1. Representation of algebras

All algebras $\mathcal{H}$ we consider are $\mathbb{C}$-algebras of countable or finite dimension. An $\mathcal{H}$-module $M$ is irreducible if it does not have proper $\mathcal{H}$-submodules.

**Lemma 0.1 (Schur’s Lemma).** If an algebra $\mathcal{H}$ has countable dimension then $\text{End}_{\mathcal{H}}(M) = \mathbb{C}$ for any irreducible $M \in \mathcal{M}(\mathcal{H})$.

**Proof.** Since $M$ is irreducible, $A := \text{End}_{\mathcal{H}} M$ is a skew-field. Since the algebra $\mathcal{H}$ is of countable dimension over $\mathbb{C}$ the space $M$ is also of countable dimension. Since $M$ is irreducible the map $A \to M, a \to am_0$ is an imbedding for any non-zero $m_0 \in M$ and therefore $A$ is also of countable dimension. So the Schur’s Lemma would follows from the following result.

**Claim 0.2.** If $A$ is a skew-field of countable dimension over $\mathbb{C}$, then $A = \mathbb{C}$.

**Proof.** We have to show that $a \in \mathbb{C}$ for any $a \in A$. Suppose that $a - \lambda \neq 0$ for all $\lambda \in \mathbb{C}$. Since $A$ has countable dimension, the elements $\{(a - \lambda)^{-1}, \lambda \in \mathbb{C}\}$ are linearly dependent. Thus, there exist non-zero complex numbers $c_i, 1 \leq i \leq k$ so that

$$\sum_{i=1}^{k} c_i (a - \lambda_i)^{-1} = 0.$$  

Multiplying through by $\prod_{i=1}^{k} (a - \lambda_i)$, we see that there exists a non-zero polynomial $Q(x) \in \mathbb{C}[x]$ such that $Q(a) = 0$. Factoring this polynomial, we see that there are $\mu_j \in \mathbb{C}$ so that

$$\prod_{j} (a - \mu_j) = 0.$$  

Since $A$ is a skew-field one of these factors must be equal to zero because. Hence $a \in \mathbb{C}$. □

**Definition 0.3.**

1. We denote by $Z(\mathcal{H})$ the the center the algebra $\mathcal{H}$.

2. By the Schur Lemma for an irreducible representation $M$ of an algebra $\mathcal{H}$ of countable dimension and $z \in Z(\mathcal{H})$ there exists $\chi_M(z) \in \mathbb{C}$ such that $z_M = \chi_M(z) Id_M$ where $z_M$ is the action of $z$ on $M$. We say that the map $\chi_M : Z(\mathcal{H}) \to \mathbb{C}$ is the central character of the module $M$.

**Remark 0.4.** The statement is false if we drop the condition that a skew-field $A$ has countable dimension. Really take $A = M = \mathbb{C}(x)$.

**Lemma 0.5.** Let $\mathcal{H}$ be an algebra of countable dimension over $\mathbb{C}$ with unit $e$. Then for any non-nilpotent element $a \in \mathcal{H}$ there exists a simple $\mathcal{H}$-module $M$ such that $a|_M \neq 0$.  


Proof. The proof is similar to that of Schur’s lemma. By adding the unit we may assume that the algebra $H$ is unital. First we establish the following result.

Claim 0.6. There exists $\lambda \in \mathbb{C} \setminus 0$ such that $a - \lambda$ is not invertible in $H$.

Proof. If $a \in \mathbb{C}$, this is trivial. Otherwise, by countable-dimensionality of $H$, the elements the $(a - \mu)^{-1}$ are linearly dependent. Thus there exists $c_i \in \mathbb{C}, 1 \leq i \leq k$ so that
\[
\sum_{i=1}^{k} c_i (a - \lambda_i)^{-1} = 0.
\]
Multiplying through by $\prod_{i=1}^{k} (a - \lambda_i)$, we get a non-zero polynomial over $\mathbb{C}$ with $a$ as a root. Thus, there are $\lambda_j \in \mathbb{C} \setminus 0$ and $n_j \geq 0$ so that
\[
a^{n_0} \prod_{j} (a - \lambda_j)^{n_j} = 0.
\]
But this is not possible since $(a - \lambda_j)$ are invertible and $a$ is not nilpotent.

Choose $\lambda \in \mathbb{C} \setminus 0$ such that $a - \lambda$ is not invertible in $H$. Then the quotient $H$-module $N := H/(a - \lambda)$ is not trivial. Let $\bar{e} \in N$ be the image of $e$ and $N_0 \subset N$ be a maximal submodule which does not contain $\bar{e}$.

Problem 0.7.
The quotient $M := N/N_0$ satisfies the conditions of the Lemma.

