CHAPTER 2
Character theory

2.1 The character of a representation

Let V be a vector space having a basis \((e_i)\) of \(n\) elements, and let \(a\) be a linear map of V into itself, with matrix \((a_{ij})\). By the trace of \(a\) we mean the scalar

\[
\text{Tr}(a) = \sum_i a_{ii}.
\]

It is the sum of the eigenvalues of \(a\) (counted with their multiplicities), and does not depend on the choice of basis \((e_i)\).

Now let \(\rho: G \to GL(V)\) be a linear representation of a finite group \(G\) in the vector space \(V\). For each \(s \in G\), put:

\[
\chi_\rho(s) = \text{Tr}(\rho(s)).
\]

The complex valued function \(\chi_\rho\) on \(G\) thus obtained is called the character of the representation \(\rho\); the importance of this function comes primarily from the fact that it characterizes the representation \(\rho\) (cf. 2.3).

**Proposition 1.** If \(\chi\) is the character of a representation \(\rho\) of degree \(n\), we have:

(i) \(\chi(1) = n\),
(ii) \(\chi(s^{-1}) = \chi(s)^*\) for \(s \in G\),
(iii) \(\chi(sts^{-1}) = \chi(s)\) for \(s, t \in G\).

(If \(z = x + iy\) is a complex number, we denote the conjugate \(x - iy\) either by \(z^*\) or \(\bar{z}\).)

We have \(\rho(1) = 1\), and \(\text{Tr}(1) = n\) since \(V\) has dimension \(n\); hence (i).

For (ii) we observe that \(\rho_s\) has finite order; consequently the same is true
of its eigenvalues $\lambda_1, \ldots, \lambda_n$ and so these have absolute value equal to 1 (this is also a consequence of the fact that $\rho_s$ can be defined by a unitary matrix, cf. 1.3). Thus

$$\chi(s)^* = \text{Tr}(\rho_s)^* = \sum\lambda_i^* = \sum\lambda_i^{-1} = \text{Tr}(\rho_s^{-1}) = \text{Tr}(\rho_{s^{-1}}) = \chi(s^{-1}).$$

Formula (iii) can also be written $\chi(uv) = \chi(uv)$, putting $u = ts, v = t^{-1}$; hence it follows from the well known formula

$$\text{Tr}(ab) = \text{Tr}(ba),$$

valid for two arbitrary linear mappings $a$ and $b$ of $V$ into itself. \hfill \Box

**Remark.** A function $f$ on $G$ satisfying identity (iii), or what amounts to the same thing, $f(uv) = f(vu)$, is called a class function. We will see in 2.5 that each class function is a linear combination of characters.

**Proposition 2.** Let $\rho^1: G \to \text{GL}(V_1)$ and $\rho^2: G \to \text{GL}(V_2)$ be two linear representations of $G$, and let $\chi_1$ and $\chi_2$ be their characters. Then:

(i) The character $\chi$ of the direct sum representation $V_1 \oplus V_2$ is equal to $\chi_1 + \chi_2$.

(ii) The character $\psi$ of the tensor product representation $V_1 \otimes V_2$ is equal to $\chi_1 \cdot \chi_2$.

Let us be given $\rho^1$ and $\rho^2$ in matrix form: $R_1^1, R_2^1$. The representation $V_1 \oplus V_2$ is then given by

$$R_s = \begin{pmatrix} R_1^1 & 0 \\ 0 & R_2^2 \end{pmatrix}$$

whence $\text{Tr}(R_s) = \text{Tr}(R_1^1) + \text{Tr}(R_2^2)$, that is $\chi(s) = \chi_1(s) + \chi_2(s)$.

We proceed likewise for (ii): with the notation of 1.5, we have

$$\chi_1(s) = \sum_{i_1} \eta_{i_1}(s), \quad \chi_2(s) = \sum_{i_2} \eta_{i_2}(s),$$

$$\psi(s) = \sum_{i_1, i_2} \eta_{i_1}(s)\eta_{i_2}(s) = \chi_1(s) \cdot \chi_2(s). \hfill \Box$$

**Proposition 3.** Let $\rho: G \to \text{GL}(V)$ be a linear representation of $G$, and let $\chi$ be its character. Let $\chi_s^2$ be the character of the symmetric square $\text{Sym}^2(V)$ of $V$ (cf. 1.6), and let $\chi_s^2$ be that of $\text{Alt}^2(V)$. For each $s \in G$, we have

$$\chi_s^2(s) = \frac{1}{2}(\chi(s)^2 + \chi(s^2))$$

$$\chi_s^2(s) = \frac{1}{2}(\chi(s)^2 - \chi(s^2))$$

and $\chi_s^2 + \chi_s^2 = \chi^2$. 

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Let \( s \in G \). A basis \((e_i)\) of \( V \) can be chosen consisting of eigenvectors for \( \rho_s \); this follows for example from the fact that \( \rho_s \) can be represented by a unitary matrix, cf. 1.3. We have then \( \rho_s e_i = \lambda_i e_i \) with \( \lambda_i \in \mathbb{C} \), and so

\[
\chi(s) = \sum \lambda_i, \quad \chi(x^2) = \sum \lambda_i^2.
\]

On the other hand, we have

\[
(\rho_s \otimes \rho_s)(e_i \cdot e_j + e_j \cdot e_i) = \lambda_i \lambda_j \cdot (e_i \cdot e_j + e_j \cdot e_i),
\]

\[
(\rho_s \otimes \rho_s)(e_i \cdot e_j - e_j \cdot e_i) = \lambda_i \lambda_j \cdot (e_i \cdot e_j - e_j \cdot e_i),
\]

hence

\[
\chi^2(s) = \sum_{i,j} \lambda_i \lambda_j = \sum \lambda_i^2 + \sum \lambda_i \lambda_j = \frac{1}{2} (\sum \lambda_i)^2 + \frac{1}{2} \sum \lambda_i^2
\]

\[
\chi^2(x^2) = \sum_{i,j} \lambda_i \lambda_j = \frac{1}{2} (\sum \lambda_i)^2 - \frac{1}{2} \sum \lambda_i^2.
\]

The proposition follows.

