CHAPTER 2

Character theory

2.1 The character of a representation

Let V be a vector space having a basis (e_i) of n elements, and let a be a linear map of V into itself, with matrix (a_{ij}). By the *trace* of a we mean the scalar

$$\mathrm{Tr}(a) = \sum_i a_{ii}.$$

It is the *sum of the eigenvalues of a* (counted with their multiplicities), and does not depend on the choice of basis (e_i).

Now let ρ: $G \to \mathbf{GL}(V)$ be a linear representation of a finite group G in the vector space V. For each $s \in G$, put:

$$\chi_\rho(s) = \mathrm{Tr}(\rho_s).$$

The complex valued function χ_ρ on G thus obtained is called the *character* of the representation ρ; the importance of this function comes primarily from the fact that it *characterizes* the representation ρ (cf. 2.3).

Proposition 1. *If χ is the character of a representation ρ of degree n, we have*:

(i) $\chi(1) = n$,
(ii) $\chi(s^{-1}) = \chi(s)^*$ *for* $s \in G$,
(iii) $\chi(tst^{-1}) = \chi(s)$ *for* $s, t \in G$.

(If $z = x + iy$ is a complex number, we denote the conjugate $x - iy$ either by z^* or $\bar{z}$.)

We have $\rho(1) = 1$, and $\mathrm{Tr}(1) = n$ since V has dimension n; hence (i).

For (ii) we observe that ρ_s has finite order; consequently the same is true of its eigenvalues $\lambda_1, \ldots, \lambda_n$ and so these have absolute value equal to 1 (this is also a consequence of the fact that ρ_s can be defined by a unitary matrix, cf. 1.3). Thus

$$\chi(s)^* = \mathrm{Tr}(\rho_s)^* = \sum \lambda_i^* = \sum \lambda_i^{-1} = \mathrm{Tr}(\rho_s^{-1}) = \mathrm{Tr}(\rho_{s^{-1}}) = \chi(s^{-1}).$$

Formula (iii) can also be written $\chi(vu) = \chi(uv)$, putting $u = ts$, $v = t^{-1}$; hence it follows from the well known formula

$$\mathrm{Tr}(ab) = \mathrm{Tr}(ba),$$

valid for two arbitrary linear mappings a and b of V into itself. □

Remark. A function f on G satisfying identity (iii), or what amounts to the same thing, $f(uv) = f(vu)$, is called a *class function* We will see in 2.5 that each class function is a linear combination of characters.

Proposition 2. *Let* $\rho^1\colon \mathrm{G} \to \mathbf{GL}(\mathrm{V}_1)$ *and* $\rho^2\colon \mathrm{G} \to \mathbf{GL}(\mathrm{V}_2)$ *be two linear representations of* G, *and let* χ_1 *and* χ_2 *be their characters. Then*:

(i) *The character* χ *of the direct sum representation* $\mathrm{V}_1 \oplus \mathrm{V}_2$ *is equal to* $\chi_1 + \chi_2$.
(ii) *The character* ψ *of the tensor product representation* $\mathrm{V}_1 \otimes \mathrm{V}_2$ *is equal to* $\chi_1 \cdot \chi_2$.

Let us be given ρ^1 and ρ^2 in matrix form: R_s^1, R_s^2. The representation $\mathrm{V}_1 \oplus \mathrm{V}_2$ is then given by

$$\mathrm{R}_s = \begin{pmatrix} \mathrm{R}_s^1 & 0 \\ 0 & \mathrm{R}_s^2 \end{pmatrix}$$

whence $\mathrm{Tr}(\mathrm{R}_s) = \mathrm{Tr}(\mathrm{R}_s^1) + \mathrm{Tr}(\mathrm{R}_s^2)$, that is $\chi(s) = \chi_1(s) + \chi_2(s)$.

We proceed likewise for (ii): with the notation of 1.5, we have

$$\chi_1(s) = \sum_{i_1} r_{i_1 i_1}(s), \qquad \chi_2(s) = \sum_{i_2} r_{i_2 i_2}(s),$$

$$\psi(s) = \sum_{i_1, i_2} r_{i_1 i_1}(s) r_{i_2 i_2}(s) = \chi_1(s) \cdot \chi_2(s). \qquad \square$$

Proposition 3. *Let* $\rho\colon \mathrm{G} \to \mathbf{GL}(\mathrm{V})$ *be a linear representation of* G, *and let* χ *be its character. Let* χ_σ^2 *be the character of the symmetric square* $\mathbf{Sym}^2(\mathrm{V})$ *of* V (cf. 1.6), *and let* χ_α^2 *be that of* $\mathbf{Alt}^2(\mathrm{V})$. *For each* $s \in \mathrm{G}$, *we have*

$$\chi_\sigma^2(s) = \frac{1}{2}(\chi(s)^2 + \chi(s^2))$$

$$\chi_\alpha^2(s) = \frac{1}{2}(\chi(s)^2 - \chi(s^2))$$

and $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$.