Definition 0.8. (1) For any $\mathbb{C}$-algebra $H$ we denote by $H^{op}$ the opposite algebra.
(2) For any $H$-module $M$ we define the action of $H^{op}$ on $M^\vee := \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$ by
\[
h\lambda(m) := \lambda(mh), m \in M, \lambda \in M^\vee, h \in H
\]
and define an action of $H \otimes H^{op}$ on $\text{End}_{\mathbb{C}}(M)$ by
\[
h' \otimes h''(A) := h'Ah''
\]
(3) We denote by $\alpha_M$ [or simply $\alpha$] the $H \otimes H^{op}$-morphism
\[
M \otimes M^\vee \to \text{End}_{\mathbb{C}}(M)
\]
given by
\[
\alpha(v \otimes \lambda)(w) := \lambda(w)v, v, w \in M, \lambda \in M^\vee
\]
(4) We say that a simple finite-dimensional $H$-module $M$ is compact if the action map $a_M : H \to \text{End}_{\mathbb{C}}(M)$ of $H \otimes H^{op}$-modules splits. That is there exists an imbedding $\mu_M : \text{End}_{\mathbb{C}}(M) \to H$ of $H \otimes H^{op}$-modules such that $a_M \circ \mu_M = Id$. 

Lemma 0.9. For any simple finite-dimensional compact \( H \)-module \( M \) we have a direct sum algebra decomposition

\[ H = H_M \oplus H_M^\perp \]

where \( H_M = \text{Im}(\mu_M) \), \( H_M^\perp = \text{Ker}(a_M) \).

Proof. Since we have direct sum algebra decomposition \( H = H_M \oplus H_M^\perp \) of \( H \) as a vector space and by the definition \( H_M^\perp = \text{Ker}(a_M) \subset H \) is a subalgebra we have to check that

(a) the map \( \mu_M \) is an algebra homomorphism and
(b) \( h_M h_M^\perp = 0 \) for all \( h_M \in H_M, h_M^\perp \in H_M^\perp \).

Proof of (a). Choose any \( r', r'' \in \text{End}_C(M) \). Since the map \( \mu_M \) is an morphism of \( H \otimes H \) -modules the product \( \mu_M(r') \mu_M(r'') \) is equal to \( \mu_M(r) \) for some \( r \in \text{End}_C(M) \) and \( a_M(r) = a_M(r') a_M(r'') \). Since \( a_M \circ \mu_M = \text{Id} \) we see that \( r = r'r'' \).

A proof of (b) is completely analogous.

Problem 0.10. (1) \( \alpha_M \) is an imbedding.
(2) Let \( M \) be a simple finite-dimensional compact \( H \)-module. The any \( H \)-module \( N \) admits unique decomposition \( N = N_0 \oplus N_1 \) such that \( N_0 \) is a multiple of \( M \) and \( N_1 \) does not have subquotients isomorphic to \( M \).

1.1. Idempotented Algebras.

Definition 0.11. (1) An algebra \( H \) is idempotented if for every finite collection \( \{ f_i \}, i \in I \) of elements of \( H \) there exists an idempotent \( e \in H \) such that \( ef_i = f_i e = f_i \) for all \( i \in I \).

Let \( H \) be an idempotented algebra.
(2) For any \( H \) -module \( M \) we define \( M_{sm} = H M \).
(3) A module \( M \) of an idempotented algebra \( H \) is called non-degenerate if \( M = M_{sm} \).
(4) We denote by \( \mathcal{M}(H) \) the abelian category of non-degenerate \( H \)-modules.
(5) For any \( M \in \mathcal{M}(H) \) we define \( \tilde{M} \in \mathcal{M}(H^{op}) \) by \( \tilde{M} := (M^\vee)_{sm} \).
(6) For any \( M \in \text{Ob}(\mathcal{M}(H)), N \in \text{Ob}(\mathcal{M}(H^{op})) \) we denote by \( \langle N, M \rangle \) the space of bilinear maps \( \phi : N \times M \to \mathbb{C} \) such that \( \phi(nh, m) \equiv \phi(n, hm) \).
(7) For any \( M \in \text{Ob}(\mathcal{M}(H)), N \in \text{Ob}(\mathcal{M}(H^{op})) \) we denote by

\[ \kappa_{M,N} : \text{Hom}_H(M, \tilde{N}) \to \langle N, M \rangle \]

the map given by

\[ \kappa_{M,N}(A)(n, m) := A(m)(n), m \in M, n \in N, A \in \text{Hom}_H(M, \tilde{N}) \]

and denote by \( \kappa_{N,M} : \text{Hom}_{H^{op}}(N, \tilde{M}) \to \langle N, M \rangle \) the map given by

\[ \kappa_{N,M}(B)(n, m) := B(n)(m), m \in M, n \in N, B \in \text{Hom}_H(N, \tilde{M}). \]
A non-degenerate $\mathcal{H}$-module $M$ is admissible if $\dim_{\mathbb{C}}(eM) < \infty$ for any idempotent $e \in \mathcal{H}$.