(Observable the equality \( \chi^2_0 + \chi^2_n = \chi^2 \), which reflects the fact that \( V \otimes V \) is the direct sum of \( \text{Sym}^2(V) \) and \( \text{Alt}^2(V) \)).

\[\square\]

Exercises

2.1. Let \( \chi \) and \( \chi' \) be the characters of two representations. Prove the formulas:

\[
(x + x')^2 = x^2 + x'^2 + xx'.
\]

\[
(x + x')^2 = x^2 + x'^2 +xx'.
\]

2.2. Let \( X \) be a finite set on which \( G \) acts, let \( \rho \) be the corresponding permutation representation [cf. 1.2, example (c)], and \( \chi_X \) be the character of \( \rho \). Let \( s \in G \); show that \( \chi_X(s) \) is the number of elements of \( X \) fixed by \( s \).

2.3. Let \( \rho : G \to \text{GL}(V) \) be a linear representation with character \( \chi \) and let \( V' \) be the dual of \( V \), i.e., the space of linear forms on \( V \). For \( x \in V \), \( x' \in V' \) let \( \langle x, x' \rangle \) denote the value of the linear form \( x' \) at \( x \). Show that there exists a unique linear representation \( \rho' : G \to \text{GL}(V') \), such that

\[
\langle \rho_s x, \rho'_s x' \rangle = \langle x, x' \rangle \quad \text{for} \ s \in G, \ x \in V, \ x' \in V'.
\]

This is called the contragredient (or dual) representation of \( \rho \); its character is \( \chi^* \).

2.4. Let \( \rho_1 : G \to \text{GL}(V_1) \) and \( \rho_2 : G \to \text{GL}(V_2) \) be two linear representations with characters \( \chi_1 \) and \( \chi_2 \). Let \( W = \text{Hom}(V_1, V_2) \), the vector space of linear mappings \( f : V_1 \to V_2 \). For \( s \in G \) and \( f \in W \) let \( \rho_s f = \rho_s \circ f \circ \rho_s^{-1} \); so \( \rho_s f \in W \). Show that this defines a linear representation \( \rho : G \to \text{GL}(W) \), and that its character is \( \chi^*_1 \cdot \chi_2 \). This representation is isomorphic to \( \rho_1 \otimes \rho_2 \).
where \( \rho_j \) is the contragredient of \( \rho_i \), cf. ex. 2.3.

### 2.2 Schur's lemma; basic applications

**Proposition 4 (Schur's lemma).** Let \( \rho^1 : G \to \text{GL}(V_1) \) and \( \rho^2 : G \to \text{GL}(V_2) \) be two irreducible representations of \( G \), and let \( f \) be a linear mapping of \( V_1 \) into \( V_2 \) such that \( \rho^2_s \circ f = f \circ \rho^1_s \) for all \( s \in G \). Then:

1. If \( \rho^1 \) and \( \rho^2 \) are not isomorphic, we have \( f = 0 \).
2. If \( V_1 = V_2 \) and \( \rho^1 = \rho^2 \), \( f \) is a homothety (i.e., a scalar multiple of the identity).

The case \( f = 0 \) is trivial. Suppose now \( f \neq 0 \) and let \( W_1 \) be its kernel (that is, the set of \( x \in V_1 \) such that \( fx = 0 \)). For \( x \in W_1 \) we have \( f\rho^1_s x = \rho^2_s fx = 0 \), whence \( \rho^1_s x \in W_1 \), and \( W_1 \) is stable under \( G \). Since \( V_1 \) is irreducible, \( W_1 \) is equal to \( V_1 \) or \( 0 \); the first case is excluded, as it implies \( f = 0 \). The same argument shows that the image \( W_2 \) of \( f \) (the set of \( fx \), for \( x \in V_1 \)) is equal to \( V_2 \). The two properties \( W_1 = 0 \) and \( W_2 = V_2 \) show that \( f \) is an isomorphism of \( V_1 \) onto \( V_2 \), which proves assertion (1).

Suppose now that \( V_1 = V_2 \), \( \rho^1 = \rho^2 \), and let \( \lambda \) be an eigenvalue of \( f \); there exists at least one, since the field of scalars is the field of complex numbers. Put \( f' = f - \lambda \). Since \( \lambda \) is an eigenvalue of \( f \), the kernel of \( f' \) is \( \neq 0 \); on the other hand, we have \( \rho^2_s \circ f' = f' \circ \rho^1_s \). The first part of the proof shows that these properties are possible only if \( f' = 0 \), that is, if \( f \) is equal to \( \lambda \).

Let us keep the hypothesis that \( V_1 \) and \( V_2 \) are irreducible, and denote by \( g \) the order of the group \( G \).