Let $s \in G$. A basis (e_i) of V can be chosen consisting of *eigenvectors* for ρ_s; this follows for example from the fact that ρ_s can be represented by a *unitary* matrix, cf. 1.3. We have then $\rho_s e_i = \lambda_i e_i$ with $\lambda_i \in \mathbf{C}$, and so

$$\chi(s) = \sum \lambda_i, \qquad \chi(s^2) = \sum \lambda_i^2.$$

On the other hand, we have

$$(\rho_s \otimes \rho_s)(e_i \cdot e_j + e_j \cdot e_i) = \lambda_i \lambda_j \cdot (e_i \cdot e_j + e_j \cdot e_i),$$
$$(\rho_s \otimes \rho_s)(e_i \cdot e_j - e_j \cdot e_i) = \lambda_i \lambda_j \cdot (e_i \cdot e_j - e_j \cdot e_i),$$

hence

$$\chi_\sigma^2(s) = \sum_{i \leqslant j} \lambda_i \lambda_j = \sum \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}(\sum \lambda_i)^2 + \frac{1}{2} \sum \lambda_i^2$$

$$\chi_\alpha^2(s) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}(\sum \lambda_i)^2 - \frac{1}{2} \sum \lambda_i^2.$$

The proposition follows.

(Observe the equality $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$, which reflects the fact that $V \otimes V$ is the *direct sum* of $\mathbf{Sym}^2(V)$ and $\mathbf{Alt}^2(V)$). □

Exercises

2.1. Let χ and χ' be the characters of two representations. Prove the formulas:

$$(\chi + \chi')_\sigma^2 = \chi_\sigma^2 + \chi_\sigma'^2 + \chi\chi',$$
$$(\chi + \chi')_\alpha^2 = \chi_\alpha^2 + \chi_\alpha'^2 + \chi\chi'.$$

2.2. Let X be a finite set on which G acts, let ρ be the corresponding permutation representation [cf. 1.2, example (c)], and χ_X be the character of ρ. Let $s \in G$; show that $\chi_X(s)$ is the number of elements of X fixed by s.

2.3. Let $\rho\colon G \to \mathbf{GL}(V)$ be a linear representation with character χ and let V′ be the dual of V, i.e., the space of linear forms on V. For $x \in V$, $x' \in V'$ let $\langle x, x' \rangle$ denote the value of the linear form x' at x. Show that there exists a unique linear representation $\rho'\colon G \to \mathbf{GL}(V')$, such that

$$\langle \rho_s x, \rho'_s x' \rangle = \langle x, x' \rangle \quad \text{for } s \in G,\ x \in V,\ x' \in V'.$$

This is called the *contragredient* (or *dual*) representation of ρ; its character is χ^*.

2.4. Let $\rho_1\colon G \to \mathbf{GL}(V_1)$ and $\rho_2\colon G \to \mathbf{GL}(V_2)$ be two linear representations with characters χ_1 and χ_2. Let $W = \mathrm{Hom}(V_1, V_2)$, the vector space of linear mappings $f\colon V_1 \to V_2$. For $s \in G$ and $f \in W$ let $\rho_s f = \rho_{2,s} \circ f \circ \rho_{1,s}^{-1}$; so $\rho_s f \in W$. Show that this defines a linear representation $\rho\colon G \to \mathbf{GL}(W)$, and that its character is $\chi_1^* \cdot \chi_2$. This representation is isomorphic to $\rho'_1 \otimes \rho_2$, where ρ_1' is the contragredient of ρ_1, cf. ex. 2.3.

2.2 Schur's lemma; basic applications

Proposition 4 (*Schur's lemma*). *Let $\rho^1: G \to \mathbf{GL}(V_1)$ and $\rho^2: G \to \mathbf{GL}(V_2)$ be two irreducible representations of* G, *and let f be a linear mapping of* V_1 *into* V_2 *such that $\rho_s^2 \circ f = f \circ \rho_s^1$ for all $s \in G$. Then*:

(1) *If ρ^1 and ρ^2 are not isomorphic, we have $f = 0$.*

(2) *If $V_1 = V_2$ and $\rho^1 = \rho^2$, f is a homothety (i.e., a scalar multiple of the identity).*

The case $f = 0$ is trivial. Suppose now $f \neq 0$ and let W_1 be *its kernel* (that is, the set of $x \in V_1$ such that $fx = 0$). For $x \in W_1$ we have $f\rho_s^1 x = \rho_s^2 fx = 0$, whence $\rho_s^1 \chi \in W_1$, and W_1 is stable under G. Since V_1 is irreducible, W_1 is equal to V_1 or 0; the first case is excluded, as it implies $f = 0$. The same argument shows that the *image* W_2 of f (the set of fx, for $x \in V_1$) is equal to V_2. The two properties $W_1 = 0$ and $W_2 = V_2$ show that f is an isomorphism of V_1 onto V_2, which proves assertion (1).

Suppose now that $V_1 = V_2$, $\rho^1 = \rho^2$, and let λ be an *eigenvalue* of f: there exists at least one, since the field of scalars is the field of complex numbers. Put $f' = f - \lambda$. Since λ is an eigenvalue of f, the kernel of f' is $\neq 0$; on the other hand, we have $\rho_s^2 \circ f' = f' \circ \rho_s^1$. The first part of the proof shows that these properties are possible only if $f' = 0$, that is, if f is equal to λ. □

Let us keep the hypothesis that V_1 and V_2 are irreducible, and denote by g the *order* of the group G.

