1. Introduction

1.1. **The Dirichlet theorem.** In this course we will study the theory of representaions of finite groups. The theory of representaions of finite groups originated in the work of Dirichlet on primes in arithmetic progressions and I start with a review of this work of Dirichlet

Let q be a positive number and a a positive number prime to q.

Theorem 1.1. *T:D For any number a prime to q the arithmetic progression $a, a+q, a+2q, \dots$ contains an infinite numbers of primes.*

In the case when $q = 1$ or $q = 2$ the result was know to Euclead but for general q the theorem was proven by Lejeune Dirichlet in 1837. I will not present a complete proof but outline the main idea of the proof.

Let $\mathbb{Z}/q\mathbb{Z}$ be the ring of residues mod q and $G = (\mathbb{Z}/q\mathbb{Z})^\star \subset \mathbb{Z}/q\mathbb{Z}$ be the multiplicative group of invertable elements. [In other words $G = \{\bar{a}\}$ where $a \in \mathbb{Z}$ is prime to q where we denote by $\bar{a} \in \mathbb{Z}/q\mathbb{Z}$ the residue of a mod q].

Definition 1.2. a) A *character* of G is a function $\chi : G \to \mathbb{C}^\star$ such that

$$\chi(\bar{a}\bar{b}) = \chi(\bar{a})\chi(\bar{b}) \forall \bar{a}, \bar{b} \in G$$

We denote by $\chi_0 \in G^\vee$ the *trivial* character $\chi_0(\bar{a}) \equiv 1$.

b) We denote by $G^\vee$ the set of characters of the group G and for any $\chi \in G^\vee$ denote by $\tilde{\chi} : \mathbb{Z} \to \mathbb{C}$ the function such that $\tilde{\chi}(a) = 0$ if $(a, q) \neq 1$ and $\tilde{\chi}(a) := \chi(\bar{a})$ where $\bar{a} \in \mathbb{Z}/q\mathbb{Z}$ the residue of a mod q if n is prime to q.

c) For any $\chi \in G^\vee$ we define the Dirichlet *L-function* $L_\chi(s)$ by

$$L_\chi(s) = \sum_{n \geq 1} \tilde{\chi}(n)/n^s$$

It is easy to see that this series is convergent for $s > 1$.

Claim 1.3. *C:1 a) For any $\bar{a}, \bar{b} \in G^\vee$ we have*

$$\sum_{\chi \in G^\vee} \chi^{-1}(\bar{a})\chi(\bar{b}) = 0$$

if $\bar{a} \neq \bar{b}$ and

$$\sum_{\chi \in G^\vee} \chi^{-1}(\bar{a})\chi(\bar{b}) = |G|$$

if $\bar{b} = \bar{a}$.

b) We have $\sum_{\bar{a}\in G}\chi(\bar{a})=0$ *for all* $\chi\in G^\vee,\chi\neq\chi_0$ *and*

$$\sum_{\bar{a}\in G}\chi_0(\bar{a})=|G|$$

if $\chi\equiv 1$.

We will later prove Claim 1.3 for arbitrary finite abelian groups.

Lemma 1.4. *L:con For any* $\chi\in G^\vee,\chi\neq\chi_0$ *the series* $L_\chi(s)=\sum_{n\geq 1}\tilde{\chi}(n)/n^s$ *is convergent for any* $s\in\mathbb{C}$ *with* $Re(s)>0$.

Proof. The proof is based on the following result from Calculs

Claim 1.5. *C:2 Let* $a_n,n\geq 1$ *monotonely decreasing sequence of positive numbers such that* $\lim_{n\to\infty}a_n=0$ *and* $b_n,n\geq 1$ *a sequence of complex numbers such that the sequence* $B_n:=\sum_{k=1}^n b_k$ *is bounded. Then the series* $\sum_{n\geq 1}a_nb_n$ *is convergent.*

Take now $a_n=1/n^s$ and $b_n=\tilde{\chi}(n)$. As follows from Corollary 1.5 for any $d>0$ we have $\sum_{i=1}^q b_{d+i}=0$. So the sequence $B_n:=\sum_{k=1}^n b_k$ is bounded. □

Proposition 1.6. *P:D* $L_\chi(1)\neq 0$ *for any* $\chi\in G^\vee,\chi\neq\chi_0$

The proof of Proposition 1.6 is not very difficult [see for example the book of Ram Murty "Problems in Amalytic Number Theory" pp.24-29] but it would take us away from out theme. So I'll not prove this propostion but only show how to deduce the Dirichlet theorem from Proposition 1.6.