**Corollary 1.** Let \( h \) be a linear mapping of \( V_1 \) into \( V_2 \); and put:

\[
h^0 = \frac{1}{g} \sum_{s \in G} (\rho^2_s)^{-1} h \rho^1_s.
\]

Then:

1. If \( \rho^1 \) and \( \rho^2 \) are not isomorphic, we have \( h^0 = 0 \).
2. If \( V_1 = V_2 \) and \( \rho^1 = \rho^2 \), \( h^0 \) is a homothety of ratio \((1/n)\text{Tr}(h)\), with \( n = \dim(V_1) \).

We have \( \rho^2_s h^0 = h^0 \rho^1_s \). Indeed:

\[
(\rho^2_s)^{-1} h^0 \rho^1_s = \frac{1}{g} \sum_{t \in G} (\rho^2_t)^{-1} (\rho^1_t)^{-1} h \rho^1_t \rho^1_s
\]

\[
= \frac{1}{g} \sum_{t \in G} (\rho^2_t)^{-1} h \rho^1_t = h^0.
\]

Applying prop. 4 to \( f = h^0 \), we see in case (1) that \( h^0 = 0 \), and in case (2) that \( h^0 \) is equal to a scalar \( \lambda \). Moreover, in the latter case, we have:
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\[ \text{Tr}(h^0) = \frac{1}{g} \sum_{r \in G} \text{Tr}((\rho^0_r)^{-1} h \rho^0_r) = \text{Tr}(h), \]

and since \( \text{Tr}(\lambda) = n \cdot \lambda \), we get \( \lambda = (1/n) \text{Tr}(h) \).

Now we rewrite corollary 1 assuming that \( \rho^1 \) and \( \rho^2 \) are given in matrix form:

\[ \rho^1 = (\eta_{i_h}(t)), \quad \rho^2 = (\eta_{j_h}(t)). \]

The linear mapping \( h \) is defined by a matrix \( (x_{i'h}) \) and likewise \( h^0 \) is defined by \( (x_{j'h}^0) \). We have by definition of \( h^0 \):

\[ x_{i'h}^0 = \frac{1}{g} \sum_{i,j,j_2} \eta_{i,j_2}(t^{-1}) \delta_{j_1 h} \delta_{j_1 h} \eta_{i}(t). \]

The right hand side is a linear form with respect to \( x_{i'h}^0 \); in case (1) this form vanishes for all systems of values of the \( x_{i'h}^0 \); thus its coefficients are zero. Whence:

**Corollary 2.** In case (1), we have:

\[ \frac{1}{g} \sum_{i \in G} \eta_{i,j_2}(t^{-1}) \delta_{j_1 h} \eta_{i}(t) = 0 \]

for arbitrary \( i_1, j_2, j_1, j_2 \).

In case (2) we have similarly \( h^0 = \lambda \), i.e., \( x_{i'h}^0 = \lambda \delta_{i'h} \) (\( \delta_{i'h} \) denotes the Kronecker symbol, equal to 1 if \( i_1 = i_2 \) and 0 otherwise), with \( \lambda = (1/n) \text{Tr}(h) \), that is, \( \lambda = (1/n) \sum \delta_{j_1 h} x_{j'h} \). Hence the equality:

\[ \frac{1}{g} \sum_{i \in G} \eta_{i,j_2}(t^{-1}) x_{j_2 h} \delta_{j_1 h} \delta_{j_1 h} \eta_{i}(t) = \frac{1}{n} \sum_{j_1,j_2} \delta_{i_1 h} \delta_{j_2 h} x_{j_2 h}. \]

Equating coefficients of the \( x_{j_2 h} \), we obtain as above:

**Corollary 3.** In case (2) we have:

\[ \frac{1}{g} \sum_{i \in G} \eta_{i,j_2}(t^{-1}) \delta_{j_1 h} \delta_{j_2 h} \eta_{i}(t) = \begin{cases} \frac{1}{n}, & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ 0, & \text{otherwise}. \end{cases} \]

**Remarks**

1. If \( \phi \) and \( \psi \) are functions on \( G \), set

\[ \langle \phi, \psi \rangle = \frac{1}{g} \sum_{r \in G} \phi(r^{-1}) \psi(r) = \frac{1}{g} \sum_{r \in G} \phi(r) \psi(r^{-1}). \]

We have \( \langle \phi, \psi \rangle = \langle \psi, \phi \rangle \). Moreover \( \langle \phi, \psi \rangle \) is linear in \( \phi \) and in \( \psi \). With this notation, corollaries 2 and 3 become, respectively

\[ \langle \eta_{i,j_2} \delta_{j_1 h}, \delta_{j_2 h} \rangle = 0 \quad \text{and} \quad \langle \eta_{i,j_2} \delta_{j_1 h}, \delta_{j_2 h} \rangle = \frac{1}{n} \delta_{i_1 h} \delta_{j_2 h}. \]
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(2) Suppose that the matrices \( g_j(t) \) are unitary (this can be realized by a suitable choice of basis, cf. 1.3). We have then \( g_j(t^{-1}) = g_j(t)^* \) and corollaries 2 and 3 are just orthogonality relations for the scalar product \( \langle \phi, \psi \rangle \) defined in the following section.

2.3 Orthogonality relations for characters

We begin with a notation. If \( \phi \) and \( \psi \) are two complex-valued functions on \( G \), put

\[
\langle \phi, \psi \rangle = \frac{1}{g} \sum_{t \in G} \phi(t) \overline{\psi(t)}, \quad g \text{ being the order of } G.
\]

This is a scalar product: it is linear in \( \phi \), semilinear in \( \psi \), and we have \( \langle \phi, \phi \rangle > 0 \) for all \( \phi \neq 0 \).

