

## CHAPTER 3

### Subgroups, products, induced representations

All the groups considered below are assumed to be finite.

#### 3.1 Abelian subgroups

Let  $G$  be a group. One says that  $G$  is *abelian* (or *commutative*) if  $st = ts$  for all  $s, t \in G$ . This amounts to saying that each conjugacy class of  $G$  consists of a single element, also that each function on  $G$  is a class function. The linear representations of such a group are particularly simple:

**Theorem 9.** *The following properties are equivalent:*

- (i)  $G$  is abelian.
- (ii) All the irreducible representations of  $G$  have degree 1.

Let  $g$  be the order of  $G$ , and let  $(n_1, \dots, n_h)$  be the degrees of the distinct irreducible representations of  $G$ ; we know, cf. Ch. 2, that  $h$  is the number of classes of  $G$ , and that  $g = n_1^2 + \dots + n_h^2$ . Hence  $g$  is equal to  $h$  if and only if all the  $n_i$  are equal to 1, which proves the theorem.  $\square$

**Corollary.** *Let  $A$  be an abelian subgroup of  $G$ , let  $a$  be its order and let  $g$  be that of  $G$ . Each irreducible representation of  $G$  has degree  $\leq g/a$ .*

(The quotient  $g/a$  is the *index* of  $A$  in  $G$ .)

Let  $\rho: G \rightarrow \mathbf{GL}(V)$  be an irreducible representation of  $G$ . Through *restriction* to the subgroup  $A$ , it defines a representation  $\rho_A: A \rightarrow \mathbf{GL}(V)$  of  $A$ . Let  $W \subset V$  be an irreducible subrepresentation of  $\rho_A$ ; by th. 9, we have  $\dim(W) = 1$ . Let  $V'$  be the vector subspace of  $V$  generated by the images  $\rho_s W$  of  $W$ ,  $s$  ranging over  $G$ . It is clear that  $V'$  is stable under  $G$ ; since  $\rho$  is irreducible, we thus have  $V' = V$ . But, for  $s \in G$  and  $t \in A$  we have

$$\rho_{st} W = \rho_s \rho_t W = \rho_s W.$$

It follows that the number of distinct  $\rho_s W$  is at most equal to  $g/a$ , hence the desired inequality  $\dim(V) \leq g/a$ , since  $V$  is the sum of the  $\rho_s W$ .  $\square$

EXAMPLE. A dihedral group contains a cyclic subgroup of index 2. Its irreducible representations thus have degree 1 or 2; we will determine them later (5.3).

### EXERCISES

- 3.1. Show directly, using Schur's lemma, that each irreducible representation of an abelian group, finite or not, has degree 1.
- 3.2. Let  $\rho$  be an irreducible representation of  $G$  of degree  $n$  and character  $\chi$ ; let  $C$  be the center of  $G$  (i.e., the set of  $s \in G$  such that  $st = ts$  for all  $t \in G$ ), and let  $c$  be its order.
  - (a) Show that  $\rho_s$  is a homothety for each  $s \in C$ . [Use Schur's lemma.] Deduce from this that  $|\chi(s)| = n$  for all  $s \in C$ .
  - (b) Prove the inequality  $n^2 \leq g/c$ . [Use the formula  $\sum_{s \in G} |\chi(s)|^2 = g$ , combined with (a).]
  - (c) Show that, if  $\rho$  is faithful (i.e.,  $\rho_s \neq 1$  for  $s \neq 1$ ), the group  $C$  is cyclic.
- 3.3. Let  $G$  be an abelian group of order  $g$ , and let  $\hat{G}$  be the set of irreducible characters of  $G$ . If  $\chi_1, \chi_2$  belong to  $\hat{G}$ , the same is true of their product  $\chi_1 \chi_2$ . Show that this makes  $\hat{G}$  an abelian group of order  $g$ ; the group  $\hat{G}$  is called the dual of the group  $G$ . For  $x \in G$  the mapping  $\chi \mapsto \chi(x)$  is an irreducible character of  $\hat{G}$  and so an element of the dual  $\hat{\hat{G}}$  of  $\hat{G}$ . Show that the map of  $G$  into  $\hat{\hat{G}}$  thus obtained is an injective homomorphism; conclude (by comparing the orders of the two groups) that it is an *isomorphism*.