Corollary 1. *Let h be a linear mapping of* V_1 *into* V_2, *and put*:

$$h^0 = \frac{1}{g} \sum_{t \in G} (\rho_t^2)^{-1} h \rho_t^1.$$

Then:

(1) *If ρ^1 and ρ^2 are not isomorphic, we have $h^0 = 0$.*

(2) *If $V_1 = V_2$ and $\rho^1 = \rho^2$, h^0 is a homothety of ratio $(1/n)\mathrm{Tr}(h)$, with $n = \dim(V_1)$.*

We have $\rho_s^2 h^0 = h^0 \rho_s^1$. Indeed:

$$(\rho_s^2)^{-1} h^0 \rho_s^1 = \frac{1}{g} \sum_{t \in G} (\rho_s^2)^{-1} (\rho_t^2)^{-1} h \rho_t^1 \rho_s^1$$

$$= \frac{1}{g} \sum_{t \in G} (\rho_{ts}^2)^{-1} h \rho_{ts}^1 = h^0.$$

Applying prop. 4 to $f = h^0$, we see in case (1) that $h^0 = 0$, and in case (2) that h^0 is equal to a scalar λ. Moreover, in the latter case, we have:

$$\operatorname{Tr}(h^0) = \frac{1}{g}\sum_{t\in G}\operatorname{Tr}((\rho_t^1)^{-1}h\rho_t^1) = \operatorname{Tr}(h),$$

and since $\operatorname{Tr}(\lambda) = n\cdot\lambda$, we get $\lambda = (1/n)\operatorname{Tr}(h)$. □

Now we rewrite corollary 1 assuming that ρ^1 and ρ^2 are given *in matrix form*:

$$\rho_t^1 = (r_{i_1j_1}(t)),\ \rho_t^2 = (r_{i_2j_2}(t)).$$

The linear mapping h is defined by a matrix $(x_{i_2i_1})$ and likewise h^0 is defined by $(x^0_{i_2i_1})$. We have by definition of h^0:

$$x^0_{i_2i_1} = \frac{1}{g}\sum_{t,j_1,j_2} r_{i_2j_2}(t^{-1})x_{j_2j_1}r_{j_1i_1}(t).$$

The right hand side is a linear form with respect to $x_{j_2j_1}$; in case (1) this form vanishes for all systems of values of the $x_{j_2j_1}$; thus its coefficients are zero. Whence:

Corollary 2. *In case* (1), *we have*:

$$\frac{1}{g}\sum_{t\in G} r_{i_2j_2}(t^{-1})r_{j_1i_1}(t) = 0$$

for arbitrary i_1, i_2, j_1, j_2.

In case (2) we have similarly $h^0 = \lambda$, i.e., $x^0_{i_2i_1} = \lambda\delta_{i_2i_1}$ ($\delta_{i_2i_1}$ denotes the Kronecker symbol, equal to 1 if $i_1 = i_2$ and 0 otherwise), with $\lambda = (1/n)\operatorname{Tr}(h)$, that is, $\lambda = (1/n)\sum\delta_{j_2j_1}x_{j_2j_1}$. Hence the equality:

$$\frac{1}{g}\sum_{t,j_1,j_2} r_{i_2j_2}(t^{-1})x_{j_2j_1}r_{j_1i_1}(t) = \frac{1}{n}\sum_{j_1,j_2}\delta_{i_2i_1}\delta_{j_2j_1}x_{j_2j_1}.$$

Equating coefficients of the $x_{j_2j_1}$, we obtain as above:

Corollary 3. *In case* (2) *we have*:

$$\frac{1}{g}\sum_{t\in G} r_{i_2j_2}(t^{-1})r_{j_1i_1}(t) = \frac{1}{n}\delta_{i_2i_1}\delta_{j_2j_1} = \begin{cases} 1/n & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ 0 & \text{otherwise}. \end{cases}$$

Remarks

(1) If ϕ and ψ are functions on G, set

$$\langle\phi,\psi\rangle = \frac{1}{g}\sum_{t\in G}\phi(t^{-1})\psi(t) = \frac{1}{g}\sum_{t\in G}\phi(t)\psi(t^{-1}).$$

We have $\langle\phi,\psi\rangle = \langle\psi,\phi\rangle$. Moreover $\langle\phi,\psi\rangle$ is linear in ϕ and in ψ. With this notation, corollaries 2 and 3 become, respectively

$$\langle r_{i_2j_2}, r_{j_1i_1}\rangle = 0 \quad\text{and}\quad \langle r_{i_2j_2}, r_{j_1i_1}\rangle = \frac{1}{n}\delta_{i_2i_1}\delta_{j_2j_1}.$$

(2) Suppose that the matrices $(r_{ij}(t))$ are *unitary* (this can be realized by a suitable choice of basis, cf. 1.3). We have then $r_{ij}(t^{-1}) = r_{ji}(t)^*$ and corollaries 2 and 3 are just *orthogonality relations* for the scalar product $(\phi|\psi)$ defined in the following section.