If \( \tilde{\psi} \) is the function defined by the formula \( \tilde{\psi}(t) = \psi(t^{-1})^* \), we have

\[
\langle \phi, \tilde{\psi} \rangle = \frac{1}{g} \sum_{t \in G} \phi(t) \overline{\psi(t^{-1})} = \langle \phi, \psi \rangle,
\]

cf. 2.2, remark 1. In particular, if \( \chi \) is the character of a representation of \( G \), we have \( \tilde{\chi} = \chi \) (prop. 1), so that \( \langle \phi, \chi \rangle = \langle \phi, \tilde{\chi} \rangle \) for all functions \( \phi \) on \( G \). So we can use at will \( \langle \phi, \chi \rangle \) or \( \langle \phi, \tilde{\chi} \rangle \), so long as we are concerned with characters.

Theorem 3

(i) If \( \chi \) is the character of an irreducible representation, we have \( \langle \chi, \chi \rangle = 1 \) (i.e., \( \chi \) is "of norm 1").

(ii) If \( \chi \) and \( \chi' \) are the characters of two nonisomorphic irreducible representations, we have \( \langle \chi, \chi' \rangle = 0 \) (i.e. \( \chi \) and \( \chi' \) are orthogonal).

Let \( \rho \) be an irreducible representation with character \( \chi \), given in matrix form \( \rho_i = (g_j(t)) \). We have \( \chi(t) = \sum g_j(t) \), hence

\[
\langle \chi, \chi \rangle = \langle \chi, \chi \rangle = \sum_{i,j} \langle g_i, g_j \rangle.
\]

But according to cor. 3 to prop. 4, we have \( \langle g_i, g_j \rangle = \delta_{ij}/n \), where \( n \) is the degree of \( \rho \). Thus

\[
\langle \chi, \chi \rangle = \left( \sum_{i,j} \delta_{ij} \right)/n = n/n = 1,
\]

since the indices \( i,j \) each take \( n \) values. (ii) is proved in the same way, by applying cor. 2 instead of cor. 3.

\[ \square \]

Remark. A character of an irreducible representation is called an irreducible character. Theorem 3 shows that the irreducible characters form an orthonormal system; this result will be completed later (2.5, th. 6).
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Theorem 4. Let $V$ be a linear representation of $G$, with character $\phi$, and suppose $V$ decomposes into a direct sum of irreducible representations:

$$V = W_1 \oplus \cdots \oplus W_k.$$  

Then, if $W$ is an irreducible representation with character $\chi$, the number of $W_i$ isomorphic to $W$ is equal to the scalar product $(\phi|\chi) = \langle \phi, \chi \rangle$.

Let $\chi_i$ be the character of $W_i$. By prop. 2, we have

$$\phi = \chi_1 + \cdots + \chi_k.$$  

Thus $(\phi|\chi) = (\chi_1|\chi) + \cdots + (\chi_k|\chi)$. But, according to the preceding theorem, $(\chi_i|\chi)$ is equal to 1 or 0, depending on whether $W_i$ is, or is not, isomorphic to $W$. The result follows. \qed

Corollary 1. The number of $W_i$ isomorphic to $W$ does not depend on the chosen decomposition.

(This number is called the "number of times that $W$ occurs in $V$", or the "number of times that $W$ is contained in $V$."

Indeed, $(\phi|\chi)$ does not depend on the decomposition. \qed

Remark. It is in this sense that one can say that there is uniqueness in the decomposition of a representation into irreducible representations. We shall return to this in 2.6.

Corollary 2. Two representations with the same character are isomorphic.

Indeed, cor. 1 shows that they contain each given irreducible representation the same number of times.

The above results reduce the study of representations to that of their characters. If $\chi_1, \ldots, \chi_k$ are the distinct irreducible characters of $G$, and if $W_1, \ldots, W_k$ denote corresponding representations, each representation $V$ is isomorphic to a direct sum

$$V = m_1 W_1 \oplus \cdots \oplus m_k W_k \quad m_i \text{ integers } \geq 0.$$  

The character $\phi$ of $V$ is equal to $m_1 \chi_1 + \cdots + m_k \chi_k$, and we have $m_i = (\phi|\chi_i)$. [This applies notably to the tensor product $W_i \otimes W_j$ of two irreducible representations, and shows that the product $\chi_i \cdot \chi_j$ decomposes into $\chi_i \chi_j = \sum m_h^i \chi_h$, the $m_h^i$ being integers $\geq 0$.] The orthogonality relations among the $\chi_i$ imply in addition:

$$(\phi|\phi) = \sum_{i=1}^{i=h} m_i^2,$$  

whence:

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Theorem 5. If \( \phi \) is the character of a representation \( V \), \( (\phi, \phi) \) is a positive integer and we have \( (\phi, \phi) = 1 \) if and only if \( V \) is irreducible.

Indeed, \( \sum m_i^2 \) is only equal to 1 if one of the \( m_i \)'s is equal to 1 and the others to 0, that is, if \( V \) is isomorphic to one of the \( W_r \).

We obtain thus a very convenient irreducibility criterion.

EXERCISES

2.5. Let \( \rho \) be a linear representation with character \( \chi \). Show that the number of times that \( \rho \) contains the unit representation is equal to \( (\chi|1) = (1/|G|) \sum_{x \in G} \chi(x) \).

2.6. Let \( X \) be a finite set on which \( G \) acts, let \( \rho \) be the corresponding permutation representation (1.2) and let \( \chi \) be its character.

