1. Examples

Definition 1.1. a) For any finite set X we denote by $\mathbb{C}[X]$ the space of complex-valued functions on X.

b) We define the Hermitian scalar product $(,)$ on $\mathbb{C}[X]$ by

$$(f', f'') := 1/|X| \sum_{x \in X} f'(x) \bar{f}''(x)$$

c) If a group G acts on the set X we define a function $\rho : G \to Aut(\mathbb{C}[X])$ by

$$\rho(g) f(x) := f(xg), f \in \mathbb{C}[X], x \in X$$

Please check that $\rho : G \to Aut(\mathbb{C}[X])$ is a representation of G on $(\mathbb{C}[X]$ which preserves the scalar product $(,)$.

Remark 1.2. If $X = G$ and the action of G on X is by left shifts $(g, x) \to gx$ then the map $e_g \to \delta_g, g \in G$ where

$\delta_g(x) = 0$ if $x \neq g$ and $\delta_g(g) = 1$

defines an equivalence between the representation ρ and the regular representation of the group G. [see [S]1.2].

1.1. **Commutative groups.** Let G be a finite commutative group. Then [see Th.7 in [S]] any irreducible representation of G is one-dimensional and is given by a function $\chi : G \to \mathbb{C}^\star$ such that $\chi(g'g'') = \chi(g')\chi(g'')$ for all $g', g'' \in G$. Since a representation of G is one-dimensional the character of the representation χ is equal to χ. So for commutative groups G we can identify irreducible representation of G with its character.

Definition 1.3. For a commutative group G we denote by $\hat{G}$ the set of irreducible representations of G and for any $\chi', \chi'' \in \hat{G}$ we define *the product* $\chi' \circ \chi'' \in \hat{G}$ by

$$(\chi' \circ \chi'')(g) := \chi'(g)\chi''(g), g \in G$$

It is easy to see that the operation $\chi', \chi'' \to \chi' \circ \chi''$ defines a structure of an abelian group on the set $\hat{G}$ with the unit being the function 1. We say that $\hat{G}$ is the group *dual* to G.

Remark 1.4. The dual group $\hat{G}$ is defined only for abelian groups G.

The right action of G on itself defines a representation ρ of G on $\mathbb{C}[G]$. On the other hand the map $\rho^\vee : G \to Aut(\mathbb{C}[\hat{G}])$ given by

$$(\rho^\vee(g)\phi)(\chi) := \chi(g^{-1})\phi(\chi), \phi \in \mathbb{C}[\hat{G}], g \in G$$

defines a representation $\rho^\vee$ of G on $\mathbb{C}[\hat{G}]$) [please check]. We define the *Fourier transforms* $\mathcal{F}:\mathbb{C}[G]\to\mathbb{C}[\hat{G}]$ and $\mathcal{F}^\vee:\mathbb{C}[\hat{G}]\to\mathbb{C}[G]$ by

$$\mathcal{F}(f)(\chi) := \frac{1}{\sqrt{|G|}}\sum_{g\in G}\chi(g)f(g), \chi\in\hat{G}, f\in\mathbb{C}[G]$$

and

$$\mathcal{F}^\vee(\phi)(x) := \frac{1}{\sqrt{|G|}}\sum_{\chi\in\hat{G}}\chi^{-1}(x)\phi(\chi), \phi\in\mathbb{C}[\hat{G}]), x\in G$$

Proposition 1.5. *P:fou a) $\mathcal{F}\circ\rho(g)=\rho(g)^\vee\circ\mathcal{F}$.*

b) $\mathcal{F}$ and $\mathcal{F}^\vee$ are unitary linear maps which are inverse to each other.

