Definition 0.1. Given a group G, a subgroup $H \subset G$ and a representation $\rho : H \to GL(W)$ we say that a pair (σ, α) where $\sigma : G \to GL(V)$ is a representation of G and $\alpha \in Hom_H(W, V)$ is *a representation induced* from ρ if for any representation $\pi : G \to GL(L)$ of G the linear map

$$Hom_G(V, L) \to Hom_H(W, L), a \to a \circ \alpha$$

is an isomorphism.

Remark 0.2. An *induced representation* is a pair (σ, α) but we will often say that σ is an induced representation.

Lemma 0.3. *If (σ, α) and (σ', α') are two representations induced from ρ then there exists unique $r \in Hom_G(V, V')$ such that $r \circ \alpha = \alpha'$ and it is an isomorphism.*

Proof. Since the pair (σ, α) is an induced representation the map

$Hom_G(V, V') \to Hom_H(W, V'), a \to \alpha \circ a$ is a bijection. This proves the existence and uniqueness of r.

The same arguments show that there exists unique $r' \in Hom_G(V', V)$ such that $r' \circ \alpha' = \alpha$. I claim that $r' \circ r = Id_V, r \circ r' = Id_{V'}$. Really consider $s := r' \circ r$. Then $s \in Hom_G(V, V)$ is such that $s \circ \alpha = \alpha$. But we also have $Id_V \circ \alpha = \alpha$. It follow now from the first part of the Lemma that $s = Id_V$. □

Remark 0.4. We see that an induced representation is essentially unique. So we often say *the induced representation*.

Now we give an explicit construction of an induced representation. Let $\rho : H \to GL(W)$ be a representation of a subgroup $H \subset G$ and $F(G, W)$ be the space of functions on G with values in W. Consider the subspace

$$V := \{f \in F(V, W) | f(hg) = \rho(h)f(g) \forall g \in G, h \in H\}$$

and define a map $\sigma : G \to GL(V)$ by

$$(\sigma(x)f)(g) := f(gx), x, g \in G$$

We define $\alpha \in Hom(W, V)$ by

$\alpha(w)(g) = 0$ if $g \notin H$ and $\alpha(w)(g) = \rho(g)w$ if $g \in H$.

it is clear that $\alpha \in Hom_H(W, V)$.

Lemma 0.5. *a) $\sigma : G \to GL(V)$ is a representation.*

b) The pair (σ, α) is an representation induced from ρ.

Proof. a) We have $\sigma(x)\sigma(y)(f) = \sigma(x)\phi, \phi(g) = f(gy), g \in G$. So

$$\sigma(x)\sigma(y)(f)(g) = \sigma(x)(\phi)(g) = \phi(gx) = f((gx)y) = f(g(xy)) = \sigma(xy)(f)$$

b) Let $\pi : G :\to GL(L)$ be a representation of G and r be the linear map

$$r : Hom_G(V, L) \to Hom_H(W, L), r(a) := \alpha \circ a$$

To show that r is an isomorphism consider a map $s : Hom_H(W, L) \to Hom(V, L)$ given by $b \to s(b), b \in Hom_H(W, L)$ where

$$s(b)(f) = 1/|H| \sum_{g \in G} \pi^{-1}(g) b(f(g))$$

I claim that $s(b) \in Hom_G(V, L)$. Really $s(b)(\sigma(x)(f) = s(b)(\phi), \phi(g) = f(gx)$. So

$$s(b)(\sigma(x)(f) = 1/|H| \sum_{g \in G} \pi^{-1}(g) b(\phi(g)) = 1/|H| \sum_{g \in G} \pi(g^{-1}) b(f(gx)) =$$

$$= 1/|H| \sum_{g' \in G} \pi^{-1}(g'x^{-1}) b(f(g')) = \pi(x) 1/|H| \sum_{g' \in G} \pi^{-1}(g') b(f(g')) = \pi(x) \circ s(b)(f)$$

I'll leave for you to check that the map s is the inverse to r. □

Remark 0.6. a) For any $f \in V, g \in G, h \in H$ and $b \in Hom_H(W, L)$ we have $\pi^{-1}(hg) b(f(hg)) = \pi^{-1}(g) b(f(g))$. So we can write

$$s(b)(f) = \sum_{\bar{g} \in H \backslash G} \pi^{-1}(\bar{g}) b(f(\bar{g}))$$

In this form the construction works over an arbitrary field.

b) We denote the representation of G induced from a representation $\rho : H \to GL(W)$ of $H \subset G$ by $ind_H^G(\rho)$.

Let's now compute the character $(\pi, V) = ind_H^G(\rho, W)$. The group G acts naturally on the set $X := H\backslash G, (g, x) \to xg$. For any $g \in G$ we define $X^g := \{x \in X | xg = g\}$. Choose $a \in x \subset G$. Since $x \in X^g$ there exists $h_a \in H$ such that $ag = h_a a$. It is clear that $h_y a = y^{-1} h_a y$ for any $y \in H$. Therefore $\chi_\rho(h_a)$ does not depend on a choice of $a \in x$. We denote this value by $\chi_\rho(x)$

Lemma 0.7. $\chi_\pi(g) = \sum_{x \in X^g} \chi_\rho(x)$

Proof. For any $x \in X := H\backslash G$ we denote by $V_x \subset V$ the subspace of functions with support in $x \subset G$. It is easy to see that for any $g \in G$ we have $\pi(g) V_x = V_{xg}$. Therefore $Tr(\pi(g)) = \sum_{x \in X^g} Tr(\pi(g)_x)$ where $\pi(g)_x \in GL(V_x)$ is the restriction of $\pi(g)$ on the subspace $V_x \subset V$.