**Lemma 0.12.** The functor $M \to M_{sm}$ from the category of $\mathcal{H}$-modules to $\mathcal{M}(\mathcal{H})$ is exact.

**Proof.** It is clear that for any imbedding $M \hookrightarrow N$ of $\mathcal{H}$-modules the induced map $M_{sm} \to N_{sm}$ is an imbedding. So the functor $M \to M_{sm}$ is left exact.

To show that this functor is right exact we have to check that for any surjection $p : M \to N$ and any $n \in N_{sm}$ we can find $m \in M_{sm}$ such that $n = p(m)$.

Since $n \in N_{sm}$ there exists an idempotent $e \in \mathcal{H}$ such that $en = n$. On the other hand since the map $p : M \to N$ is surjective there exists $m' \in M$ such that $n = p(m')$. Take $m := em' \in M_{sm}$. Then $p(m) = p(em') = ep(m') = en = n$.

**Problem 0.13.**

1. Let $M_\infty$ be the algebra of matrices $(m_{i,j})$, $1 \leq i, j < \infty$ such that $m_{i,j} = 0$ for almost all pairs $i, j$. Show that the algebra $M_\infty$ is idempotented.

2. For any totally disconnected topological space $X$ the algebra $S(X)$ of local constant functions with compact support is idempotented.

3. For any $M \in \text{Ob}(\mathcal{M}(\mathcal{H})), N \in \text{Ob}(\mathcal{M}(\mathcal{H}^{op}))$ the maps $\kappa_{M,N}$ and $\kappa_{N,M}$ are bijections.

4. For any $M \in \text{Ob}(\mathcal{M}(\mathcal{H})), N \in \text{Ob}(\mathcal{M}(\mathcal{H}^{op}))$ we have a canonical isomorphism $\text{Hom}_{\mathcal{H}}(M, \tilde{N}) \to \text{Hom}_{\mathcal{H}^{op}}(N, \tilde{M})$.

5. Show that the correspondence $M \to \tilde{M}$ defines an exact contravariant functor from $\mathcal{M}(\mathcal{H})$ to $\mathcal{M}(\mathcal{H}^{op})$.

6. Construct a canonical morphism $V \to \tilde{V}$ and show that it is an imbedding.

### 1.2. Projective and Injective modules.

Recall that an object $P$ of an abelian category $\mathcal{M}$ is *projective* if the functor

$$\mathcal{M} \to \text{Ab} \quad \text{given by} \quad X \mapsto \text{Hom}_\mathcal{M}(P, X)$$

is exact.

Analogously an object $P$ is *injective* if the functor

$$\mathcal{M} \to \text{Ab} \quad \text{given by} \quad X \mapsto \text{Hom}_\mathcal{M}(X, I)$$

is exact.

**Lemma 0.14.** For any projective object $P \in \text{Ob}(\mathcal{M}(\mathcal{H}))$ the object $\tilde{P} \in \mathcal{M}(\mathcal{H}^{op})$ is injective.
PROOF. We must show that the functor $X \mapsto \text{Hom}(X, \tilde{P})$ on $\mathcal{M}(\mathcal{H}^{\text{op}})$ is exact. As follows from the previous Problem we have isomorphisms
\[ \text{Hom}(X, \tilde{P}) = \text{Hom}(P, \tilde{X}) \]

Since $P$ is projective and the functor $X \to \tilde{X}$ is exact we see that $\tilde{P}$ is injective.

THEOREM 0.15. For any idempotented algebra $\mathcal{H}$ the category $\mathcal{M}(\mathcal{H})$ has enough projective and injective objects.

PROOF. We start with the proof for the existence of a projective cover $P \to M$ for an $\mathcal{H}$-module $M$. For any idempotent $e \in \mathcal{H}$ the functor $X \to \text{Hom}(P_e, X) = eX$ is exact and therefore the $\mathcal{H}$-module $P_e = \mathcal{H}e$ is projective. Since direct sums of projective objects are projective direct sums of any collection of the modules of the form $P_e$ are also projective.

If $X \in \text{Ob} \mathcal{M}(\mathcal{H})$ and $\xi \in X$, then it follows from non-degeneracy that there exists an idempotent $e$ so that $e\xi = \xi$. Then $\xi$ is in the image of the map $P_e \to X$ given by $he \mapsto h\xi$. Taking the direct sum over all $\xi \in X$ of the associated $P_e$, we see that $X$ is a quotient of a projective object.

Now we want to construct an imbedding $M \hookrightarrow I$ of an $\mathcal{H}$-module $M$ into an injective object. As we have enough projectives, there is an epimorphism $P \to M$. Now consider the composition
\[ M \hookrightarrow \tilde{M} \hookrightarrow \tilde{P} \]
where $\tilde{P}$ is injective by Lemma 0.14.