(a) The set \( G \times X \) of images under \( G \) of an element \( x \in X \) is called an orbit.

Let \( e \) be the number of distinct orbits. Show that \( e \) is equal to the number of times that \( \rho \) contains the unit representation 1; deduce from this that \( (\chi|1) = e \). In particular, if \( G \) is transitive (i.e., if \( e = 1 \)), \( \rho \) can be decomposed into \( 1 \oplus \theta \) and \( \theta \) does not contain the unit representation.

If \( \psi \) is the character of \( \theta \), we have \( \chi = 1 + \psi \) and \( (\psi|1) = 0 \).

(b) Let \( G \) act on the product \( X \times X \) of \( X \) by itself by means of the formula \( g(x, y) = (gx, gy) \). Show that the character of the corresponding permutation representation is equal to \( \chi^2 \).

(c) Suppose that \( G \) is transitive on \( X \) and that \( X \) has at least two elements.

We say that \( G \) is doubly transitive if, for all \( x, y, x', y' \in X \) with \( x \neq y \) and \( x' \neq y' \), there exists \( s \in G \) such that \( x' = sx \) and \( y' = sy \). Prove the equivalence of the following properties:

(i) \( G \) is doubly transitive.

(ii) The action of \( G \) on \( X \times X \) has two orbits, the diagonal and its complement.

(iii) \( (\chi^2|1) = 2 \).

(iv) The representation \( \theta \) defined in (a) is irreducible.

[The equivalence (i) \( \iff \) (ii) is immediate; (ii) \( \iff \) (iii) follows from (a) and (b). If \( \psi \) is the character of \( \theta \), we have \( 1 + \psi = \chi \) and \( (1|1) = 1 \), \( (\psi|1) = 0 \), which shows that (iii) is equivalent to \( (\psi^2|1) = 1 \), i.e., \( (1/|G|) \sum_{x \in G} \psi(x) = 1 \); since \( \psi \) is real-valued, this indeed means that \( \theta \) is irreducible, cf. th. 5.]

2.4 Decomposition of the regular representation

Notation. For the rest of Ch. 2, the irreducible characters of \( G \) are denoted \( \chi_1, \ldots, \chi_n \); their degrees are written \( n_1, \ldots, n_n \); we have \( n_i = \chi_i(1) \), cf. prop. 1.

Let \( R \) be the regular representation of \( G \). Recall (cf. 1.2) that it has a basis \( (e_i)_{i \in G} \) such that \( \rho_s e_t = e_{st} \). If \( s \neq 1 \), we have \( st \neq t \) for all \( t \), which
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shows that the diagonal terms of the matrix of \( \rho_s \) are zero; in particular we have \( \text{Tr}(\rho_s) = 0 \). On the other hand, for \( s = 1 \), we have

\[
\text{Tr}(\rho_1) = \text{Tr}(1) = \dim(R) = g.
\]

Whence:

**Proposition 5.** The character \( r_G \) of the regular representation is given by the formulas:

\[
\begin{align*}
r_G(1) &= g, & \text{order of } G, \\
r_G(s) &= 0 & \text{if } s \neq 1.
\end{align*}
\]

**Corollary 1.** Every irreducible representation \( \mathcal{W}_i \) is contained in the regular representation with multiplicity equal to its degree \( n_i \).

According to th. 4, this number is equal to \( \langle r_G, \chi_i \rangle \), and we have

\[
\langle r_G, \chi_i \rangle = \frac{1}{g} \sum_{s \in G} r_G(s^{-1}) \chi_i(s) = \frac{1}{g} g \cdot \chi_i(1) = \chi_i(1) = n_i.
\]

**Corollary 2.**

(a) The degrees \( n_i \) satisfy the relation \( \sum_{i=1}^{m} n_i^2 = g \).

(b) If \( s \in G \) is different from \( 1 \), we have \( \sum_{i=1}^{m} n_i \chi_i(s) = 0 \).

By cor. 1, we have \( r_G(s) = \sum n_i \chi_i(s) \) for all \( s \in G \). Taking \( s = 1 \) we obtain (a), and taking \( s \neq 1 \), we obtain (b).

**Remarks**

(1) The above result can be used in determining the irreducible representations of a group \( G \): suppose we have constructed some mutually nonisomorphic irreducible representations of degrees \( n_1, \ldots, n_k \); in order that they be all the irreducible representations of \( G \) (up to isomorphism), it is necessary and sufficient that \( n_1^2 + \cdots + n_k^2 = g \).

(2) We will see later (Part II, 6.5) another property of the degrees \( n_i \); they divide the order \( g \) of \( G \).

**EXERCISE**

2.7. Show that each character of \( G \) which is zero for all \( s \neq 1 \) is an integral multiple of the character \( r_G \) of the regular representation.