Proof. a) $\mathcal{F}\circ\rho(g_0)(g)(\chi)=\frac{1}{\sqrt{|G|}}\sum_{g\in G}\chi(g)(\rho(g_0)f(g))=$

$\frac{1}{\sqrt{|G|}}\sum_{g\in G}\chi(g)f(gg_0)=\frac{1}{\sqrt{|G|}}\sum_{g'\in G}\chi(gg_0^{-1})f(g))=$

$\chi(g_0^{-1})\frac{1}{\sqrt{|G|}}\sum_{g'\in G}\chi(g)f(g)=\rho^\vee(g_0)(\mathcal{F}(f))$

b) To prove that $\mathcal{F}^\vee\circ\mathcal{F}=Id_{\mathbb{C}[G]}$ it is sufficient to show that $\mathcal{F}^\vee\circ\mathcal{F}(\delta_g)=\delta_g$ for all $g\in G$. By the definition for all $\chi\in\hat{G}$ we have

$$\mathcal{F}(\delta_g)(\chi)=\frac{1}{\sqrt{|G|}}\chi(g)$$

Therefore

$$\mathcal{F}^\vee\circ\mathcal{F}(\delta_g)(x)=1/|G|\sum_{\chi\in\hat{G}}\chi^{-1}(x)\chi(g)=1/|G|\sum_{\chi\in\hat{G}}\chi(x^{-1}g)$$

Now the equality follows from Proposition 7 in [S].

The proof of the equlity $\mathcal{F}\circ\mathcal{F}^\vee=Id_{\mathbb{C}[\hat{G}]}$ is completely analogous.

Since functions $\{|G|\delta_x\}, x\in G$ is an orthonormal basis of the space $\mathbb{C}[G]$ it is sufficient to check that for any $x\neq y\in G$ we have $(\mathcal{F}(\delta_x),\mathcal{F}(\delta_y))=0$ and that $(\mathcal{F}(\delta_x),\mathcal{F}(\delta_x))=1/|G|^2$ for all $x\in G$. But this follows from the equlity $\mathcal{F}(\delta_g)(\chi)=\frac{1}{\sqrt{|G|}}\chi(g), g\in G$ and Theorem 3 in [S]. □

1.2. **One-dimensional representations.** Let G be a finite group, $[G,G]\subset G$ the subgroup generated by elements of the form $aba^{-1}b^{-1}$ for $a,b\in G$. As you know $[G,G]$ is a normal subgroup of G and the quotient group $\bar{G}=G/[G,G]$ is commutative. Let $p:G\to\bar{G}$ be the natural projection. For any one-dimensional representation χ of the group $\bar{G}, \chi\circ p$ is an one-dimensional representation of the group G.

Lemma 1.6. *L:com Any one-dimensional representation π of the group G has a form $\chi \circ p$ where χ is an one-dimensional representation of the group $\bar{G}$.*

Proof. Since the representation π is one-dimensional we have

$$\pi(aba^{-1}b^{-1}) = \pi(a)\pi(b)\pi(a^{-1})\pi(b^{-1}) = 1$$

for all $a, b \in G$. So $\pi_{[G,G]} \equiv 1$. Since π is a representation we see that $\pi(g\gamma) = \pi(g)\forall g \in G, \gamma \in [G, G]$. But this show thatwTherefore there exists a function $\chi : \bar{G} \to \mathbb{C}^\star$ such that $\pi = \chi \circ p$. Since π is a representation of G, χ is also a representation of G. □

1.3. **The additive group.** Let p be a prime number $q = p^r, r > 0$ and $\mathbb{F}_q$ be the field of order q. We will study representations of different groups over $\mathbb{F}_q$. Let will denote the additive group of the field $\mathbb{F}_q$ simply by $\mathbb{F}_q$. This is a commutative group of order q. Since the group $\mathbb{F}_q$ is commutative all it irreducible representations are one-dimensional. As we know there exist q one-dimensional representations of the group $\mathbb{F}_q$. We fix one non-trivial such representation $\psi : \mathbb{F}_q \to \mathbb{C}^\star$. By the definition $\psi : \mathbb{F}_q \to \mathbb{C}^\star$ is a function such that $\psi(x+y) = \psi(x)\psi(y)$ for all $x, y \in \mathbb{F}_q$. For any $c \in \mathbb{F}_q$ we denote by $\psi_c : \mathbb{F}_q \to \mathbb{C}^\star$ the function given by

$$\psi_c(x) := \psi(cx), x \in \mathbb{F}_q$$

It is clear that for any $c \in \mathbb{F}_q$ the function ψ_c defines an one-dimensional representations of the group $\mathbb{F}_q$.

