G$ can be written uniquely in the form $s = s_1 s_2$ with $s_1 \in G_1$ and $s_2 \in G_2$.

(ii) For $s_1 \in G_1$ and $s_2 \in G_2$, we have $s_1 s_2 = s_2 s_1$.
The product of two elements \( s = s_1 s_2, \ t = t_1 t_2 \) can then be written
\[
st = s_1 s_2 t_1 t_2 = (s_1 t_1)(s_2 t_2).
\]
It follows that, if we let \( (s_1, s_2) \in G_1 \times G_2 \) correspond to the element \( s_1, s_2 \)
of \( G \), we obtain an isomorphism of \( G_1 \times G_2 \) onto \( G \). In this case, we also say that \( G \) is the product (or the direct product) of its subgroups \( G_1 \) and \( G_2 \), and we identify it with \( G_1 \times G_2 \).

Now let \( \rho^1 : G_1 \to \text{GL}(V_1) \) and \( \rho^2 : G_2 \to \text{GL}(V_2) \) be linear representations of \( G_1 \) and \( G_2 \) respectively. We define a linear representation \( \rho^1 \otimes \rho^2 \) of \( G_1 \times G_2 \) into \( V_1 \otimes V_2 \) by a procedure analogous to \ref{1.5} by setting
\[
(\rho^1 \otimes \rho^2)(s_1, s_2) = \rho^1(s_1) \otimes \rho^2(s_2).
\]
This representation is called the tensor product of the representations \( \rho^1 \) and \( \rho^2 \). If \( \chi_i \) is the character of \( \rho_i (i = 1, 2) \), the character \( \chi \) of \( \rho^1 \otimes \rho^2 \) is given by:
\[
\chi(s_1, s_2) = \chi_1(s_1) \cdot \chi_2(s_2).
\]
When \( G_1 \) and \( G_2 \) are equal to the same group \( G \), the representation \( \rho^1 \otimes \rho^2 \) defined above is a representation of \( G \times G \). When restricted to the diagonal subgroup of \( G \times G \) (consisting of \( (s, s) \), where \( s \) ranges over \( G \)), it gives the representation of \( G \) denoted \( \rho^1 \otimes \rho^2 \) in \ref{1.5}; in spite of the identity of notations, it is important to distinguish these two representations.

**Theorem 10**

(i) If \( \rho^1 \) and \( \rho^2 \) are irreducible, \( \rho^1 \otimes \rho^2 \) is an irreducible representation of \( G_1 \times G_2 \).

(ii) Each irreducible representation of \( G_1 \times G_2 \) is isomorphic to a representation \( \rho^1 \otimes \rho^2 \), where \( \rho^i \) is an irreducible representation of \( G_i \) \((i = 1, 2)\).

If \( \rho^1 \) and \( \rho^2 \) are irreducible, we have (cf. 2.3):
\[
\frac{1}{|G_1|} \sum_{s_1} |\chi_1(s_1)|^2 = 1, \quad \frac{1}{|G_2|} \sum_{s_2} |\chi_2(s_2)|^2 = 1.
\]
By multiplication, this gives:
\[
\frac{1}{|G|} \sum_{s_1, s_2} |\chi(s_1, s_2)|^2 = 1
\]
which shows that \( \rho^1 \otimes \rho^2 \) is irreducible (th. 5). In order to prove (ii), it suffices to show that each class function \( f \) on \( G_1 \times G_2 \), which is orthogonal to the characters of the form \( \chi_1(s_1) \chi_2(s_2) \), is zero. Suppose then that we have:
\[
\sum_{s_1, s_2} f(s_1, s_2) \chi_1(s_1)^* \chi_2(s_2)^* = 0.
\]
Chapter 3: Subgroups, products, induced representations

Fixing $\chi_2$ and putting $g(s_1) = \sum_{s_2} f(s_1, s_2)\chi_2(s_2)^*$ we have:

$$\sum_{s_1} g(s_1)\chi_1(s_1)^* = 0 \text{ for all } \chi_1.$$ 

Since $g$ is a class function, this implies $g = 0$, and, since the same is true for each $\chi_2$, we conclude by the same argument that $f(s_1, s_2) = 0$. □

[It is also possible to prove (ii) by computing the sum of the squares of the degrees of the representations $\rho^1 \otimes \rho^2$ and applying 2.4.]