2.5 Number of irreducible representations

Recall (cf. 2.1) that a function \( f \) on \( G \) is called a class function if \( f(st^{-1}) = f(s) \) for all \( s, t \in G \).
Proposition 6. Let $f$ be a class function on $G$, and let $\rho: G \to \text{GL}(V)$ be a linear representation of $G$. Let $\rho_f$ be the linear mapping of $V$ into itself defined by:

$$\rho_f = \sum_{t \in G} f(t)\rho_t.$$ 

If $V$ is irreducible of degree $n$ and character $\chi$, then $\rho_f$ is a homothety of ratio $\lambda$ given by:

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t)\chi(t) = \frac{g}{n} \langle f|\chi^* \rangle.$$ 

Let us compute $\rho_s^{-1}\rho_f\rho_s$. We have:

$$\rho_s^{-1}\rho_f\rho_s = \sum_{t \in G} f(t)\rho_s^{-1}\rho_t\rho_s = \sum_{t \in G} f(t)\rho_{s^{-1}ts}.$$ 

Putting $u = s^{-1}ts$, this becomes:

$$\rho_s^{-1}\rho_f\rho_s = \sum_{u \in G} f(sus^{-1})\rho_u = \sum_{u \in G} f(u)\rho_u = \rho_f.$$ 

So we have $\rho_f\rho_s = \rho_s\rho_f$. By the second part of prop. 4, this shows that $\rho_f$ is a homothety $\lambda$. The trace of $\lambda$ is $n\lambda$; that of $\rho_f$ is $\sum_{t \in G} f(t)\text{Tr}(\rho_t) = \sum_{t \in G} f(t)\chi(t)$. Hence $\lambda = (1/n) \sum_{t \in G} f(t)\chi(t) = \langle g/n \rangle \langle f|\chi^* \rangle$. 

We introduce now the space $H$ of class functions on $G$; the irreducible characters $\chi_1, \ldots, \chi_h$ belong to $H$.

Theorem 6. The characters $\chi_1, \ldots, \chi_h$ form an orthonormal basis of $H$.

Theorem 3 shows that the $\chi_t$ form an orthonormal system in $H$. It remains to prove that they generate $H$, and for this it is enough to show that every element of $H$ orthogonal to the $\chi_t^*$ is zero. Let $f$ be such an element. For each representation $\rho$ of $G$, put $\rho_f = \sum_{t \in G} f(t)\rho_t$. Since $f$ is orthogonal to the $\chi_t^*$, prop. 6 above shows that $\rho_f$ is zero so long as $\rho$ is irreducible; from the direct sum decomposition we conclude that $\rho_f$ is always zero. Applying this to the regular representation $R$ (cf. 2.4) and computing the image of the basis vector $e_1$ under $\rho_f$, we have

$$\rho_f e_1 = \sum_{t \in G} f(t)\rho_t e_1 = \sum_{t \in G} f(t) e_t.$$ 

Since $\rho_f$ is zero, we have $\rho_f e_1 = 0$ and the above formula shows that $f(t) = 0$ for all $t \in G$; hence $f = 0$, and the proof is complete.

Recall that two elements $t$ and $t'$ of $G$ are said to be conjugate if there exists $s \in G$ such that $t' = sts^{-1}$; this is an equivalence relation, which partitions $G$ into classes (also called conjugacy classes).

Theorem 7. The number of irreducible representations of $G$ (up to isomorphism) is equal to the number of classes of $G$. 

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Let $C_1, \ldots, C_k$ be the distinct classes of $G$. To say that a function $f$ on $G$ is a class function is equivalent to saying that it is constant on each of $C_1, \ldots, C_k$; it is thus determined by its values $\lambda_i$ on the $C_i$, and these can be chosen arbitrarily. Consequently, the dimension of the space $H$ of class functions is equal to $k$. On the other hand, this dimension is, by th. 6, equal to the number of irreducible representations of $G$ (up to isomorphism). The result follows.

Here is another consequence of th. 6:

**Proposition 7.** Let $s \in G$, and let $c(s)$ be the number of elements in the conjugacy class of $s$.

(a) We have $\sum_{i=1}^{\text{dim} H} \chi_i(s)^* \chi_i(s) = g/c(s)$.

(b) For $t \in G$ not conjugate to $s$, we have $\sum_{i=1}^{\text{dim} H} \chi_i(s)^* \chi_i(t) = 0$.
(For $s = 1$, this yields cor. 2 to prop. 5.)

Let $f_s$ be the function equal to 1 on the class of $s$ and equal to 0 elsewhere. Since it is a class function, it can, by th. 6, be written

$$f_s = \sum_{i=1}^{\text{dim} H} \lambda_i \chi_i,$$
with $\lambda_i = (f_s | \chi_i) = \frac{c(s)}{g} \chi_i(s)^*$.

We have then, for each $t \in G$,

$$f_s(t) = \frac{c(s)}{g} \sum_{i=1}^{\text{dim} H} \chi_i(s)^* \chi_i(t).$$

This gives (a) if $t = s$, and (b) if $t$ is not conjugate to $s$.

**Example.** Take for $G$ the group of permutations of three letters. We have $g = 6$, and there are three classes: the element 1, the three transpositions, and the two cyclic permutations. Let $t$ be a transposition and $c$ a cyclic permutation. We have $t^2 = 1$, $c^3 = 1$, $tc = c^2 t$; whence there are just two characters of degree 1: the unit character $\chi_1$ and the character $\chi_2$ giving the signature of a permutation. Theorem 7 shows that there exists one other irreducible character $\theta$; if $n$ is its degree we must have $1 + 1 + n^2 = 6$, hence $n = 2$. The values of $\theta$ can be deduced from the fact that $\chi_1 + \chi_2 + 2\theta$ is the character of the regular representation of $G$ (cf. prop. 5). We thus get the character table of $G$:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$t$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\theta$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
We obtain an irreducible representation with character \( \theta \) by having \( G \) permute the coordinates of elements of \( \mathbb{C}^3 \) satisfying the equation \( x + y + z = 0 \) (cf. ex. 2.6c)).