Lemma 1.7. *L:ad Any irreducible representation of the group $\mathbb{F}_q$ is equal to ψ_c for some $c \in \mathbb{F}_q$.*

Proof. Since [please check] all the functions $\psi_c, c \in \mathbb{F}_q$ are distinct we obtain q distinct irreducible representations of the group $\mathbb{F}_q$. Since we know the group $\mathbb{F}_q$ has q distinct irreducible representations [see Th.7 in [S]] the Lemma is proven. □

Remark 1.8. Since Since $\psi_{c'}\psi_{c''} = \psi_{c'+c''}$ for any $c', c'' \in \mathbb{F}_q$ [please check] we see that a choice of non-trivial elements $\psi \in \mathbb{F}_q^\vee$ defines an isomorphsim $c \to \psi_c$ between the groups $\mathbb{F}_q$ and $\mathbb{F}_q^\vee$.

Let L be a finite-dimensional $\mathbb{F}_q$-vector space. We can consider L as a finite abelian group. How to describe the dual group? Let $L^\vee$ be the space of $\mathbb{F}_q$-linear functionals $\lambda : L \to \mathbb{F}_q$. If we fix a non-trivial character $\psi : \mathbb{F}_q \to \mathbb{C}^\star$ we can associate to any $\lambda \in L^\vee$ a function $\tilde{\lambda} : L \to \mathbb{C}^\star$ where $\tilde{\lambda}(l) := \psi(\lambda(l))$.

Problem 1.9. *P:1 a) For any* $\lambda \in L^\vee$ *the function* $\tilde{\lambda} : L \to \mathbb{C}^\star$ *is a character of the group* L.

b) The map $\lambda \to \tilde{\lambda}$ *defines an isomorphism between the groups* $L^\vee$ *and* $\hat{L}$.

As follows from Proposition 1.9 we can consider the Fourier transform $\mathcal{F} : \mathbb{C}[L] \to \mathbb{C}[\hat{L}]$ as a map from $\mathcal{F} : \mathbb{C}[L] \to \mathbb{C}[L^\vee]$.

Definition 1.10. a) For any bilinear form $B : L \times L \to \mathbb{F}_q$ we denote by $\tilde{B} : L \to L^\vee$ the $\mathbb{F}_q$-linear map given by

$$\tilde{B}(l)(l') := B(l, l'), l, l' \in L$$

b) A bilinear form B is *non-degenerate* if $\tilde{B} : L \to L^\vee$ is an isomorphism of vector spaces.

c) For any non-degenerate bilinear form $B : L \times L \to \mathbb{F}_q$ we define a linear map $\mathbb{F}_B : \mathbb{C}[L] \to \mathbb{C}[L]$ by

$$\mathbb{F}_B(f)(l) := \mathbb{F}(f)(\tilde{B}(l)), f \in \mathbb{C}[L], l \in L$$

In other words

$$\mathbb{F}_B(f)(l) := \frac{1}{\sqrt{|L|}} \sum_{l' \in L} \psi(B(l, l'))f(l')$$

Problem 1.11. *P:2 a) If the form* B *is symmetric then for any* $f \in \mathbb{C}[L], l \in L$ *we have* $\mathbb{F}_B(f)(l) = f(-l)$.

b) If the form B *is anti-symmetric then* $\mathbb{F}_B(f)(l) = f(l)$ *for any* $f \in \mathbb{C}[L], l \in L$.

We denote simply by $\mathcal{F} : \mathbb{C}[\mathbb{F}_q] \to \mathbb{C}[\mathbb{F}_q]$ the Fourier transform corresponding to the bilinear form $(x, y) \to -xy, x, y \in \mathbb{F}_q$. In other words

$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}_q} \psi(-xy)f(y)$$

1.4. **The group of affine transformation of a line.** Consider now the group P of 2×2-matricies over $\mathbb{F}_q$ of the form

$$p_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad , a \in \mathbb{F}_q^\star, b \in \mathbb{F}_q$$

We define $U := \{p_{1,b}\}, b \in \mathbb{F}_q$ and $H := \{p_{a,0}\}, a \in \mathbb{F}_q^\star$.