For any idempotented algebra $\mathcal{H}$ we define $\hat{\mathcal{H}} := \text{End}_{\mathcal{H}^{\text{op}}}(\mathcal{H})$. Since $\mathcal{H}$ is associative we have a natural imbedding $\mathcal{H} \hookrightarrow \hat{\mathcal{H}}$.

LEMMA 0.16. For any idempotented algebras $\mathcal{H}$ and a non-degenerate $\mathcal{H}$-module $M$ the action of $\mathcal{H}$ on $M$ extends uniquely to an action of $\hat{\mathcal{H}}$ on $M$.

PROOF. To construct a map $\alpha : \hat{\mathcal{H}} \to \text{End}_{\mathbb{C}}(M)$ we have to define $\alpha(\hat{h})(m)$ for $\hat{h} \in \hat{\mathcal{H}}$, $m \in M$.

We choose an idempotent $e \in \mathcal{H}$ such that $em = m$ [this is possible since $M$ is non-degenerate] and write $\alpha_e(\hat{h})m := \hat{h}(e)m$.

Let us show that $\alpha_e(\hat{h})m$ does not depend on a choice of an idempotent $e \in \mathcal{H}$ such that $em = m$. Since for any two idempotents $e', e'' \in \mathcal{H}$ there exists an idempotent $e \in \mathcal{H}$ such that $ee'e = e', ee''e = e''$ it is sufficient to show that $\alpha_e(\hat{h})(m) = \alpha_{e'}(\hat{h})(m)$ for idempotents $e', e$ such that $ee' = e'$. But in the case we have
\[ \alpha_{e'}(\hat{h})(m) = \hat{h}(e')(m) = \hat{h}(ee')(m) = \hat{h}(e)e'm = \hat{h}(e)m = \alpha_e(\hat{h})(m) \]

Since the element $\alpha_e(\hat{h})m$ does not depend on a choice of an idempotent $e \in \mathcal{H}$ such that $em = m$ we will write $\alpha(\hat{h})(m)$ instead of $\alpha_e(\hat{h})(m)$.

The uniqueness of an extension $\hat{\mathcal{H}} \to \text{End}_{\mathbb{C}}(M)$ of the action map $\alpha : \mathcal{H} \to \text{End}_{\mathbb{C}}(M)$ follows immediately from the following result.
Problem 0.17. For any \( \hat{h} \in \hat{\mathcal{H}}, h \in \mathcal{H} \subset \hat{\mathcal{H}} \) we have \( \hat{h}h = (h) \in \mathcal{H} \subset \hat{\mathcal{H}} \).

\[ \square \]

Problem 0.18. Describe the algebras \( \hat{M}_\infty \) and \( \hat{S}(X) \).

Lemma 0.19. Let \( M \) be an irreducible \( \mathcal{H} \)-module, \( e \in \mathcal{H} \) be an idempotent and \( \mathcal{H}_e := e\mathcal{H}e \). Then

1. Either \( eM = \{0\} \) or \( eM \) is an irreducible \( \mathcal{H}_e \)-module.
2. Every irreducible \( \mathcal{H}_e \)-module has a form \( eM \) for some irreducible \( M \in \mathcal{M}(\mathcal{H}) \).

Proof. (1) Assume that \( eM \neq \{0\} \). To prove the irreducibility of the \( \mathcal{H}_e \)-module \( eM \) it is sufficient to show that for any \( w, v \neq 0 \) in \( eM \) there exists \( h \in \mathcal{H}_e \) such that \( v = hw \). Since the \( \mathcal{H} \)-module \( M \) is irreducible there exists \( h \in \mathcal{H} \) such that \( v = hw \). But now we can take \( \bar{h} = ehe \in \mathcal{H}_e \).

(2) Let \( \tilde{M} \) be an irreducible \( e\mathcal{H}e \)-module. As follows from (1) it is sufficient to show the existence of an irreducible \( \mathcal{H} \)-module \( L \) such \( \tilde{M} \) is a submodule of \( eL \). Set \( N = \mathcal{H} \otimes e\mathcal{H}e \). The imbedding \( \mathcal{H}_e \subset \mathcal{H} \) induces the inbeddings \( e\mathcal{H}_e \subset e\mathcal{H}e \). Consider the partially ordered set \( X \) of proper \( \mathcal{H}_e \)-submodules \( N_x \subset N \) where the ordering is by inclusion. Since the \( \mathcal{H}_e \)-module \( \tilde{M} \) is irreducible and \( N = \mathcal{H}\bar{N} \) we see that \( N_x \cap \tilde{M} = \{0\} \) for all \( x \in X \). Therefore the partially ordered set satisfies the conditions of the Zorn’s lemma and there exists a maximal proper submodule \( N_0 \subset N \).