2.6 Canonical decomposition of a representation

Let \( \rho: G \to \text{GL}(V) \) be a linear representation of \( G \). We are going to define a direct sum decomposition of \( V \) which is “coarser” than the decomposition into irreducible representations, but which has the advantage of being unique. It is obtained as follows:

Let \( \chi_1, \ldots, \chi_h \) be the distinct characters of the irreducible representations \( W_1, \ldots, W_h \) of \( G \) and \( n_1, \ldots, n_h \) their degrees. Let \( V = U_1 \oplus \cdots \oplus U_m \) be a decomposition of \( V \) into a direct sum of irreducible representations. For \( i = 1, \ldots, h \) denote by \( V_i \) the direct sum of those of the \( U_1, \ldots, U_m \) which are isomorphic to \( W_i \). Clearly we have:

\[
V = V_1 \oplus \cdots \oplus V_h.
\]

(In other words, we have decomposed \( V \) into a direct sum of irreducible representations and collected together the isomorphic representations.)

This is the canonical decomposition we had in mind. Its properties are as follows:

**Theorem 8**

(i) The decomposition \( V = V_1 \oplus \cdots \oplus V_h \) does not depend on the initially chosen decomposition of \( V \) into irreducible representations.

(ii) The projection \( p_i \) of \( V \) onto \( V_i \) associated with this decomposition is given by the formula:

\[
p_i = \frac{n_i}{g} \sum_{\alpha \in G} \chi_i(\alpha)^* \rho_i.
\]

We prove (ii). Assertion (i) will follow because the projections \( p_i \) determine the \( V_i \). Put

\[
q_i = \frac{n_i}{g} \sum_{\alpha \in G} \chi_i(\alpha)^* \rho_i.
\]

Proposition 6 shows that the restriction of \( q_i \) to an irreducible representation \( W \) with character \( \chi \) and of degree \( \nu \) is a homothety of ratio \((n_i/\nu)(\chi_i/\chi)\); it is thus 0 if \( \chi \neq \chi_i \) and 1 if \( \chi = \chi_i \). In other words \( q_i \) is the identity on an irreducible representation isomorphic to \( W_i \) and is zero on the others. In view of the definition of the \( V_i \), it follows that \( q_i \) is the identity on \( V_i \) and is 0 on \( V_j \) for \( j \neq i \). If we decompose an element \( x \in V \) into its components \( x_i \in V_i \):

\[
x = x_1 + \cdots + x_h,
\]

we have then \( q_i(x) = q_i(x_1) + \cdots + q_i(x_h) = x_i \). This means that \( q_i \) is equal to the projection \( p_i \) of \( V \) onto \( V_i \).

\( \square \)
Thus the decomposition of a representation $V$ can be done in two stages. First the canonical decomposition $V_1 \oplus \cdots \oplus V_n$ is determined; this can be done easily using the formulas giving the projections $p_i$. Next, if needed, one chooses a decomposition of $V_i$ into a direct sum of irreducible representations each isomorphic to $W_j$:

$$V_i = W_j \oplus \cdots \oplus W_j.$$ 

This last decomposition can in general be done in an infinity of ways (cf. section 2.7, as well as ex. 2.8 below); it is just as arbitrary as the choice of a basis in a vector space.

**Example.** Take for $G$ the group of two elements \{1, s\} with $s^2 = 1$. This group has two irreducible representations of degree 1, $W^+$ and $W^-$, corresponding to $\rho_1 = +1$ and $\rho_s = -1$. The canonical decomposition of a representation $V$ is $V = V^+ \oplus V^-$, where $V^+$ (resp. $V^-$) consists of the elements $x \in V$ which are symmetric (resp. antisymmetric), i.e., which satisfy $\rho_1 x = x$ (resp. $\rho_s x = -x$). The corresponding projections are:

$$p^+ x = \frac{1}{2} (x + \rho_s x), \quad p^- x = \frac{1}{2} (x - \rho_s x).$$

To decompose $V^+$ and $V^-$ into irreducible components means simply to decompose these spaces into a direct sum of lines.

**Exercise**

2.8. Let $H_i$ be the vector space of linear mappings $h: W_i \to V$ such that $\rho_s h = h \rho_s$ for all $s \in G$. Each $h \in H_i$ maps $W_j$ into $V_j$.

(a) Show that the dimension of $H_i$ is equal to the number of times that $W_j$ appears in $V$, i.e., $\dim V / \dim W_j$. [Reduce to the case where $V = W_j$ and use Schur’s lemma].

(b) Let $G$ act on $H_i \otimes W_j$ through the tensor product of the trivial representation of $G$ on $H_i$ and the given representation on $W_j$. Show that the map

$$F: H_i \otimes W_j \to V_i$$

defined by the formula

$$F(\sum h_n \cdot w_n) = \sum h_n (w_n)$$

is an isomorphism of $H_i \otimes W_j$ onto $V_i$. [Same method].

(c) Let $(h_1, \ldots, h_k)$ be a basis of $H_i$ and form the direct sum $W_j \oplus \cdots \oplus W_j$ of $k$ copies of $W_j$. The system $(h_1, \ldots, h_k)$ defines in an obvious way a linear mapping $h$ of $W_j \oplus \cdots \oplus W_j$ into $V_i$; show that it is an isomorphism of representations and that each isomorphism is thus obtainable [apply (b), or argue directly]. In particular, to decompose $V_i$ into a direct sum of representations isomorphic to $W_j$ amounts to choosing a basis for $H_i$. 