Problem 1.12. *P:3 a)* U *is a commutative normal subgroup of* P *isomorphic to* $\mathbb{F}_q$. *Moreover* $U = [P, P]$

b) The group $\bar{P} := P/[P,P]$ *is isomorphic to* $\mathbb{F}_q^\star$ *and the map* $p : P \to \bar{P} := P/[P,P]$ *is given by* $p_{a,b} \to a$*. The map* p *defines an isomorphism* $H \to \bar{P}$*.*

c) Any $p \in P$ *can be written in the form* $p = hu, h \in H, u \in U$ *and such a decomposition is unique.*

As follows from Lemma 1.6 the number of one-dimensional representation of the group P is equal to the number of one-dimensional representation of the commutative group $\bar{P} =$ has $q-1$ one-dimensional representations of the group P which have the form $\chi \circ p$ where $\chi \in (\mathbb{F}_q^\star)^\vee$.

Lemma 1.13. *L:P The group* P *has one equivalence classe of irreducible representations of dimension* > 1 *and they have dimension* $q-1$*.*

Proof. I'll give two proofs of Lemma 1.13. The first work for finite field [the only ones we study in this course] and the second works also for real and p-adic fields.

The first proof. We start with the following result

Claim 1.14. *a) If two elements* $p_{a,b}, p_{a',b'} \in P$ *are conjugate then* $a = a'$

b) If $a \neq 1$ *then all the elements* $p_{a,b}, b \in \mathbb{F}_q$ *are conjugate.*
c) All elements $p_{1,b}, b \in \mathbb{F}_q^\star$ *are conjugate.*

Proof. The result follows from an explict formula

$$p_{x,y}^{-1} p_{a,b} p_{x,y} = p_{a,c}, c = x^{-1}[(a-1)y + by]$$

□

So we see that the group P has q conjugacy classes. Therefore (by Theorem 7 in [S]) the group P has q equivalence classes of irreducible representations. Since P has $q-1$ distinct 1-dimensional representations we see that it has unique equivalence class of $[\rho]$ of irreducible representations of dimension > 1. Let d be it's dimension. In follows from Corollary 2 to Proposition 5 in [S] that $d^2 + (q-1) = |P|$. So $d^2 + (q-1) = q(q-1)$ and $d = q-1$. □

Corollary 1.15. *C:P Any representation* π *of* P *of dimension* $q-1$ *whose restriction on the subgroup* $U \subset P$ *is not trivial is irreducible and therefore belongs to* $[\rho]$*.*

Proof. If π is reducible then all it summands have dimension smaller then $q-1$. In this case it would follow from Lemma 1.13 that all this summands were trivial on U□.

The second proof. Since U is a normal commutative subgroup of P the group H acts on the group $U^\vee$ by

$$\chi \to \chi^h, \chi^h(u) := \chi(h^{-1}uh), h \in H, u \in U$$

As follows from Lemma 1.7 the group H acts simply transitively on the set $U^\vee - \{e\}$.

Fix any non-trivial character $\chi : U \to \mathbb{C}^\star$ and consider the induced representation $\tau = ind_U^P \chi$. Since the group H acts simply transitively on $U^\vee - \{e\}$ it follows from the Corollary 0.15 in in the section of induced representations that the representation τ is irreducible and does not depend on a choice of $\chi \in U^\vee - \{e\}$.

Now we want to prove that any irreducible representation (π, V) of P of dimension > 1 is equivaent to τ. Since both π and τ are irreducible it is sufficient to prove that $Hom_P(\tau, \pi) \neq= \{0\}$.

Since U is a normal subgroup of P the subspace $V^U \subset V$ of U-invariant vectors is a P-invariant subspace of V. Since V is irreducible either $V = V^U$ or $V^U = \{0\}$. In the first case we obtain an irreducible representation of $\bar{P}$ on V. Since $\bar{P}$ is commutative this would contradict the assumption that $dim(V) > 1$. So $V^U = \{0\}$.