### 3.2 Product of two groups

Let  $G_1$  and  $G_2$  be two groups, and let  $G_1 \times G_2$  be their *product*, that is, the set of pairs  $(s_1, s_2)$ , with  $s_1 \in G_1$  and  $s_2 \in G_2$ .

Putting

$$(s_1, s_2) \cdot (t_1, t_2) = (s_1 t_1, s_2 t_2),$$

we define a group structure on  $G_1 \times G_2$ ; endowed with this structure,  $G_1 \times G_2$  is called the *group product* of  $G_1$  and  $G_2$ . If  $G_1$  has order  $g_1$  and  $G_2$  has order  $g_2$ ,  $G_1 \times G_2$  has order  $g = g_1 g_2$ . The group  $G_1$  can be identified with the subgroup of  $G_1 \times G_2$  consisting of elements  $(s_1, 1)$ , where  $s_1$  ranges over  $G_1$ ; similarly,  $G_2$  can be identified with a subgroup of  $G_1 \times G_2$ . With these identifications, each element of  $G_1$  commutes with each element of  $G_2$ .

Conversely, let  $G$  be a group containing  $G_1$  and  $G_2$  as subgroups, and suppose the following two conditions are satisfied:

- (i) Each  $s \in G$  can be written uniquely in the form  $s = s_1 s_2$  with  $s_1 \in G_1$  and  $s_2 \in G_2$ .
- (ii) For  $s_1 \in G_1$  and  $s_2 \in G_2$ , we have  $s_1 s_2 = s_2 s_1$ .

The product of two elements  $s = s_1 s_2$ ,  $t = t_1 t_2$  can then be written

$$st = s_1 s_2 t_1 t_2 = (s_1 t_1)(s_2 t_2).$$

It follows that, if we let  $(s_1, s_2) \in G_1 \times G_2$  correspond to the element  $s_1 s_2$  of  $G$ , we obtain an *isomorphism of  $G_1 \times G_2$  onto  $G$* . In this case, we also say that  $G$  is the *product* (or the *direct product*) of its subgroups  $G_1$  and  $G_2$ , and we identify it with  $G_1 \times G_2$ .

Now let  $\rho^1: G_1 \rightarrow \text{GL}(V_1)$  and  $\rho^2: G_2 \rightarrow \text{GL}(V_2)$  be linear representations of  $G_1$  and  $G_2$  respectively. We define a linear representation  $\rho^1 \otimes \rho^2$  of  $G_1 \times G_2$  into  $V_1 \otimes V_2$  by a procedure analogous to 1.5 by setting

$$(\rho^1 \otimes \rho^2)(s_1, s_2) = \rho^1(s_1) \otimes \rho^2(s_2).$$

This representation is called the *tensor product* of the representations  $\rho^1$  and  $\rho^2$ . If  $\chi_i$  is the character of  $\rho_i$  ( $i = 1, 2$ ), the character  $\chi$  of  $\rho^1 \otimes \rho^2$  is given by:

$$\chi(s_1, s_2) = \chi_1(s_1) \cdot \chi_2(s_2).$$

When  $G_1$  and  $G_2$  are equal to the same group  $G$ , the representation  $\rho^1 \otimes \rho^2$  defined above is a representation of  $G \times G$ . When restricted to the *diagonal* subgroup of  $G \times G$  (consisting of  $(s, s)$ , where  $s$  ranges over  $G$ ), it gives the representation of  $G$  denoted  $\rho^1 \otimes \rho^2$  in 1.5; in spite of the identity of notations, it is important to distinguish these two representations.