The above theorem completely reduces the study of representations of $G_1 \times G_2$ to that of representations of $G_1$ and of representations of $G_2$.

3.3 Induced representations

Left cosets of a subgroup

Recall the following definition: Let $H$ be a subgroup of a group $G$. For $s \in G$, we denote by $sH$ the set of products $st$ with $t \in H$, and say that $sH$ is the left coset of $H$ containing $s$. Two elements $s, s'$ of $G$ are said to be congruent modulo $H$ if they belong to the same left coset, i.e., if $s^{-1}s'$ belongs to $H$; we write then $s' \equiv s \pmod{H}$. The set of left cosets of $H$ is denoted by $G/H$; it is a partition of $G$. If $G$ has $g$ elements and $H$ has $h$ elements, $G/H$ has $g/h$ elements; the integer $g/h$ is the index of $H$ in $G$ and is denoted by $(G:H)$.

If we choose an element from each left coset of $H$, we obtain a subset $R$ of $G$ called a system of representatives of $G/H$; each $s$ in $G$ can be written uniquely $s = rt$, with $r \in R$ and $t \in H$.

Definition of induced representations

Let $\rho: G \to \text{GL}(V)$ be a linear representation of $G$, and let $\rho_H$ be its restriction to $H$. Let $W$ be a subrepresentation of $\rho_H$, that is, a vector subspace of $V$ stable under the $\rho_t$, $t \in H$. Denote by $\theta: H \to \text{GL}(W)$ the representation of $H$ in $W$ thus defined. Let $s \in G$; the vector space $\rho_sW$ depends only on the left coset $sH$ of $s$; indeed, if we replace $s$ by $st$, with $t \in H$, we have $\rho_{st}W = \rho_s\rho_tW = \rho_sW$ since $\rho_tW = W$. If $s$ is a left coset of $H$, we can thus define a subspace $W_s$ of $V$ to be $\rho_sW$ for any $s \in \sigma$. It is clear that the $W_s$ are permuted among themselves by the $\rho_t$, $s \in G$. Their sum $\sum_{s \in G/H} W_s$ is thus a subrepresentation of $V$.

Definition. We say that the representation $\rho$ of $G$ in $V$ is induced by the representation $\theta$ of $H$ in $W$ if $V$ is equal to the sum of the $W_s$ ($s \in G/H$) and if this sum is direct (that is, if $V = \bigoplus_{s \in G/H} W_s$).

We can reformulate this condition in several ways:

(i) Each $x \in V$ can be written uniquely as $\sum_{s \in G/H} x_s$, with $x_s \in W_s$ for each $s$. 

28
(ii) If $R$ is a system of representatives of $G/H$, the vector space $V$ is the direct sum of the $\rho_r W$, with $r \in R$.

In particular, we have $\dim(V) = \sum_{r \in R} \dim(\rho_r W) = (G : H) \cdot \dim(W)$.

**Examples 1.** Take for $V$ the regular representation of $G$; the space $V$ has a basis $(e_s)_{s \in G}$ such that $\rho_s e_t = e_{st}$ for $s, t \in G$. Let $W$ be the subspace of $V$ with basis $(e_{\iota})_{\iota \in H}$. The representation $\theta$ of $H$ in $W$ is the regular representation of $H$, and it is clear that $\rho$ is induced by $\theta$.

2. Take for $V$ a vector space having a basis $(e_s)$ indexed by the elements $\sigma$ of $G/H$ and define a representation $\rho$ of $G$ in $V$ by $\rho_s e_{\sigma} = e_{\sigma s}$ for $s \in G$ and $\sigma \in G/H$ (this formula makes sense, because, if $\sigma$ is a left coset of $H$, so is $\sigma \iota$). We thus obtain a representation of $G$ which is the permutation representation of $G$ associated with $G/H$ [cf. 1.2, example (c)]. The vector $e_{\iota H}$ corresponding to the coset $H$ is invariant under $H$; the representation of $H$ in the subspace $C e_{H}$ is thus the unit representation of $H$, and it is clear that this representation induces the representation $\rho$ of $G$ in $V$.

3. If $\rho_1$ is induced by $\theta_1$ and if $\rho_2$ is induced by $\theta_2$, then $\rho_1 \oplus \rho_2$ is induced by $\theta_1 \oplus \theta_2$.