Since $V^U = \{0\}$ and the group U is commutative we have a decomposion $V = \sum_{\chi \in U^\vee - \{e\}} V_\chi$ where $V_\chi \subset V$ is U-invariant and $\pi(u)_{|V_\chi} = \chi(u) Id_{V_\chi}$. Choose $\chi \in U^\vee - \{e\}$ such that $V_\chi \neq \{0\}$. Since $\tau = ind_U^P \chi$ we have $Hom_P(\tau, \pi) \neq= Hom_U(\chi, V) = V_\chi$. □

To describe an other way of construction of an irreducible representation of P of dimension $q - 1$ consider the action of the group P on the set $\mathbb{F}_q$. To any element $p_{a,b} \in P$ we can associate an affine transformation $\tilde{p}_{a,b} : \mathbb{F}_q \to \mathbb{F}_q$ by

$$\tilde{p}_{a,b}(x) = ax + b, a \in \mathbb{F}_q^\star, b, x \in \mathbb{F}_q$$

Problem 1.16. *P:4 a) The map $p_{a,b} \to \tilde{p}_{a,b}$ defines an action of the group P on $\mathbb{F}_q$. In other words*

$$p_{a,b}\tilde{p}_{a',b'} = \tilde{p}_{a,b}\tilde{p}_{a',b'}$$

for any $p_{a,b}, p_{a',b'} \in P$

b) One can identify this action of P on $\mathbb{F}_q$ with the action of P on P/H.

Since the group P acts on the set $\mathbb{F}_q$ we obtain a homomorphism $\tilde{\rho} : P \to Aut\mathbb{C}[\mathbb{F}_q]$ by

$$\tilde{\rho}(p_{a,b})(f)(x) = f(\tilde{p}_{a,b}^{-1}(x), f \in \mathbb{C}[\mathbb{F}_q], x \in \mathbb{F}_q$$

[In other words $\tilde{\rho}(p_{a,b})(f)(x) = f(a^{-1}x - a^{-1}b)$.] Let $V \subset \mathbb{C}[\mathbb{F}_q]$ be the subspace of functions $f \in \mathbb{C}[\mathbb{F}_q]$ such that $\sum_{x\in\mathbb{F}_q} f(x) = 0$. Since the group P acts on the set $\mathbb{F}_q$ by permutaions the subspace $V \subset \mathbb{C}[\mathbb{F}_q]$ is P-invariant. We denote by $\rho' : P \to Aut(V)$ the corresponding subrepresentation. Since $dim(V) = q-1$ and the restriction of ρ' on U is not trivial it follows from Construction 1.15 that the representation $\rho' : P \to Aut(V)$ is irreducible and belongs to $[\rho]$.

Since the representations $\rho' : P \to Aut(V)$ and $\tau P \to GL(W)$ are equivalent and irreducible it follows from the Schur's lemma that the spaces $Hom_P(V, W)$ is one-dimensional. How to construct a non-trivial element of this space?

Since $P = HU, H \cap U = \{e\}$ we can identify W with the space of functions on H. The map $a \to p_{a,0}$ identifies H with $\mathbb{F}_q^\star$. This gives an identification of W with the space of functions on $\mathbb{F}_q^\star$.

Problem 1.17. *After this identification of W with the space of functions on $\mathbb{F}_q^\star$ we have*

$$(\tau(p_{a,b})f)(x) = \chi(bx)(f(ax), f \in W\; x \in \mathbb{F}_q^\star$$

Let $T : V' \to V''$ be the map given by $f \to \bar{f}, \bar{f}(a) := \mathcal{F}(f)(a), a \in \mathbb{F}_q^\star$.

Claim 1.18. *$T \in Hom_P(V', V'')$.*

Proof. Since elements $p_{1,b}$ and $p_{a,0}, a \in \mathbb{F}_q^\star, b \in \mathbb{F}_q$ generate the group P it is sufficient to check that $T \circ \rho'(p_{a,0}) = \rho''(p_{a,0}) \circ T$ for all $a \in \mathbb{F}_q^\star$ and $T \circ \rho'(p_{1,b}) = \rho''(p_{1,b}) \circ T$ for all $b \in \mathbb{F}_q$.