**Theorem 10**

- (i) If  $\rho^1$  and  $\rho^2$  are irreducible,  $\rho^1 \otimes \rho^2$  is an irreducible representation of  $G_1 \times G_2$ .
- (ii) Each irreducible representation of  $G_1 \times G_2$  is isomorphic to a representation  $\rho^1 \otimes \rho^2$ , where  $\rho^i$  is an irreducible representation of  $G_i$  ( $i = 1, 2$ ).

If  $\rho^1$  and  $\rho^2$  are irreducible, we have (cf. 2.3):

$$\frac{1}{g_1} \sum_{s_1} |\chi_1(s_1)|^2 = 1, \quad \frac{1}{g_2} \sum_{s_2} |\chi_2(s_2)|^2 = 1.$$

By multiplication, this gives:

$$\frac{1}{g} \sum_{s_1, s_2} |\chi(s_1, s_2)|^2 = 1$$

which shows that  $\rho^1 \otimes \rho^2$  is irreducible (th. 5). In order to prove (ii), it suffices to show that each class function  $f$  on  $G_1 \times G_2$ , which is orthogonal to the characters of the form  $\chi_1(s_1)\chi_2(s_2)$ , is zero. Suppose then that we have:

$$\sum_{s_1, s_2} f(s_1, s_2) \chi_1(s_1)^* \chi_2(s_2)^* = 0.$$

Fixing  $\chi_2$  and putting  $g(s_1) = \sum_{s_2} f(s_1, s_2) \chi_2(s_2)^*$  we have:

$$\sum_{s_1} g(s_1) \chi_1(s_1)^* = 0 \quad \text{for all } \chi_1.$$

Since  $g$  is a class function, this implies  $g = 0$ , and, since the same is true for each  $\chi_2$ , we conclude by the same argument that  $f(s_1, s_2) = 0$ .  $\square$

[It is also possible to prove (ii) by computing the sum of the squares of the degrees of the representations  $\rho^1 \otimes \rho^2$ , and applying 2.4.]

The above theorem completely reduces the study of representations of  $G_1 \times G_2$  to that of representations of  $G_1$  and of representations of  $G_2$ .

### 3.3 Induced representations

#### *Left cosets of a subgroup*

Recall the following definition: Let  $H$  be a subgroup of a group  $G$ . For  $s \in G$ , we denote by  $sH$  the set of products  $st$  with  $t \in H$ , and say that  $sH$  is the *left coset* of  $H$  containing  $s$ . Two elements  $s, s'$  of  $G$  are said to be *congruent modulo*  $H$  if they belong to the same left coset, i.e., if  $s^{-1}s'$  belongs to  $H$ ; we write then  $s' \equiv s \pmod{H}$ . The set of left cosets of  $H$  is denoted by  $G/H$ ; it is a partition of  $G$ . If  $G$  has  $g$  elements and  $H$  has  $h$  elements,  $G/H$  has  $g/h$  elements; the integer  $g/h$  is the *index* of  $H$  in  $G$  and is denoted by  $(G:H)$ .

If we choose an element from each left coset of  $H$ , we obtain a subset  $R$  of  $G$  called a system of representatives of  $G/H$ ; each  $s$  in  $G$  can be written uniquely  $s = rt$ , with  $r \in R$  and  $t \in H$ .

#### *Definition of induced representations*

Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of  $G$ , and let  $\rho_H$  be its restriction to  $H$ . Let  $W$  be a subrepresentation of  $\rho_H$ , that is, a vector subspace of  $V$  stable under the  $\rho_t$ ,  $t \in H$ . Denote by  $\theta: H \rightarrow \text{GL}(W)$  the representation of  $H$  in  $W$  thus defined. Let  $s \in G$ ; the vector space  $\rho_s W$  depends only on the left coset  $sH$  of  $s$ ; indeed, if we replace  $s$  by  $st$ , with  $t \in H$ , we have  $\rho_{st} W = \rho_s \rho_t W = \rho_s W$  since  $\rho_t W = W$ . If  $\sigma$  is a left coset of  $H$ , we can thus define a subspace  $W_\sigma$  of  $V$  to be  $\rho_s W$  for any  $s \in \sigma$ . It is clear that the  $W_\sigma$  are permuted among themselves by the  $\rho_s$ ,  $s \in G$ . Their sum  $\sum_{\sigma \in G/H} W_\sigma$  is thus a subrepresentation of  $V$ .