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2.7 Explicit decomposition of a representation

Keep the notation of the preceding section, and let

\[ V = V_1 \oplus \cdots \oplus V_n \]

be the canonical decomposition of the given representation. We have seen how one can determine the \( i \)th component \( V_i \) by means of the corresponding projection (th. 8). We now give a method for explicitly constructing a decomposition of \( V \) into a direct sum of subrepresentations isomorphic to \( W_\gamma \). Let \( W_\gamma \) be given in matrix form \(( \rho_{\alpha\beta}(s) )\) with respect to a basis \(( e_1, \ldots, e_n )\); we have \( \chi_\gamma(s) = \sum_\alpha \rho_{\alpha\gamma}(s) \) and \( n = n_\gamma = \dim W_\gamma \). For each pair of integers \( \alpha, \beta \) taken from 1 to \( n \), let \( p_{\alpha\beta} \) denote the linear map of \( V \) into \( V \) defined by

\[ p_{\alpha\beta} = \frac{1}{n} \sum_{t \in G} \rho_{\alpha\beta}(t^{-1}) \rho_t. \]

Proposition 8

(a) The map \( p_{\alpha\beta} \) is a projection; it is zero on the \( V_j, j \neq i \). Its image \( V_{i\alpha\beta} \) is contained in \( V_i \) and \( V_i \) is the direct sum of the \( V_{i\alpha\beta} \) for \( 1 \leq \alpha \leq n \).

We have \( p_i = \sum_\alpha p_{\alpha\alpha} \).

(b) The linear map \( p_{\alpha\beta} \) is zero on the \( V_j, j \neq i \), as well as on the \( V_{i\gamma} \) for \( \gamma \neq \beta \); it defines an isomorphism from \( V_{i\beta} \) onto \( V_{i\beta} \).

(c) Let \( x_\gamma \) be an element \( \neq 0 \) of \( V_{i1} \) and let \( x_\alpha = p_{\alpha1}(x_\gamma) \in V_{i\alpha} \). The \( x_\alpha \) are linearly independent and generate a vector subspace \( W(x_\gamma) \) stable under \( G \) and of dimension \( n \). For each \( s \in G \), we have

\[ \rho_s(x_\alpha) = \sum_\beta \rho_{\alpha\beta}(s) x_\beta \]

(in particular, \( W(x_\gamma) \) is isomorphic to \( W_\gamma \)).

(d) If \( \{ x_1^{(m)} \} \) is a basis of \( V_{i1} \), the representation \( V_i \) is the direct sum of the subrepresentations \( W(x_1^{(1)}), \ldots, W(x_1^{(m)}) \) defined in \( c \).

(Thus the choice of a basis of \( V_{i1} \) gives a decomposition of \( V_i \) into a direct sum of representations isomorphic to \( W_\gamma \)).

We observe first that the formula \((*) \) above allows us to define the \( p_{\alpha\beta} \) in arbitrary representations of \( G \), and in particular in the irreducible representations \( W_\gamma \). For \( W_\gamma \), we have

\[ p_{\alpha\beta}(e_\gamma) = \frac{n}{\beta} \sum_{t \in G} \rho_{\alpha\beta}(t^{-1}) \rho_t(e_\gamma) = \frac{n}{\beta} \sum_{t \in G} \rho_{\alpha\beta}(t^{-1}) e_\gamma(t) e_\beta. \]

By cor. 3 to prop. 4 we have then

\[ p_{\alpha\beta}(e_\gamma) = \begin{cases} e_\alpha & \text{if } \gamma = \beta \\ 0 & \text{otherwise} \end{cases} \]

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We get from this the fact that \( \sum_a \rho_{aa} \) is the identity map of \( W_i \), and the formulas

\[
\rho_{\alpha\beta} \circ \rho_{\gamma\delta} = \begin{cases} 
\rho_{\alpha\delta} & \text{if } \beta = \gamma \\
0 & \text{otherwise}
\end{cases}
\]

\[
\rho_\gamma \circ \rho_{\alpha\gamma} = \sum_{\beta} \eta_{\alpha\beta}(s) \rho_{\beta\gamma}.
\]

For \( W_j \) with \( j \neq i \), we use cor. 2 to prop. 4 and the same argument to show that all the \( \rho_{\alpha\beta} \) are zero.

Having done this, we decompose \( V \) into a direct sum of subrepresentations isomorphic to \( W_j \) and apply the preceding to each of these representations. Assertions (a) and (b) follow; moreover, the above formulas remain valid in \( V \). Under the hypothesis of (c), we have then

\[
\rho_\delta(x_\alpha) = \rho_\alpha \circ \rho_{\alpha\delta}(x_\lambda) = \sum_{\beta} \eta_{\alpha\beta}(s) \rho_{\delta\beta}(x_\lambda) = \sum_{\beta} \eta_{\alpha\beta}(s) x_\beta,
\]

which proves (c). Finally (d) follows from (a), (b), and (c). \( \square \)

EXERCISES

2.9. Let \( H_i \) be the space of linear maps \( h : W_i \to V \) such that \( h \circ \rho_{ij} = \rho_{ij} \circ h \), cf. ex. 2.8. Show that the map \( h \mapsto h(e_\alpha) \) is an isomorphism of \( H_i \) onto \( V_{\alpha i} \).

2.10. Let \( x \in V_i \) and let \( V(x) \) be the smallest subrepresentation of \( V \) containing \( x \). Let \( \chi^\alpha \) be the image of \( x \) under \( \rho_{\alpha i} \); show that \( V(x) \) is the sum of the representations \( W(\chi^\alpha) \), \( \alpha = 1, \ldots, n \). Deduce from this that \( V(x) \) is the direct sum of at most \( n \) subrepresentations isomorphic to \( W_i \).