Let \( L := N/N_0 \). Since \( N_0 \cap \tilde{M} = \{0\} \) the projection \( p : N \to L \) defines an imbedding \( \tilde{M} \to eL \) of \( \mathcal{H}_e \)-modules.

I claim that the \( \mathcal{H} \)-module \( L \) is irreducible. Really if \( L' \subset L \) is a non-zero proper \( \mathcal{H} \)-submodule of \( L \) the preimage \( p^{-1}(L') \subset N \) is a proper \( \mathcal{H}_e \)-submodule of \( N \) strictly bigger then \( N_0 \). But this is impossible since \( N_0 \) is a maximal proper submodule of \( N \).

We introduce a notion of admissible and compact modules which will be central for our analysis of representations of groups over local non-archimedian fields.

Definition 0.20. Let \( \mathcal{H} \) be an idempotented algebra.

1. A non-degenerate \( \mathcal{H} \)-module \( M \) is admissible if for every idempotent \( e \in \mathcal{H} \) the space \( eM \) is finite dimensional.
2. An admissible \( \mathcal{H} \)-module \( M \) is compact if for any idempotent \( e \in \mathcal{H} \) the finite dimensional \( e\mathcal{H}e \)-module \( eM \) is compact.

Problem 0.21. (1) \( M \) is admissible if the natural imbedding \( M \to \tilde{M} \) is an isomorphism.

(2) For any compact \( \mathcal{H} \)-module \( M \) the direct sum decompositions (see 0.10)

\[ \mathcal{H}_e = \mathcal{H}_M \oplus \mathcal{H}_M^\perp \]
where \( e \) run through idempotents \( e \in \mathcal{H} \) define a direct sum algebra decomposition

\[
\mathcal{H} = \mathcal{H}_M \oplus \mathcal{H}_M^\perp
\]
such that \( \mathcal{H}_M^\perp \) acts trivially on \( M \) and the restriction of the action map \( \mathcal{H} \to \text{End}(M) \) on \( \mathcal{H}_M \) is an imbedding.

**Lemma 0.22.** For any admissible \( \mathcal{H} \)-module \( M \) the imbedding [see 0.10] \( \alpha_M : M \otimes \tilde{M} \to \text{End}(M)_{\text{sm}} \) is an isomorphism where we consider \( \text{End}(M) \) as \( \mathcal{H} \otimes \mathcal{H}^{\text{op}} \)-module.

**Proof.** The statement is clear in the case when \( \text{dim}(M) < \infty \). We use the admissibility for a reduction to this case.

By 0.10 the map \( \alpha_M \) is an imbedding. So it is sufficient to show that for any idempotent \( e \in \mathcal{H} \) the map

\[
\alpha_e : eM \otimes e\tilde{M} \to e\text{End}(M)e
\]
is onto. As \( \alpha_e \) is an imbedding it is sufficient to show that

\[
\text{dim}(e\text{End}(M)e) \leq \text{dim}(eM) \times \text{dim}(e\tilde{M})
\]
Since t the restriction \( e\text{End}(M)e \to \text{End}(eM) \) is an imbedding this inequality follows from the equalities

\[
\text{dim}(e\tilde{M}) = \text{dim}(eM), \text{dim}(\text{End}(eM)) = \text{dim}^2(e\tilde{M}).
\]

**Corollary 0.23.** For any irreducible admissible \( \mathcal{H} \)-module \( M \) we have

\[
\text{dim}(\text{Hom}_{\mathcal{H} \otimes \mathcal{H}^{\text{op}}}((\mathcal{H}, M \otimes \tilde{M})) = 1
\]

**Proof.** Since we have a natural non-trivial map \( \mathcal{H} \to \text{End}(M)_{\text{sm}} \) we see that

\[
\text{dim}(\text{Hom}_{\mathcal{H} \otimes \mathcal{H}^{\text{op}}}((\mathcal{H}, M \otimes \tilde{M})) \neq 0
\]
So it is sufficient to show that \( \text{dim}(\text{Hom}_{\mathcal{H} \otimes \mathcal{H}^{\text{op}}}((\mathcal{H}, M \otimes \tilde{M})) \leq 1 \). For this it is sufficient to show that \( \text{dim}(\text{Hom}_{\mathcal{H} \otimes \mathcal{H}^{\text{op}}}((\mathcal{H}, eM \otimes e\tilde{M})) \leq 1 \) for all idempotents \( e \in \mathcal{H} \). But follows immediately from Lemma 0.22.

### 1.3. Irreducible modules and the Jordan-Holder Content.

Let \( \mathcal{H} \) be an idempotented algebra.

**Definition 0.24.** (1) We denote by \( \text{Irr}_\mathcal{H} \) be the set of equivalence classes of irreducible representations of the algebra \( \mathcal{H} \).