**Definition.** We say that the representation  $\rho$  of  $G$  in  $V$  is *induced* by the representation  $\theta$  of  $H$  in  $W$  if  $V$  is equal to the sum of the  $W_\sigma$  ( $\sigma \in G/H$ ) and if this sum is direct (that is, if  $V = \bigoplus_{\sigma \in G/H} W_\sigma$ ).

We can reformulate this condition in several ways:

- (i) Each  $x \in V$  can be written uniquely as  $\sum_{\sigma \in G/H} x_\sigma$ , with  $x_\sigma \in W_\sigma$  for each  $\sigma$ .

- (ii) If  $R$  is a system of representatives of  $G/H$ , the vector space  $V$  is the direct sum of the  $\rho_r W$ , with  $r \in R$ .

In particular, we have  $\dim(V) = \sum_{r \in R} \dim(\rho_r W) = (G:H) \cdot \dim(W)$ .

**EXAMPLES 1.** Take for  $V$  the regular representation of  $G$ ; the space  $V$  has a basis  $(e_t)_{t \in G}$  such that  $\rho_s e_t = e_{st}$  for  $s \in G, t \in G$ . Let  $W$  be the subspace of  $V$  with basis  $(e_t)_{t \in H}$ . The representation  $\theta$  of  $H$  in  $W$  is the regular representation of  $H$ , and it is clear that  $\rho$  is induced by  $\theta$ .

2. Take for  $V$  a vector space having a basis  $(e_\sigma)$  indexed by the elements  $\sigma$  of  $G/H$  and define a representation  $\rho$  of  $G$  in  $V$  by  $\rho_s e_\sigma = e_{s\sigma}$  for  $s \in G$  and  $\sigma \in G/H$  (this formula makes sense, because, if  $\sigma$  is a left coset of  $H$ , so is  $s\sigma$ ). We thus obtain a representation of  $G$  which is the permutation representation of  $G$  associated with  $G/H$  [cf. 1.2, example (c)]. The vector  $e_H$  corresponding to the coset  $H$  is invariant under  $H$ ; the representation of  $H$  in the subspace  $Ce_H$  is thus the unit representation of  $H$ , and it is clear that this representation induces the representation  $\rho$  of  $G$  in  $V$ .

3. If  $\rho_1$  is induced by  $\theta_1$  and if  $\rho_2$  is induced by  $\theta_2$ , then  $\rho_1 \oplus \rho_2$  is induced by  $\theta_1 \oplus \theta_2$ .

4. If  $(V, \rho)$  is induced by  $(W, \theta)$ , and if  $W_1$  is a stable subspace of  $W$ , the subspace  $V_1 = \sum_{r \in R} \rho_r W_1$  of  $V$  is stable under  $G$ , and the representation of  $G$  in  $V_1$  is induced by the representation of  $H$  in  $W_1$ .

5. If  $\rho$  is induced by  $\theta$ , if  $\rho'$  is a representation of  $G$ , and if  $\rho'_H$  is the restriction of  $\rho'$  to  $H$ , then  $\rho \otimes \rho'$  is induced by  $\theta \otimes \rho'_H$ .

#### Existence and uniqueness of induced representations

**Lemma 1.** Suppose that  $(V, \rho)$  is induced by  $(W, \theta)$ . Let  $\rho': G \rightarrow \text{GL}(V')$  be a linear representation of  $G$ , and let  $f: W \rightarrow V'$  be a linear map such that  $f(\theta_t w) = \rho'_t f(w)$  for all  $t \in H$  and  $w \in W$ . Then there exists a unique linear map  $F: V \rightarrow V'$  which extends  $f$  and satisfies  $F \circ \rho_s = \rho'_s \circ F$  for all  $s \in G$ .