4. If $(V, \rho)$ is induced by $(W, \theta)$, and if $W_1$ is a stable subspace of $W$, the subspace $V_1 = \sum_{r \in R} \rho_r W_1$ of $V$ is stable under $G$, and the representation of $G$ in $V_1$ is induced by the representation of $H$ in $W_1$.

5. If $\rho$ is induced by $\theta$, if $\rho'$ is a representation of $G$, and if $\rho'_{\iota H}$ is the restriction of $\rho'$ to $H$, then $\rho \otimes \rho'$ is induced by $\theta \otimes \rho'_{\iota H}$.

**Existence and uniqueness of induced representations**

**Lemma 1.** Suppose that $(V, \rho)$ is induced by $(W, \theta)$. Let $\rho': G \to GL(V')$ be a linear representation of $G$, and let $f: W \to V'$ be a linear map such that $f(\theta_r w) = \rho'_r f(w)$ for all $r \in H, w \in W$. Then there exists a unique linear map $F: V \to V'$ which extends $f$ and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$.

If $F$ satisfies these conditions, and if $x \in \rho_s W$, we have $\rho_s^{-1} x \in W$; hence

$$F(x) = F(\rho_s \rho_s^{-1} x) = \rho'_s F(\rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x).$$

This formula determines $F$ on $\rho_s W$, and so on $V$, since $V$ is the sum of the $\rho_s W$. This proves the uniqueness of $F$.

Now let $x \in W_\iota$, and choose $\sigma \in \sigma$; we define $F(x)$ by the formula

$$F(x) = \rho'_s f(\rho_s^{-1} x)$$

as above. This definition does not depend on the choice of $\sigma$ in $\sigma$; indeed, if we replace $\sigma$ by $\sigma t$, with $t \in H$, we have

$$\rho'_s f(\rho_s^{-1} x) = \rho'_s \rho'_t f(\rho_t^{-1} \rho_s^{-1} x) = \rho'_s (\theta_t \theta_t^{-1} \rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x).$$

Since $V$ is the direct sum of the $W_\iota$, there exists a unique linear map
Chapter 3: Subgroups, products, induced representations

F: \( V \to V' \) which extends the partial mappings thus defined on \( W_o \). It is easily checked that \( F \circ \rho_s = \rho_s' \circ F \) for all \( s \in G \).

\[ \square \]

**Theorem 11.** Let \((W, \theta)\) be a linear representation of \( H \). There exists a linear representation \((V, \rho)\) of \( G \) which is induced by \((W, \theta)\), and it is unique up to isomorphism.

Let us first prove the existence of the induced representation \( \rho \). In view of example 3, above, we may assume that \( \theta \) is irreducible. In this case, \( \theta \) is isomorphic to a subrepresentation of the regular representation of \( H \), which can be induced to the regular representation of \( G \) (cf. example 1). Applying example 4, we conclude that \( \theta \) itself can be induced.

It remains to prove the uniqueness of \( \rho \) up to isomorphism. Let \((V, \rho)\) and \((V', \rho')\) be two representations induced by \((W, \theta)\). Applying Lemma 1 to the injection of \( W \) into \( V' \), we see that there exists a linear map \( F: V \to V' \) which is the identity on \( W \) and satisfies \( F \circ \rho_s = \rho_s' \circ F \) for all \( s \in G \). Consequently the image of \( F \) contains all the \( \rho_s' W \), and thus is equal to \( V' \). Since \( V' \) and \( V \) have the same dimension \((G: H) \cdot \dim(W)\), we see that \( F \) is an isomorphism, which proves the theorem. (For a more natural proof of Theorem 11, see 7.1.)

\[ \square \]

**Character of an induced representation**

Suppose \((V, \rho)\) is induced by \((W, \theta)\) and let \( \chi_\theta \) and \( \chi_\rho \) be the corresponding characters of \( G \) and of \( H \). Since \((W, \theta)\) determines \((V, \rho)\) up to isomorphism, we ought to be able to compute \( \chi_\rho \) from \( \chi_\theta \). The following theorem tells how:

**Theorem 12.** Let \( h \) be the order of \( H \) and let \( R \) be a system of representatives of \( G/H \). For each \( u \in G \), we have

\[
\chi_\rho(u) = \frac{1}{h} \sum_{r \in R} \chi_\theta(r^{-1}ur) = \frac{1}{h} \sum_{s \in G} \chi_\theta(s^{-1}us).
\]

(In particular, \( \chi_\rho(u) \) is a linear combination of the values of \( \chi_\theta \) on the intersection of \( H \) with the conjugacy class of \( u \) in \( G \).)