2.3 Orthogonality relations for characters

We begin with a notation. If ϕ and ψ are two complex-valued functions on G, put

$$(\phi|\psi) = \frac{1}{g} \sum_{t \in G} \phi(t)\psi(t)^*, \quad g \text{ being the order of G}.$$

This is a *scalar product*: it is linear in ϕ, semilinear in ψ, and we have $(\phi|\phi) > 0$ for all $\phi \neq 0$.

If $\check{\psi}$ is the function defined by the formula $\check{\psi}(t) = \psi(t^{-1})^*$, we have

$$(\phi|\psi) = \frac{1}{g} \sum_{t \in G} \phi(t)\check{\psi}(t^{-1}) = \langle \phi, \check{\psi} \rangle,$$

cf. 2.2, remark 1. In particular, if χ is the *character* of a representation of G, we have $\check{\chi} = \chi$ (prop. 1), so that $(\phi|\chi) = \langle \phi, \chi \rangle$ for all functions ϕ on G. So we can use at will $(\phi|\chi)$ or $\langle \phi, \chi \rangle$, so long as we are concerned with characters.

Theorem 3

(i) *If χ is the character of an irreducible representation, we have* $(\chi|\chi) = 1$ (i.e., χ is "of norm 1").

ii) *If χ and χ' are the characters of two nonisomorphic irreducible representations, we have* $(\chi|\chi') = 0$ (i.e. χ and χ' are orthogonal).

Let ρ be an irreducible representation with character χ, given in matrix form $\rho_t = (r_{ij}(t))$. We have $\chi(t) = \sum r_{ii}(t)$, hence

$$(\chi|\chi) = \langle \chi, \chi \rangle = \sum_{i,j} \langle r_{ii}, r_{jj} \rangle.$$

But according to cor. 3 to prop. 4, we have $\langle r_{ii}, r_{jj} \rangle = \delta_{ij}/n$, where n is the degree of ρ. Thus

$$(\chi|\chi) = \Big(\sum_{i,j} \delta_{ij}\Big)/n = n/n = 1,$$

since the indices i,j each take n values. (ii) is proved in the same way, by applying cor. 2 instead of cor. 3. □

Remark. A character of an irreducible representation is called an *irreducible character*. Theorem 3 shows that the irreducible characters form an orthonormal system; this result will be completed later (2.5, th. 6).

Theorem 4. *Let* V *be a linear representation of* G, *with character* ϕ, *and suppose* V *decomposes into a direct sum of irreducible representations*:

$$V = W_1 \oplus \cdots \oplus W_k .$$

Then, if W *is an irreducible representation with character* χ, *the number of* W_i *isomorphic to* W *is equal to the scalar product* $(\phi|\chi) = \langle \phi, \chi \rangle$.

Let χ_i be the character of W_i. By prop. 2, we have

$$\phi = \chi_1 + \cdots + \chi_k .$$

Thus $(\phi|\chi) = (\chi_1|\chi) + \cdots + (\chi_k|\chi)$. But, according to the preceeding theorem, $(\chi_i|\chi)$ is equal to 1 or 0, depending on whether W_i is, or is not, isomorphic to W. The result follows. □

Corollary 1. *The number of* W_i *isomorphic to* W *does not depend on the chosen decomposition.*

(This number is called the "number of times that W occurs in V", or the "number of times that W is contained in V.")

Indeed, $(\phi|\chi)$ does not depend on the decomposition. □

Remark. It is in this sense that one can say that there is uniqueness in the decomposition of a representation into irreducible representations. We shall return to this in 2.6.

Corollary 2. *Two representations with the same character are isomorphic.*

Indeed, cor. 1 shows that they contain each given irreducible representation the same number of times.

The above results reduce the study of representations to that of their characters. If $\chi_1, \ldots, \chi_h$ are the distinct irreducible characters of G, and if $W_1, \ldots, W_h$ denote corresponding representations, each representation V is isomorphic to a direct sum

$$V = m_1 W_1 \oplus \cdots \oplus m_h W_h \quad m_i \text{ integers } \geqslant 0.$$

The character ϕ of V is equal to $m_1\chi_1 + \cdots + m_h\chi_h$, and we have $m_i = (\phi|\chi_i)$. [This applies notably to the tensor product $W_i \otimes W_j$ of two irreducible representations, and shows that the product $\chi_i \cdot \chi_j$ decomposes into $\chi_i\chi_j = \sum m_{ij}^k \chi_k$, the m_{ij}^k being integers $\geqslant 0$.] The orthogonality relations among the χ_i imply in addition:

$$(\phi|\phi) = \sum_{i=1}^{i=h} m_i^2 ,$$

whence:

Theorem 5. *If ϕ is the character of a representation* V, $(\phi|\phi)$ *is a positive integer and we have* $(\phi|\phi) = 1$ *if and only if* V *is irreducible.*

Indeed, $\sum m_i^2$ is only equal to 1 if one of the m_i's is equal to 1 and the others to 0, that is, if V is isomorphic to one of the W_i. □

We obtain thus a very convenient *irreducibility criterion.*

EXERCISES

2.5. Let ρ be a linear representation with character χ. Show that the number of times that ρ contains the unit representation is equal to $(\chi|1) = (1/g) \sum_{s \in G} \chi(s)$.