(2) If \( M \in \mathcal{M}(G) \), then the *Jordan-Holder content* of \( M \), \( JH(M) \), is the subset of \( \text{Irr}_\mathcal{H} \) consisting of all irreducible subquotients of \( M \).

**Lemma 0.25.** \( JH(M) \neq \emptyset \) for any non-zero \( \mathcal{H} \)-module \( M \).

**Proof.** Fix any non-zero \( m \in M \). By the Zorn’s lemma there exists a maximal proper submodule \( N \) of the \( \mathcal{H} \)-module \( M_0 := \mathcal{H}m \subset M \). By the construction the quotient \( M_0/N \) is irreducible. So \([M_0/N] \in JH(M)\) where \([M_0/N] \) is the class of \( M_0/N \).
Problem 0.26. (1) If $N$ is a subquotient of $M$, then $JH(N) \subset JH(M)$.
(2) If $M = \sum \alpha M_\alpha$ then $JH(M) = \cup \alpha JH(M_\alpha)$.

1.4. Decomposing Categories.

Definition 0.27. Let $\mathcal{M}$ be an abelian category such that the coproducts exists in $\mathcal{M}$ (i.e. $\mathcal{M}$ is cocomplete) and the coproduct of a family of monomorphisms is a monomorphism. [for example a category of modules over an algebra]. Given full subcategories $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{M}$ we write $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ if for any object $M \in \mathcal{M}$, there exist unique subobjects $M_i \in \mathcal{M}_i$ so that $M = M_1 \oplus M_2$.

In such a case any irreducible object either belongs to $\mathcal{M}_1$ or to $\mathcal{M}_2$ and we obtain a decomposition $\text{Irr}\mathcal{M} = \text{Irr}\mathcal{M}_1 \amalg \text{Irr}\mathcal{M}_2$.

where $\amalg$ is the ‘disjoint union’. Conversely, such a decomposition on the level of sets uniquely determines the categorical decomposition if it exists.

Lemma 0.28. If subsets $S, S' \subset \text{Irr}\mathcal{M}$ do not intersect, then the categories $\mathcal{M}(S)$ and $\mathcal{M}(S')$ are orthogonal, i.e. $M \in \mathcal{M}(S)$ and $M' \in \mathcal{M}(S')$ imply $\text{Hom}(M, M') = 0$.

Proof. Suppose $\alpha \in \text{Hom}(M, M')$. Set $N = \alpha(M)$. So, $JH(N) \subset JH(M) \subset S$ and also $JH(N) \subset JH(M') \subset S'$. But $S \cap S' = \emptyset$ so $N = 0$ by 0.25.

If $S \subset \text{Irr}\mathcal{M}$, $M \in \mathcal{M}$, we will denote by $M(S)$ the union of all subobjects of $M$ which lie in $\mathcal{M}(S)$. By the lemma, this is the maximal submodule with Jordan-Hölder content lying in $S$.

Definition 0.29. Let $S \subset \text{Irr}\mathcal{M}$ and $S' := \text{Irr}\mathcal{M} \setminus S$. We say that $S$ is a splitting subset if $\mathcal{M} = \mathcal{M}(S) \oplus \mathcal{M}(S')$ [that is, if $M = M(S) \oplus M(S')$ for each $M \in \mathcal{M}$]. In this case we say that $S$ splits $\mathcal{M}$.

Remark 0.30. A decomposition of categories $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is equivalent to a decomposition of sets $\text{Irr}\mathcal{M} = S \coprod S'$ where $S$ is a splitting subset.

Problem 0.31. Let $\mathcal{H}$ be an idempotented algebra and $M$ an admissible compact $\mathcal{H}$-module. Then $[M]$ splits $\mathcal{M}(\mathcal{H})$ where $[M] \in \text{Irr}\mathcal{M}(\mathcal{H})$ is the equivalence class of $M$.

2. $l$-Groups

2.1. Basic definitions.

Definition 0.32. (1) An $l$-space is a topological space which is Hausdorff, locally compact and 0-dimensional (i.e. totally disconnected: any point has a basis of open compact neighborhoods).
(2) For any \( l \)-space we denote by \( S(X) \) the vector space of locally constant compactly supported \( \mathbb{C} \)-valued functions on \( X \).

(3) An \( l \)-group is a Hausdorff topological group such that the identity \( e \) has a basis of neighborhoods which are open compact groups. We will always assume that \( G \) is countable at infinity [that is we assume that for any open compact subgroup \( K \) of \( G \) the quotient \( G/K \) is either countable or finite.

(4) For an \( l \)-group we denote by \( r : G \to \text{Aut}(S(G)), l : G \to \text{Aut}(S(G)) \) the right and left regular representations of \( G \) on \( S(G) \) given by \( (r(x)f)(g) := f(gx), (l(x)f)(g) = f(x^{-1}g), f \in S(G), x, g \in G. \)

Problem 0.33. Show that:

(1) For any \( F \)-variety \( X \) the topological space \( X = X(F) \) is an \( l \)-space.