To find $Tr(\pi(g)_x)$ fix $a \in x$ and consider the linear map $\phi : V_x \to W$ given by $f \to f(a), f \in V_x$. It is easy to see that ϕ defines a bijection between V_x and W and

$$\pi(g)_x = \phi^{-1} \circ \rho(h) \circ \phi$$

So $Tr(\pi(g)_x) = \chi_\rho(aga^{-1}) = \chi_\rho(x)$. □

Let $K \subset G$ be another subgroup. We will describe now the restriction $res_K(ind_H^G(\rho))$ of the representation $ind_H^G(\rho)$ to K. Fix $s \in S := H\backslash G/K$ For any $g \in s \subset G$ we define $K_g := g^{-1}Hg \cap K \subset K$ and $\tau_g := ind_{K_g}^K \rho_g$ where $\rho_g : K_g \to GL(W)$ is given by $\rho_g(k) := \rho(gkg^{-1})$.

I'll leave for you to check the following

Claim 0.8. *The equivalence class τ_s of the representation τ_g of K does not depend on a choice of $g \in s$.*

Lemma 0.9. *L:dec The representation $res_K ind_H^G(\rho)$ is isomorphic to the direct sum of representations $\tau_s, s \in S$.*

Proof. For any $s \in S$ we denote by $V_s \subset V$ the subspace of functions with support on $s \subset G$. It is clear that subspaces V_s are $K-invariant$ and $V = \oplus_{s \in S} V_s$. So it is sufficient to check that the restriction on V_s of the representation $res_K ind_H^G(\rho)$ of K is isomorphic to $\tau_s, \forall s \in S$.

Fix $g \in s$ and consider the map $\phi : V_s \to F(K, W)$ given by $\phi(f)(k) := f(gk)$. It is clear that ϕ defines an isomorphism of the representation of K on V_s with the representation $ind_{K_g}^K \rho_g$. □

Definition 0.10. a) For any two representations π, π' of a group R are we define $< \pi, \pi' >= dim Hom_R(\pi, \pi')$.

b) Two representations π, π' of a group R are *disjoint* if $< \pi, \pi' >= 0$.

Remark 0.11. We know that $< \pi, \pi' >= 1/|G| \sum_{g \in G} \chi_\pi(g) \bar{\chi}_{\pi'}(g)$. In particular $< \pi, \pi' >=< \pi', \pi >$

How to find $< ind_H^G(\rho), ind_H^G(\rho) >$. For each $g \in G$ we define $H_g := gHg^{-1} \cap H$ and denote by $\rho^g : H_g \to GL(W)$ a representation $\rho^g(h) := \rho(g^{-1}hg)$ and define $\kappa(g) =< res_{H_g} \rho \rho^g >$. I'll leave for you to check the following

Claim 0.12. *$\kappa(g)$ depends only on the double coset HgH of g.*

For any $s \in S := H\backslash G/H$ we define $\kappa(s) := \kappa(g), g \in s \subset G$.

Lemma 0.13. $< ind_H^G(\rho), ind_H^G(\rho) >= \sum_{s \in S} \kappa(s)$

Proof. Let $(\pi, V) = ind_H^G(\rho, W)$. By the definition of an induced representation we have

$$< ind_H^G(\rho), ind_H^G(\rho) >= dim(Hom_G(V, V)) = dim(Hom_H(W, res_H(V))$$

As follows from Lemma 0.9 the representation $res_H(V)$ is isomorphic to the direct sum of representations $\tau_s, s \in H\backslash G/H$. So

$$< ind_H^G(\rho), ind_H^G(\rho) >= \sum_{s \in S} < \rho, \tau_s >$$

So it is sufficient to show that $< \rho, \tau_s >= \kappa(s) \forall s \in S$.

So fix $s \in S$ and choose $g \in s$. Then $\tau_s = ind_{H_g}^H \rho^g$ and therefore

$$< \rho, \tau_s >=< \tau_s, \rho >=< ind_{H_g}^H \rho^g, \rho >= dimHom_H(ind_{H_g}^H \rho^g, \rho)$$

By the definition of an induced representation we have

$$dimHom_H(ind_{H_g}^H \rho^g, \rho) = dimHom_{H_g}(\rho^g, res_{H_g}(\rho)) = \kappa(g)$$

□

Corollary 0.14. *The representation $ind_H^G(\rho)$ is irreducible iff $\rho : H \to GL(W)$ is irreducible and for any $g \in G - H$ the representations $res_{H_g}(\rho)$ and ρ^g of the group H_g are disjoint.*

Proof. By the Schur's the representation $ind_H^G(\rho)$ is irreducible iff $< ind_H^G(\rho), ind_H^G(\rho) >= 1$.Since $\tau_e = \rho$ we see that the representation $ind_H^G(\rho)$ is irreducible iff $< \rho, \rho >= 1$ and $\kappa(g) = 0$ for all $g \in G - H$. □