2.6. Let X be a finite set on which G acts, let ρ be the corresponding permutation representation (1.2) and let χ be its character.

(a) The set Gx of images under G of an element $x \in X$ is called an *orbit.* Let c be the number of distinct orbits. Show that c is equal to the number of times that ρ contains the unit representation 1; deduce from this that $(\chi|1) = c$. In particular, if G is transitive (i.e., if $c = 1$), ρ can be decomposed into $1 \oplus \theta$ and θ does not contain the unit representation. If ψ is the character of θ, we have $\chi = 1 + \psi$ and $(\psi|1) = 0$.

(b) Let G act on the product X × X of X by itself by means of the formula $s(x, y) = (sx, sy)$. Show that the character of the corresponding permutation representation is equal to χ^2.

(c) Suppose that G is *transitive* on X and that X has at least two elements. We say that G is doubly transitive if, for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there exists $s \in G$ such that $x' = sx$ and $y' = sy$. Prove the equivalence of the following properties:

(i) G is doubly transitive.

(ii) The action of G on X × X has two orbits, the diagonal and its complement.

(iii) $(\chi^2|1) = 2$.

(iv) The representation θ defined in (a) is irreducible.

[The equivalence (i) ⇔ (ii) is immediate; (ii) ⇔ (iii) follows from (a) and (b). If ψ is the character of θ, we have $1 + \psi = \chi$ and $(1|1) = 1$, $(\psi|1) = 0$, which shows that (iii) is equivalent to $(\psi^2|1) = 1$, i.e., to $(1/g) \sum_{s \in G} \psi(s)^2 = 1$; since ψ is real-valued, this indeed means that θ is irreducible, cf. th. 5.]

2.4 Decomposition of the regular representation

Notation. For the rest of Ch. 2, the irreducible characters of G are denoted $\chi_1, \ldots, \chi_h$; their degrees are written $n_1, \ldots, n_h$, we have $n_i = \chi_i(1)$, cf. prop. 1.

Let R be the regular representation of G. Recall (cf. 1.2) that it has a basis $(e_t)_{t \in G}$ such that $\rho_s e_t = e_{st}$. If $s \neq 1$, we have $st \neq t$ for all t, which shows that the diagonal terms of the matrix of ρ_s are zero; in particular we have $\mathrm{Tr}(\rho_s) = 0$. On the other hand, for $s = 1$, we have

$$\mathrm{Tr}(\rho_s) = \mathrm{Tr}(1) = \dim(\mathrm{R}) = g.$$

Whence:

Proposition 5. *The character r_{G} of the regular representation is given by the formulas:*

$$r_{\mathrm{G}}(1) = g, \qquad \textit{order of } \mathrm{G},$$
$$r_{\mathrm{G}}(s) = 0 \qquad \textit{if } s \neq 1.$$

Corollary 1. *Every irreducible representation W_i is contained in the regular representation with multiplicity equal to its degree n_i.*

According to th. 4, this number is equal to $\langle r_{\mathrm{G}}, \chi_i \rangle$, and we have

$$\langle r_{\mathrm{G}}, \chi_i \rangle = \frac{1}{g} \sum_{s \in \mathrm{G}} r_{\mathrm{G}}(s^{-1})\chi_i(s) = \frac{1}{g} g \cdot \chi_i(1) = \chi_i(1) = n_i. \qquad \square$$

Corollary 2.

(a) *The degrees n_i satisfy the relation $\sum_{i=1}^{i=h} n_i^2 = g$.*
(b) *If $s \in \mathrm{G}$ is different from* 1, *we have $\sum_{i=1}^{i=h} n_i \chi_i(s) = 0$.*

By cor. 1, we have $r_{\mathrm{G}}(s) = \sum n_i \chi_i(s)$ for all $s \in \mathrm{G}$. Taking $s = 1$ we obtain (a), and taking $s \neq 1$, we obtain (b). $\square$

Remarks

(1) The above result can be used in determining the irreducible representations of a group G: suppose we have constructed some mutually nonisomorphic irreducible representations of degrees $n_1, \ldots, n_k$; in order that they be *all* the irreducible representations of G (up to isomorphism), it is necessary and sufficient that $n_1^2 + \cdots + n_k^2 = g$.

(2) We will see later (Part II, 6.5) another property of the degrees n_i: they divide the order g of G.

Exercise

2.7. Show that each character of G which is zero for all $s \neq 1$ is an *integral* multiple of the character r_{G} of the regular representation.

2.5 Number of irreducible representations

Recall (cf. 2.1) that a function f on G is called a *class function* if $f(tst^{-1}) = f(s)$ for all $s, t \in \mathrm{G}$.