To show that $T \circ \rho'(p_{a,0})(f)(x) = \rho''(p_{a,0}) \circ T(f)(x)$ consider the function Let $f'(x) := \rho'(p_{a,0})(f)(x) = f(ax)$. Then

$$T \circ \rho'(p_{a,0})(f)(x) = \frac{1}{\sqrt{q}} \sum_{y\in\mathbb{F}_q} \psi(-xy)(f')(y) = \frac{1}{\sqrt{q}} \sum_{y\in\mathbb{F}_q} \psi(-xy) f(ay) =$$

$$\frac{1}{\sqrt{q}} \sum_{z\in\mathbb{F}_q} \psi(-xaz) f(z) = T(f)(ax) = \rho''(p_{a,0}) \circ T$$

Analogously $T \circ \rho'(p_{1,b})(f)(x) = T(f''(x))$ where $f''(x) = f(x+b)$. So

$$T \circ \rho'(p_{1,b})(f)(x) = \frac{1}{\sqrt{q}} \sum_{y\in\mathbb{F}_q} \psi(-xy)(f'')(y) = \frac{1}{\sqrt{q}} \sum_{y\in\mathbb{F}_q} \psi(-xy) f(y+b) =$$

$$\frac{1}{\sqrt{q}} \sum_{z\in\mathbb{F}_q} \psi(-x(z-b)) f(z) = \frac{1}{\sqrt{q}} \sum_{z\in\mathbb{F}_q} \psi(xb)\psi(-xz) f(z) = \rho''(p_{1,b}) \circ T$$

□

1.5. **The Heisenberg group.** In this section we assume that q is odd and fix a non-trivial character ψ of $\mathbb{F}_q$.

Let L be a 2-dimnsional $\mathbb{F}_q$-vector space, and $<,>: L \times L \to \mathbb{F}_q$ on L be a non-zero skew-symmetric bilinear form. If you fix a basis e_1, e_2 in L then

$< ae_1 + be_2, ce_1 + de_2 >= \alpha(ad - bc)$ where $\alpha =< e_1, e_2 >$

Let H be the product $L \times \mathbb{F}_q$. We define the group structure on H by

$$(l, a) \times (l', a') \to (l + l', a + a' + 1/2 < l, l' >)$$

I'll leave for you to check the parts a)-d) of the following

Claim 1.19. *C:H a) The map m defines a group structure on H such that $(0, 0)$ is the unit.*

b) The subgroup $Z := \{(0, a)\}, a \in \mathbb{F}_q$ is the center of H.

c) For any line $R \subset L$ the subset $\tilde{R} := R \times \mathbb{F}_q \subset H$ is a normal subgroup of H and

$$(r, a)(r', a') = (r + r', a + a'), r, r' \in R, a, a' \in \mathbb{F}_q$$

We denote by Ψ be the set of characters $\tilde{\psi} : \tilde{R} \to \mathbb{C}^\star$ of $\tilde{R}$ such that $\tilde{\psi}(0, a) = \psi(a)$ for any $a \in \mathbb{F}_q$. For any $h = (l, b) \in H$ and $\tilde{\psi} \in \Psi$ we define a character $\tilde{\psi}^h : \tilde{R} \to \mathbb{C}^\star$ by

$$\tilde{\psi}^h(\tilde{r}) := \tilde{\psi}(h^{-1}\tilde{r}h), \tilde{r} \in \tilde{R}$$

d) $\tilde{\psi}^h \in \Psi$ for all $\tilde{\psi} \in \Psi, h \in H$.

e) The group H acts transitively on the set Ψ.

Proof. of e). For any two characters $\tilde{\psi}, \tilde{\psi}' \in \Psi$ the ratio $\tilde{\psi}/\tilde{\psi}'$ is character χ of $\tilde{R}$ trivial on Z. So we can consider it as a character of $R = \tilde{R}/Z$. Since $h^{-1}\tilde{r}h = (r, a+ < r, l >), \tilde{r} = (r, a), h = (l, b)$ we have

$$\tilde{\psi}^h(r) = \psi(< l, r >)\tilde{\psi}(r)$$

As follows from Lemma 1.7 the map from L to characters of R given by $l \to \chi_l, \chi_l(r) := \psi(< l, r >)$ is surjective and we can find $l \in L$ such that $\tilde{\psi}/\tilde{\psi}'(r) = \psi(< l, r >)$. But then $\tilde{\psi}' = \tilde{\psi}^h, h = (l, 0)$. □