If  $F$  satisfies these conditions, and if  $x \in \rho_s W$ , we have  $\rho_s^{-1} x \in W$ ; hence

$$F(x) = F(\rho_s \rho_s^{-1} x) = \rho'_s F(\rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x).$$

This formula determines  $F$  on  $\rho_s W$ , and so on  $V$ , since  $V$  is the sum of the  $\rho_s W$ . This proves the uniqueness of  $F$ .

Now let  $x \in W_\sigma$ , and choose  $s \in \sigma$ ; we define  $F(x)$  by the formula  $F(x) = \rho'_s f(\rho_s^{-1} x)$  as above. This definition does not depend on the choice of  $s$  in  $\sigma$ ; indeed, if we replace  $s$  by  $st$ , with  $t \in H$ , we have

$$\rho'_{st} f(\rho_{st}^{-1} x) = \rho'_s \rho'_t f(\theta_t^{-1} \rho_s^{-1} x) = \rho'_s (\theta_t \theta_t^{-1} \rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x).$$

Since  $V$  is the direct sum of the  $W_\sigma$ , there exists a unique linear map

$F: V \rightarrow V'$  which extends the partial mappings thus defined on the  $W_\theta$ . It is easily checked that  $F \circ \rho_s = \rho'_s \circ F$  for all  $s \in G$ .  $\square$

**Theorem 11.** *Let  $(W, \theta)$  be a linear representation of  $H$ . There exists a linear representation  $(V, \rho)$  of  $G$  which is induced by  $(W, \theta)$ , and it is unique up to isomorphism.*

Let us first prove the existence of the induced representation  $\rho$ . In view of example 3, above, we may assume that  $\theta$  is irreducible. In this case,  $\theta$  is isomorphic to a subrepresentation of the regular representation of  $H$ , which can be induced to the regular representation of  $G$  (cf. example 1). Applying example 4, we conclude that  $\theta$  itself can be induced.

It remains to prove the uniqueness of  $\rho$  up to isomorphism. Let  $(V, \rho)$  and  $(V', \rho')$  be two representations induced by  $(W, \theta)$ . Applying Lemma 1 to the injection of  $W$  into  $V'$ , we see that there exists a linear map  $F: V \rightarrow V'$  which is the identity on  $W$  and satisfies  $F \circ \rho_s = \rho'_s \circ F$  for all  $s \in G$ . Consequently the image of  $F$  contains all the  $\rho'_s W$ , and thus is equal to  $V'$ . Since  $V'$  and  $V$  have the same dimension  $(G:H) \cdot \dim(W)$ , we see that  $F$  is an *isomorphism*, which proves the theorem. (For a more natural proof of Theorem 11, see 7.1.)  $\square$

#### Character of an induced representation

Suppose  $(V, \rho)$  is induced by  $(W, \theta)$  and let  $\chi_\rho$  and  $\chi_\theta$  be the corresponding characters of  $G$  and of  $H$ . Since  $(W, \theta)$  determines  $(V, \rho)$  up to isomorphism, we ought to be able to compute  $\chi_\rho$  from  $\chi_\theta$ . The following theorem tells how:

**Theorem 12.** *Let  $h$  be the order of  $H$  and let  $R$  be a system of representatives of  $G/H$ . For each  $u \in G$ , we have*

$$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(r^{-1}ur) = \frac{1}{h} \sum_{\substack{s \in G \\ s^{-1}us \in H}} \chi_\theta(s^{-1}us).$$

(In particular,  $\chi_\rho(u)$  is a linear combination of the values of  $\chi_\theta$  on the intersection of  $H$  with the conjugacy class of  $u$  in  $G$ .)