Proof. To start with we consider the Riemann zeta function

$$\zeta(s)=\sum_{n/\geq 1}1/n^s$$

which is convergent for $s>1$. I claim that

$$\lim_{s\to 1^+}(s-1)\zeta(s)=1$$

Really it follows from the inequalities

$$\int_n^{n+1}x^{-s}ds<n^{-s}<\int_{n-1}^n x^{-s}ds$$

that the difference $\zeta(s)-1/(s-1)$ is bounded when $s\to 1^+$.

Corollary 1.7. *C:in* $\lim_{s\to 1^+}(s-1)L(\chi_0(s)\neq 0$

Proof. Since $L(\chi_0(s) = \prod_{p\in P_q}(1-p^{-s})\zeta(s)$ we see that

$$\lim_{s\to 1^+}(s-1)L(\chi_0(s) = \prod_{p\in P_q}(1-1/p) \neq 0$$

□

The proof of Theorem 1.1 is based on the possibility to write the L-function $L_\chi(s)$ as the Euler product.

Claim 1.8. *For any $s>1$ and $\chi \in G^\vee$ we have*

$$L_\chi(s) = \prod_{p\in P-P_q}(1-\chi(p)/p^s)$$

The Claim follows immediately from the uniqueness of the decomposition to prime factors.

Corollary 1.9. *C:D a)* $\lim_{s\to 1^+}\sum_{p\in P-P_q}(\tilde{\chi}_0(p)/p^s) = \infty$.

b) For any $\chi \in G^\vee, \chi \neq \chi_0$ the sum $\sum_{p\in P-P_q}(\tilde{\chi}(p)/p^s)$ which is absolutely convergent for $s>1$ extends to a smooth function on $s\geq 1$.

Proof. a) Consider the function $\ln(L_{\chi 0}(s)), \chi \in G^\vee$. Using the presentation of $L_{\chi_0}(s)$ as the Euler product we see that

$$\ln(L_{\chi_0}(s) = \sum_{p\in P-P_q}\ln(1-1/p^s) =$$

$$\sum_{p\notin P}(\tilde{\chi}(p)/p^s - 1/2(\tilde{\chi}(p)/p^s)^2+\dots) = \sum_{p\in P}(\tilde{\chi}(p)/p^s) + f_\chi(s)$$

for all $s>1$. It is easy to check that the series for $f_\chi(s)$ are absolutely convergent for $s>1/2$. Since by Corollary 1.7 $\lim_{s\to 1^+}\ln(L_{\chi_0}(s) = \infty$ we see that $\lim_{s\to 1^+}\sum_{p\in P}(\tilde{\chi}_0(p)/p^s) = \infty$.

b) For any $\chi \in G^\vee, \chi \neq \chi_0$ it follows from Lemma 1.4 that $L_\chi(s)$ is a smooth complex-valued function on the ray $s>1/2$ such that $\lim_{s\to\infty}L_\chi(s) = 1$. For $s>1$ the Euler product $L_\chi(s) = \prod_{p\in P-P_q}(1-\chi(p)/p^s)$ is absolutely convergent and therefore $L_\chi(s)\neq 0$ for $s>1$. Since by Proposition 1.6 that $L_\chi(1)\neq 0$ we see that function $\ln(L_\chi(s)$ defined for $s>1$ extends to a smooth function on $s\geq 1$.

On the other hand

$$\ln(L_\chi(s) = \sum_{p\in P-P_q}\ln(1-\tilde{\chi}(p)/p^s) =$$

$$\sum_{p\in P-P_q}(\tilde{\chi}(p)/p^s - 1/2(\tilde{\chi}(p)/p^s)^2+\dots) = \sum_{p\in P-P_q}(\tilde{\chi}(p)/p^s) + f_\chi(s)$$

for all $s > 1$. It is easy to check that the series for $f_\chi(s)$ are absolutely convergent for $s > 1/2$. $\square$

Now we can prove Theorem 1.1.Fix any $a \in \mathbb{Z}$ prime to q. To show that the arithmetic progression $a + nq, n > 0$ contains an infinite number of primes consider the function

$$\phi_a(s) := \sum_{p \in P, p \equiv a mod(q)} 1/p^s$$

which is absolutely convergent for $s > 1$. It is sufficient to show that $\lim_{s \to 1^+} \phi_a(s) = \infty$.

But it follows from Claim 1.3 that for $s > 1$ we have

$$\phi_a(s) = \sum_{\chi \in G^\vee} \chi^{-1}(a) L_\chi(s)$$

Therefore the equality $\lim_{s \to 1^+} \phi_a(s) = \infty$ follows from Corollary 1.9. $\square$