Proposition 6. *Let f be a class function on G, and let $\rho\colon G \to \mathbf{GL}(V)$ be a linear representation of G. Let ρ_f be the linear mapping of V into itself defined by:*

$$\rho_f = \sum_{t \in G} f(t)\rho_t.$$

If V is irreducible of degree n and character χ, then ρ_f is a homothety of ratio λ given by:

$$\lambda = \frac{1}{n}\sum_{t \in G} f(t)\chi(t) = \frac{g}{n}(f|\chi^*).$$

Let us compute $\rho_s^{-1}\rho_f\rho_s$. We have:

$$\rho_s^{-1}\rho_f\rho_s = \sum_{t \in G} f(t)\rho_s^{-1}\rho_t\rho_s = \sum_{t \in G} f(t)\rho_{s^{-1}ts}.$$

Putting $u = s^{-1}ts$, this becomes:

$$\rho_s^{-1}\rho_f\rho_s = \sum_{u \in G} f(sus^{-1})\rho_u = \sum_{u \in G} f(u)\rho_u = \rho_f.$$

So we have $\rho_f\rho_s = \rho_s\rho_f$. By the second part of prop. 4, this shows that ρ_f is a homothety λ. The trace of λ is $n\lambda$; that of ρ_f is $\sum_{t \in G} f(t)\mathrm{Tr}(\rho_t) = \sum_{t \in G} f(t)\chi(t)$. Hence $\lambda = (1/n)\sum_{t \in G} f(t)\chi(t) = (g/n)(f|\chi^*)$. □

We introduce now the space H of class functions on G; the irreducible characters $\chi_1, \dots, \chi_h$ belong to H.

Theorem 6. *The characters $\chi_1, \dots, \chi_h$ form an orthonormal basis of* H.

Theorem 3 shows that the χ_i form an orthonormal system in H. It remains to prove that they *generate* H, and for this it is enough to show that every element of H orthogonal to the χ_i^* is zero. Let f be such an element. For each representation ρ of G, put $\rho_f = \sum_{t \in G} f(t)\rho_t$. Since f is orthogonal to the χ_i^*, prop. 6 above shows that ρ_f is zero so long as ρ is irreducible; from the direct sum decomposition we conclude that ρ_f is always zero. Applying this to the regular representation R (cf. 2.4) and computing the image of the basis vector e_1 under ρ_f, we have

$$\rho_f e_1 = \sum_{t \in G} f(t)\rho_t e_1 = \sum_{t \in G} f(t)e_t.$$

Since ρ_f is zero, we have $\rho_f e_1 = 0$ and the above formula shows that $f(t) = 0$ for all $t \in G$; hence $f = 0$, and the proof is complete. □

Recall that two elements t and t' of G are said to be conjugate if there exists $s \in G$ such that $t' = sts^{-1}$; this is an equivalence relation, which partitions G into *classes* (also called *conjugacy classes*).

Theorem 7. *The number of irreducible representations of* G *(up to isomorphism) is equal to the number of classes of* G.

Let $C_1, \ldots, C_k$ be the distinct classes of G. To say that a function f on G is a class function is equivalent to saying that it is constant on each of $C_1, \ldots, C_k$; it is thus determined by its values λ_i on the C_i, and these can be chosen arbitrarily. Consequently, the dimension of the space H of class functions is equal to k. On the other hand, this dimension is, by th. 6, equal to the number of irreducible representations of G (up to isomorphism). The result follows. □

Here is another consequence of th. 6:

Proposition 7. *Let $s \in G$, and let $c(s)$ be the number of elements in the conjugacy class of s.*

(a) *We have* $\sum_{i=1}^{i=h} \chi_i(s)^* \chi_i(s) = g/c(s)$.

(b) *For $t \in G$ not conjugate to s, we have* $\sum_{i=1}^{i=h} \chi_i(s)^* \chi_i(t) = 0$.

(For $s = 1$, this yields cor. 2 to prop. 5.)

Let f_s be the function equal to 1 on the class of s and equal to 0 elsewhere. Since it is a class function, it can, by th. 6, be written

$$f_s = \sum_{i=1}^{i=h} \lambda_i \chi_i, \qquad \text{with } \lambda_i = (f_s | \chi_i) = \frac{c(s)}{g} \chi_i(s)^*.$$

We have then, for each $t \in G$,

$$f_s(t) = \frac{c(s)}{g} \sum_{i=1}^{i=h} \chi_i(s)^* \chi_i(t).$$

This gives (a) if $t = s$, and (b) if t is not conjugate to s. □

Example. Take for G the *group of permutations of three letters*. We have $g = 6$, and there are three classes: the element 1, the three transpositions, and the two cyclic permutations. Let t be a transposition and c a cyclic permutation. We have $t^2 = 1$, $c^3 = 1$, $tc = c^2 t$; whence there are just two characters of degree 1: the unit character χ_1 and the character χ_2 giving the signature of a permutation. Theorem 7 shows that there exists one other irreducible character θ; if n is its degree we must have $1 + 1 + n^2 = 6$, hence $n = 2$. The values of θ can be deduced from the fact that $\chi_1 + \chi_2 + 2\theta$ is the character of the regular representation of G (cf. prop. 5). We thus get the *character table* of G:

	1	t	c
χ_1	1	1	1
χ_2	1	−1	1
θ	2	0	−1

We obtain an irreducible representation with character θ by having G permute the coordinates of elements of $\mathbf{C}^3$ satisfying the equation $x + y + z = 0$ (cf. ex. 2.6c)).