The space  $V$  is the direct sum of the  $\rho_r W$ ,  $r \in R$ . Moreover  $\rho_u$  permutes the  $\rho_r W$  among themselves. More precisely, if we write  $ur$  in the form  $r_u t$  with  $r_u \in R$  and  $t \in H$ , we see that  $\rho_u$  sends  $\rho_r W$  into  $\rho_{r_u} W$ . To determine  $\chi_\rho(u) = \text{Tr}_V(\rho_u)$ , we can use a basis of  $V$  which is a union of bases of the  $\rho_r W$ . The indices  $r$  such that  $r_u \neq r$  give zero diagonal terms; the others give the trace of  $\rho_u$  on the  $\rho_r W$ . We thus obtain:

$$\chi_\rho(u) = \sum_{r \in R_u} \text{Tr}_{\rho_r W}(\rho_{u,r}),$$

where  $R_u$  denotes the set of  $r \in R$  such that  $r_u = r$  and  $\rho_{u,r}$  is the restriction of  $\rho_u$  to  $\rho_r W$ . Observe that  $r$  belongs to  $R_u$  if and only if  $ur$  can be written  $rt$ , with  $t \in H$ , i.e., if  $r^{-1}ur$  belongs to  $H$ .

It remains to compute  $\text{Tr}_{\rho_r W}(\rho_{u,r})$ , for  $r \in R_u$ . To do this, note that  $\rho_r$  defines an isomorphism of  $W$  onto  $\rho_r W$ , and that we have

$$\rho_r \circ \theta_t = \rho_{u,r} \circ \rho_r, \quad \text{with } t = r^{-1}ur \in H.$$

The trace of  $\rho_{u,r}$  is thus equal to that of  $\theta_t$ , that is, to  $\chi_\theta(t) = \chi_\theta(r^{-1}ur)$ . We indeed obtain:

$$\chi_\rho(u) = \sum_{r \in R_u} \chi_\theta(r^{-1}ur).$$

The second formula given for  $\chi_\rho(u)$  follows from the first by noting that all elements  $s$  of  $G$  in the left coset  $rH$  ( $r \in R_u$ ) satisfy  $\chi_\theta(s^{-1}us) = \chi_\theta(r^{-1}ur)$ .  $\square$

The reader will find other properties of induced representations in part II. Notably:

(i) The *Frobenius reciprocity formula*

$$(f_H | \chi_\theta)_H = (f | \chi_\rho)_G$$

where  $f$  is a class function of  $G$ , and  $f_H$  is its restriction to  $H$ , and the scalar products are calculated on  $H$  and  $G$  respectively.

(ii) *Mackey's criterion*, which tells us when an induced representation is irreducible.

(iii) *Artin's theorem* (resp. *Brauer's theorem*), which says that each character of a group  $G$  is a linear combination with rational (resp. integral) coefficients of characters of representations induced from cyclic subgroups (resp. from "elementary" subgroups) of  $G$ .

#### EXERCISES

- 3.4. Show that each irreducible representation of  $G$  is contained in a representation induced by an irreducible representation of  $H$ . [Use the fact that an irreducible representation is contained in the regular representation.] Obtain from this another proof of the cor. to th. 9.
- 3.5. Let  $(W, \theta)$  be a linear representation of  $H$ . Let  $V$  be the vector space of functions  $f: G \rightarrow W$  such that  $f(tu) = \theta_t f(u)$  for  $u \in G, t \in H$ . Let  $\rho$  be the representation of  $G$  in  $V$  defined by  $(\rho_s f)(u) = f(us)$  for  $s, u \in G$ . For  $w \in W$  let  $f_w \in V$  be defined by  $f_w(t) = \theta_t w$  for  $t \in H$  and  $f_w(s) = 0$  for  $s \notin H$ . Show that  $w \mapsto f_w$  is an isomorphism of  $W$  onto the subspace  $W_0$  of  $V$  consisting of functions which vanish off  $H$ . Show that, if we identify  $W$  and  $W_0$  in this way, the representation  $(V, \rho)$  is induced by the representation  $(W, \theta)$ .
- 3.6. Suppose that  $G$  is the direct product of two subgroups  $H$  and  $K$  (cf. 3.2). Let  $\rho$  be a representation of  $G$  induced by a representation  $\theta$  of  $H$ . Show that  $\rho$  is isomorphic to  $\theta \otimes r_K$ , where  $r_K$  denotes the regular representation of  $K$ .