The space \( V \) is the direct sum of the \( \rho_s W \), \( r \in R \). Moreover \( \rho_u \) permutes the \( \rho_s W \) among themselves. More precisely, if we write \( ur \) in the form \( r' t \) with \( r' \in R \) and \( t \in H \), we see that \( \rho_u \) sends \( \rho_s W \) into \( \rho_s' W \). To determine \( \chi_\rho(u) = Tr_V(\rho_u) \), we can use a basis of \( V \) which is a union of bases of the \( \rho_s W \). The indices \( r \) such that \( r' \neq r \) give zero diagonal terms; the others give the trace of \( \rho_u \) on the \( \rho_s W \). We thus obtain:

\[
\chi_\rho(u) = \sum_{r \in R_u} Tr_{\rho_s W}(\rho_u r).
\]
where $R_u$ denotes the set of $r \in R$ such that $\tau_r = r$ and $\rho_{ur}$ is the restriction of $\rho_u$ to $\rho_r W$. Observe that $r$ belongs to $R_u$, if and only if $ur$ can be written $rt$, with $t \in H$, i.e., if $r^{-1}ur$ belongs to $H$.

It remains to compute $Tr_{\rho_r W}(\rho_{ur})$, for $r \in R_u$. To do this, note that $\rho_r$ defines an isomorphism of $\tilde{W}$ onto $\rho_r W$, and that we have

$$\rho_r \circ \theta_i = \rho_{ur} \circ \rho_r, \quad \text{with} \quad t = r^{-1}ur \in H.$$

The trace of $\rho_{ur}$ is thus equal to that of $\theta_i$, that is, to $\chi_\theta(i) = \chi_\theta(r^{-1}ur)$. We indeed obtain:

$$\chi_\theta(u) = \sum_{r \in R_u} \chi_\theta(r^{-1}ur).$$

The second formula given for $\chi_\theta(u)$ follows from the first by noting that all elements $s$ of $G$ in the left coset $rH$ ($r \in R_u$) satisfy $\chi_\theta(s^{-1}ur) = \chi_\theta(r^{-1}ur)$. \(\Box\)

The reader will find other properties of induced representations in part II. Notably:

(i) The Frobenius reciprocity formula

$$\langle f_1 | \chi_\theta \rangle_H = \langle f \chi_\theta \rangle_G$$

where $f$ is a class function of $G$, and $f_1$ is its restriction to $H$, and the scalar products are calculated on $H$ and $G$ respectively.

(ii) Mackey's criterion, which tells us when an induced representation is irreducible.

(iii) Artin's theorem (resp. Brauer's theorem), which says that each character of a group $G$ is a linear combination with rational (resp. integral) coefficients of characters of representations induced from cyclic subgroups (resp. from "elementary" subgroups) of $G$.

**EXERCISES**

3.4. Show that each irreducible representation of $G$ is contained in a representation induced by an irreducible representation of $H$. [Use the fact that an irreducible representation is contained in the regular representation.] Obtain from this another proof of the cor. to th. 9.

3.5. Let $(W, \theta)$ be a linear representation of $H$. Let $V$ be the vector space of functions $f: G \to W$ such that $f(ux) = \theta_x f(u)$ for $u \in G$, $x \in H$. Let $\rho$ be the representation of $G$ in $V$ defined by $(\rho_f)(u) = f(ux)$ for $x, u \in G$. For $w \in W$ let $f_w \in V$ be defined by $f_w(t) = \theta_t w$ for $t \in H$ and $f_w(s) = 0$ for $s \not\in H$. Show that $w \mapsto f_w$ is an isomorphism of $W$ onto the subspace $W_0$ of $V$ consisting of functions which vanish off $H$. Show that, if we identify $W$ and $W_0$ in this way, the representation $(V, \rho)$ is induced by the representation $(W, \theta)$.

3.6. Suppose that $G$ is the direct product of two subgroups $H$ and $K$ (cf. 3.2). Let $\rho$ be a representation of $G$ induced by a representation $\theta$ of $H$. Show that $\rho$ is isomorphic to $\theta \otimes \pi_K$, where $\pi_K$ denotes the regular representation of $K$. 31