2.6 Canonical decomposition of a representation

Let ρ: G $\rightarrow$ **GL**(V) be a linear representation of G. We are going to define a direct sum decomposition of V which is "coarser" than the decomposition into irreducible representations, but which has the advantage of being *unique*. It is obtained as follows:

Let $\chi_1, \ldots, \chi_h$ be the distinct characters of the irreducible representations $W_1, \ldots, W_h$ of G and $n_1, \ldots, n_h$ their degrees. Let $V = U_1 \oplus \cdots \oplus U_m$ be a decomposition of V into a direct sum of irreducible representations. For $i = 1, \ldots, h$ denote by V_i the direct sum of those of the $U_1, \ldots, U_m$ which are isomorphic to W_i. Clearly we have:

$$V = V_1 \oplus \cdots \oplus V_h.$$

(In other words, we have decomposed V into a direct sum of irreducible representations and *collected together* the isomorphic representations.)

This is the *canonical decomposition* we had in mind. Its properties are as follows:

Theorem 8

(i) *The decomposition* $V = V_1 \oplus \cdots \oplus V_h$ *does not depend on the initially chosen decomposition of* V *into irreducible representations.*

(ii) *The projection* p_i *of* V *onto* V_i *associated with this decomposition is given by the formula*:

$$p_i = \frac{n_i}{g} \sum_{t \in G} \chi_i(t)^* \rho_t.$$

We prove (ii). Assertion (i) will follow because the projections p_i determine the V_i. Put

$$q_i = \frac{n_i}{g} \sum_{t \in G} \chi_i(t)^* \rho_t.$$

Proposition 6 shows that the restriction of q_i to an irreducible representation W with character χ and of degree n is a homothety of ratio $(n_i/n)(\chi_i|\chi)$; it is thus 0 if $\chi \neq \chi_i$ and 1 if $\chi = \chi_i$. In other words q_i is the identity on an irreducible representation isomorphic to W_i, and is zero on the others. In view of the definition of the V_i, it follows that q_i is the identity on V_i and is 0 on V_j for $j \neq i$. If we decompose an element $x \in V$ into its components $x_i \in V_i$:

$$x = x_1 + \cdots + x_h,$$

we have then $q_i(x) = q_i(x_1) + \cdots + q_i(x_h) = x_i$. This means that q_i is equal to the projection p_i of V onto V_i. □

Thus the decomposition of a representation V can be done in two stages. First the canonical decomposition $V_1 \oplus \cdots \oplus V_n$ is determined; this can be done easily using the formulas giving the projections p_i. Next, if needed, one chooses a decomposition of V_i into a direct sum of irreducible representations each isomorphic to W_i:

$$V_i = W_i \oplus \cdots \oplus W_i.$$

This last decomposition can in general be done in an infinity of ways (cf. section 2.7, as well as ex. 2.8 below); it is just as arbitrary as the choice of a basis in a vector space.

EXAMPLE. Take for G the group of two elements $\{1, s\}$ with $s^2 = 1$. This group has two irreducible representations of degree 1, W^+ and W^-, corresponding to $\rho_s = +1$ and $\rho_s = -1$. The canonical decomposition of a representation V is $V = V^+ \oplus V^-$, where V^+ (resp. V^-) consists of the elements $x \in V$ which are symmetric (resp. antisymmetric), i.e., which satisfy $\rho_s x = x$ (resp. $\rho_s x = -x$). The corresponding projections are:

$$p^+ x = \frac{1}{2}(x + \rho_s x), \qquad p^- x = \frac{1}{2}(x - \rho_s x).$$

To decompose V^+ and V^- into irreducible components means simply to decompose these spaces into a direct sum of *lines*.

EXERCISE

2.8. Let H_i be the vector space of linear mappings $h: W_i \to V$ such that $\rho_s h = h\rho_s$ for all $s \in G$. Each $h \in H_i$ maps W_i into V_i.

(a) Show that the dimension of H_i is equal to the number of times that W_i appears in V, i.e., to $\dim V_i / \dim W_i$ [Reduce to the case where $V = W_i$ and use Schur's lemma].

(b) Let G act on $H_i \otimes W_i$ through the tensor product of the trivial representation of G on H_i and the given representation on W_i. Show that the map

$$F: H_i \otimes W_i \to V_i$$

defined by the formula

$$F(\sum h_\alpha \cdot w_\alpha) = \sum h_\alpha(w_\alpha)$$

is an isomorphism of $H_i \otimes W_i$ onto V_i. [Same method.]

(c) Let $(h_1, \ldots, h_k)$ be a basis of H_i and form the direct sum $W_i \oplus \cdots \oplus W_i$ of k copies of W_i. The system $(h_1, \ldots, h_k)$ defines in an obvious way a linear mapping h of $W_i \oplus \cdots \oplus W_i$ into V_i; show that it is an isomorphism of representations and that each isomorphism is thus obtainable [apply (*b*), or argue directly]. In particular, *to decompose* V_i *into a direct sum of representations isomorphic to* W_i *amounts to choosing a basis for* H_i.