Corollary 1.20. *C:ind Let π be representation of H such that $\pi^R_\psi(0, a) = \psi(a)Id$ for any $a \in \mathbb{F}_q$. Then the restriction of π on $\tilde{R}$ contains $\tilde{\psi}$ for any $\tilde{\psi} \in \Psi$.*

Proof. Consider the restriction of π to $\tilde{R}$. Since the group $\tilde{R}$ is commutative there exists a character $\tilde{\psi}_0$ of $\tilde{R}$ which is contained in $res_{\tilde{R}}(\pi)$. Since $\pi_\psi^R(0,a) = \psi(a)Id$ for any $a \in \mathbb{F}_q$ we see that $\tilde{\psi}_0 \in \Psi$. The result follows now from the part e) of the Claim. □

Proposition 1.21. *P:Hei a) The induced representation $\pi_\psi^R = ind_{\tilde{R}}^H \tilde{\psi}$ does not depend on a choice of $\tilde{\psi} \in \Psi$, it is irreducible.*

b) $\pi_\psi^R(0,a) = \psi(a)Id$ for any $a \in \mathbb{F}_q$.

c) Any irreducible representation π of H such that $\pi_\psi^R(0,a) = \psi(a)Id$ for any $a \in \mathbb{F}_q$ is equivalent to π_ψ^R.

Proof. The part a) follows from the Collorary 0.15 in the section of induced representations and Claim 1.19. The part b) is clear since Z is [in] the center of H.

c) Let (π, V) be an irreducible representation of H such that $\pi_\psi^R(0,a) = \psi(a)Id$ for all $a \in \mathbb{F}_q$. We want to show that π is equivalent to $ind_{\tilde{R}}^H \tilde{\psi}$. Since both representations are irreducible it sufficient to show that $Hom_H(ind_{\tilde{R}}^H \tilde{\psi}, \pi) \neq \{0\}$. By the definition of an induced representation we have $Hom_H(ind_{\tilde{R}}^H \tilde{\psi}, \pi) = Hom_{\tilde{R}}(\tilde{\psi}, res_{\tilde{R}}\pi)$. But by Corollary 1.20 there is an $\tilde{R}$-invariant subspace $W \subset V$ such that $\tilde{R}$ acts on W by the multiplication by a character $\tilde{\psi}$. □

Example 1.22. Let's describe the construction of the representation π of H more explicitely. We choose a basis e_1, e_2 in L such that $< e_1, e_2 >= 1$. Then we identify elements of L with pairs $(x,y), x, y \in \mathbb{F}_q, < (x,y), (x',y') >= xy' - x'y$ and identify elements of H with triples $(x,y;a), x, y, a \in \mathbb{F}_q$ and $(x,y;a)(x',y';a') = (x+x', y+y'; a+a'+xy' - x'y/2)$. Let

$$R := \{(0,y)\}, y \in \mathbb{F}_q \subset L, \tilde{R} = \{(0,y;a)\}, y, a \in \mathbb{F}_q \subset H, S := \{(x,0;0)\}, x, a \in \mathbb{F}_q \subset H$$

Then $\tilde{R}S = H$ and $\tilde{R} \cap S = \{e\}$. Let V be the space of the induced represenation $ind_{\tilde{R}}^H \tilde{\psi}$. We can identify V with the space of functions $f : H \to \mathbb{C}$ such that $f((0,y';a')(x,y;a)) = \psi(a')f(x,y;a)$. The map $r : V \to \mathbb{C}[\mathbb{F}_q], r(f)(u) := f(u,0,0)$ identifies the space V with the space of functions of $\mathbb{F}_q$ and we obtain a representation $\pi : H \to Gl(\mathbb{C}[\mathbb{F}_q])$.

Claim 1.23. *C:rH* $\pi(x,y;a)(\phi)(u) = \psi(xy/2 + yu + a)\phi(x+u)$

Proof. Let $f = r^{-1}(\phi) \in V$. Since $(x,y;a) = (0,y;a+xy/2)(x,0;0)$ we have $f(x,y;a) = \psi(a+xy/2)\phi(x)$. So we have

$$\pi(x,y;a)(f)(u,0;0) = f((u,0;0)(x,y;a)) =$$

$$f(x+u, y, a+uy/2) = \psi(a+xy/2+yu)\phi(x+u)$$

□