(2) The space \( \mathbb{P}^n(F) \) is compact.

(3) For any \( n > 0 \) the group \( GL(n, F) \) is an \( l \)-group and \( GL(n, \mathcal{O}) \) is an open compact subgroup of \( GL(n, F) \). Moreover \( GL(n, \mathcal{O}) \) is a profinite group.

(4) For any \( F \)-group \( G \) the group \( G = G(F) \) is an \( l \)-group.

(5) For any regular tree \( T \) we define a topology on the group \( G \) of automorphisms of \( T \) in such way that for any finite subset \( X \) of \( T \) the subgroup \( G_X \subset G \) of automorphisms of \( T \) fixing all points of \( X \) is open and shifts of \( G_X \) give a basis of the open sets on \( G \). Show that \( G \) is an \( l \)-group.

(6) Any compact \( l \)-group is a profinite group.

Lemma 0.34. Let \( G \) be an \( l \)-group. Then

(1) There exists a linear functional \( \int_r \) on \( S(G) \) such that \( \int_r r(g)f = \int_r f \) for all \( f \in S(G), g \in G. \) Such functional is unique up to a multiplication by a scalar. Moreover we can choose a linear functional \( \int_r \) to take positive values on non-zero non-negative functions.

(2) There exists a character \( \Delta : G \to \mathbb{R}_+ \) such that

\[
\int_r l(x)f = \Delta(x) \int_r f
\]

for any \( x \in G \) and any \( f \in S(G) \). In particular \( \int_r \chi_A = \Delta(x) \int \chi_A \) for any \( x \in G \) and any compact open subset \( A \) of \( G \) where \( \chi_A \) is the characteristic function of \( A \).

(3) The functional \( f \to \int_r \Delta^{-1} f \) is left-invariant.
Proof. To construct a linear functional \( \int_r \) we fix an open compact subgroup \( K_0 \) and define

\[
\int_r f := \frac{1}{|K_0/K|} \sum_{r \in K \setminus G} f(r)
\]

where \( K \subset K_0 \) is any open subgroup such that the function \( f \) is invariant under left shifts by \( k \in K \). I’ll leave for you to show that \( \int f \) does not depend on a choice of a subgroup \( K \subset K_0 \) and the map \( f \to \int_r f \) defines a right-invariant linear functional which is positive on non-zero non-negative functions.

Let \( K \subset K_0 \) be any open subgroup and \( \chi_K, \chi_{K_0} \subset \mathcal{S}(G) \) the characteristic functions of \( K \) and \( K_0 \). Since \( \chi_{K_0} = \sum_{x \in K_0/K} r(x) \chi_K \) we see that \( \lambda(\chi_K) = \frac{1}{|K_0/K|} \lambda(\chi_{K_0}) \) for any right-invariant linear functional \( \lambda \) on \( \mathcal{S}(G) \). This proves the uniqueness of \( \int_r \).

Let us fix a right-invariant linear functional \( \lambda \) on \( \mathcal{S}(G) \) and for any \( x \in G \) consider the linear functional \( \lambda_x \) given by

\[
\lambda_x(f) := \lambda(l(x)(f)).
\]

By the construction the functional \( \lambda_x \) on \( \mathcal{S}(G) \) is also right-invariant and therefore there exists constant \( \Delta(x) \) such that \( \lambda_x(f) = \Delta(x) \lambda(f) \) for all \( f \in \mathcal{S}(G) \).

Since we may assume that \( \lambda \) is positive on non-zero non-negative functions we see that \( \Delta(x) \in \mathbb{R}_+ \). It is clear from the definition that the function \( \Delta : G \to \mathbb{R}_+ \) is a character.

The left-invariantness of the functional \( f \to \int_r \Delta^{-1} f \) is clear.

Problem 0.35.

(1) Let \( \mathbb{G}_a = F \) be the additive group. Then there exists a Haar measure \( dg_a \) on \( \mathbb{G}_a \) such that \( \int_{F^r} dg = q^{-r}, r \in \mathbb{Z} \).

(2) Let \( \mathbb{G}_m = F^* \) be the multiplicative group. Then \( dg_m := dg_a/\|\| \) is a Haar measure on \( \mathbb{G}_m \).

(3) Let \( H \subset GL(2, F) \) be the group of upper triangular \( 2 \times 2 \) matrices of the form

\[
h = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}
\]

Then the right -invariant Haar measure on \( H \) measure is equal to

\[
dg_m(\alpha)dg_m(\beta)dg_a(\gamma)/\|\|\]

(4)

\[
\Delta\left(\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}\right) = \|\alpha\|/\|\beta\|
\]

Definition 0.36.

(1) The function \( \Delta \) is called the modular character.