2.7 Explicit decomposition of a representation

Keep the notation of the preceding section, and let

$$V = V_1 \oplus \cdots \oplus V_h$$

be the *canonical decomposition* of the given representation. We have seen how one can determine the ith component V_i by means of the corresponding projection (th. 8). We now give a method for explicitly constructing a *decomposition* of V_i into a direct sum of subrepresentations isomorphic to W_i. Let W_i be given in matrix form $(r_{\alpha\beta}(s))$ with respect to a basis $(e_1, \ldots, e_n)$; we have $\chi_i(s) = \sum_\alpha r_{\alpha\alpha}(s)$ and $n = n_i = \dim W_i$. For each pair of integers α, β taken from 1 to n, let $p_{\alpha\beta}$ denote the linear map of V into V defined by

$$(*) \qquad p_{\alpha\beta} = \frac{n}{g} \sum_{t \in G} r_{\beta\alpha}(t^{-1})\rho_t.$$

Proposition 8

(a) *The map $p_{\alpha\alpha}$ is a projection; it is zero on the V_j, $j \neq i$. Its image $V_{i,\alpha}$ is contained in V_i, and V_i is the direct sum of the $V_{i,\alpha}$ for $1 \leqslant \alpha \leqslant n$. We have $p_i = \sum_\alpha p_{\alpha\alpha}$.*

(b) *The linear map $p_{\alpha\beta}$ is zero on the V_j, $j \neq i$, as well as on the $V_{i,\gamma}$ for $\gamma \neq \beta$; it defines an isomorphism from $V_{i,\beta}$ onto $V_{i,\alpha}$.*

(c) *Let x_1 be an element $\neq 0$ of $V_{i,1}$ and let $x_\alpha = p_{\alpha 1}(x_1) \in V_{i,\alpha}$. The x_α are linearly independent and generate a vector subspace $W(x_1)$ stable under G and of dimension n. For each $s \in G$, we have*

$$\rho_s(x_\alpha) = \sum_\beta r_{\beta\alpha}(s) x_\beta$$

(in particular, $W(x_1)$ is isomorphic to W_i).

(d) *If $(x_1^{(1)}, \ldots, x_1^{(m)})$ is a basis of $V_{i,1}$, the representation V_i is the direct sum of the subrepresentations $W(x_1^{(1)}), \ldots, W(x_1^{(m)})$ defined in c).*

(Thus the choice of a basis of $V_{i,1}$ gives a decomposition of V_i into a direct sum of representations isomorphic to W_i.)

We observe first that the formula $(*)$ above allows us to define the $p_{\alpha\beta}$ in arbitrary representations of G, and in particular in the irreducible representations W_j. For W_i, we have

$$p_{\alpha\beta}(e_\gamma) = \frac{n}{g} \sum_{t \in G} r_{\beta\alpha}(t^{-1})\rho_t(e_\gamma) = \frac{n}{g} \sum_\delta \sum_{t \in G} r_{\beta\alpha}(t^{-1}) r_{\delta\gamma}(t) e_\delta.$$

By cor. 3 to prop. 4 we have then

$$p_{\alpha\beta}(e_\gamma) = \begin{cases} e_\alpha & \text{if } \gamma = \beta \\ 0 & \text{otherwise}. \end{cases}$$

We get from this the fact that $\sum_{\alpha} p_{\alpha\alpha}$ is the identity map of W_i, and the formulas

$$p_{\alpha\beta} \circ p_{\gamma\delta} = \begin{cases} p_{\alpha\delta} & \text{if } \beta = \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_s \circ p_{\alpha\gamma} = \sum_{\beta} r_{\beta\alpha}(s) p_{\beta\gamma}.$$

For W_j with $j \neq i$, we use cor. 2 to prop. 4 and the same argument to show that all the $p_{\alpha\beta}$ are *zero.*

Having done this, we decompose V into a direct sum of subrepresentations isomorphic to W_j and apply the preceding to each of these representations. Assertions (a) and (b) follow; moreover, the above formulas remain valid in V. Under the hypothesis of (c), we have then

$$\rho_s(x_\alpha) = \rho_s \circ p_{\alpha 1}(x_1) = \sum_{\beta} r_{\beta\alpha}(s) p_{\beta 1}(x_1) = \sum_{\beta} r_{\beta\alpha}(s) x_\beta,$$

which proves (c). Finally (d) follows from (a), (b), and (c). □

Exercises

2.9. Let H_i be the space of linear maps h: $W_i \to V$ such that $h \circ \rho_s = \rho_s \circ h$, cf. ex. 2.8. Show that the map $h \mapsto h(e_\alpha)$ is an isomorphism of H_i onto $V_{i,\alpha}$.

2.10. Let $x \in V_i$, and let $V(x)$ be the smallest subrepresentation of V containing x. Let x_1^α be the image of x under $p_{1\alpha}$; show that $V(x)$ is the sum of the representations $W(x_1^\alpha)$, $\alpha = 1, \ldots, n$. Deduce from this that $V(x)$ is the direct sum of at most n subrepresentations isomorphic to W_i.