(2) A group is unimodular if \( \Delta \equiv 1 \). In this case right -invariant measures are also left-invariant. We denote such a measure by \( dg \) and call a Haar measure.

(3) Let \( U \) be a unimodular locally compact \( l \)-group \( du \) a Haar measure on \( U \) and \( \sigma \) an automorphism of \( U \). Then \( \sigma_*(du) \) is also a Haar measure on \( U \) and we define \( \text{mod}_U(\sigma) \in \mathbb{R}_+ \) by \( \sigma_*(du) = \text{mod}_U^{-1}U(\sigma)du \).
If \( P = M \ltimes U \) we write \( \text{mod}_U(m) := \text{mod}_U(\sigma_m) \) where \( \sigma_m(u) := mu_m^{-1} \).

(5) If a group \( G \) is compact we normalize a Haar measure \( dg \) in such a way that \( \int_G dg = 1 \).

Let \( G \) be a unimodular \( l \)-group with a Haar measure \( dg \).

**Problem 0.37.**

1. If \( P = M \ltimes U \) where both \( M \) and \( U \) are unimodular then \( \Delta_P(mu) = \text{mod}_U(m) \).

2. Let \( K \) be a compact \( l \)-group, \( K', K'' \subset K \) closed subgroups such that \( K'K'' = K \). Then for any \( f \in \mathcal{H}(K) \) we have
   \[
   \int_K f(k)dk = \int_{K'} f'(k')dk' \quad \text{where} \quad f'(k') = \int_{K''} f(k'k'')dk''.
   \]

3. Find the modular character for the group of \( n \times n \)-upper triangular matrices with coefficients in \( \mathbb{Q}_p \).

4. Prove that the group \( \text{GL}(n, F) \) is unimodular and describe the Haar measure \( dg \) on \( \text{GL}(n, F) \) such that \( \int_{\text{GL}(n, F)} dg = 1 \).

Let \( H \) be an \( l \)-group and \( \lambda \) be a left-invariant linear functional on \( \mathcal{S}(H) \). Let \( X \) be a principle homogeneous \( H \)-space [that is \( H \) is acting on an \( l \)-space \( X, (h, x) \rightarrow hx \) simply transitively]. Then \( H \) acts on the space \( \mathcal{S}(X), (h, f) \rightarrow f_h, f_h(x) := f(h^{-1}x) \) and we denote by \( \mathcal{L} \) the space of coinvariants. So \( \mathcal{L} = \mathcal{S}(X)/\mathcal{S}_0(X) \) where \( \mathcal{S}_0(X) \subset \mathcal{S}(X) \) is the span of \( \{f - f_h \}, f \in \mathcal{S}(X), h \in H \).

**Lemma 0.38.**

1. The space \( \mathcal{L} \) is one-dimensional.

2. The space \( \mathcal{L} \) is canonically isomorphic to the space \( \mathcal{L}_\Delta \) of functions \( r \) on \( X \) such that \( r(hx) \equiv \Delta(h)r(x), h \in H, x \in X \).

**Proof.** Any point \( x \in X \) defines a bijection \( \phi_x : H \rightarrow X, h \rightarrow hx \) which induces an isomorphism \( \phi_x^* : \mathcal{S}(X) \rightarrow \mathcal{S}(H) \) and the first claim follows from 0.34.

Consider now the linear map \( \kappa : \mathcal{S}(X) \rightarrow \mathcal{S}(X) \) given by
\[
\kappa(f)(x) := \lambda(\phi_x^*(f))
\]
It is clear that \( \kappa \) is trivial on the subspace \( \mathcal{S}_0(X) \subset \mathcal{S}(X) \) and it follows from 0.34 that the image of \( \kappa \) lies in \( \mathcal{L}_\Delta \). Therefore \( \kappa \) defines an isomorphism \( \mathcal{L} \rightarrow \mathcal{L}_\Delta \).

**Remark 0.39.** To give a more conceptual explanation of this result consider the group \( \text{Aut}_H(X) \) of automorphisms of \( X \) which commute with the action of \( H \). It is clear that the right shifts define a canonical isomorphism of the group \( H \) with \( \text{Aut}_H(H) \). Therefore for any point \( x \in X \) the isomorphism \( \phi_x : H \rightarrow X \) defines an isomorphism \( \phi_x^* : \mathcal{L} \rightarrow \mathcal{L}_\Delta \) and it is easy to see that for any two points \( x, y \in X \) the isomorphisms \( \phi_x, \phi_y : H \rightarrow \text{Aut}_H(X) \) differ by the conjugation by \( xy^{-1} \). Since the character \( \Delta \) is invariant under the conjugation we can consider it as a character of the group \( \text{Aut}_H(X) \). By the definition the group \( \text{Aut}_H(X) \) acts on the space \( \mathcal{L} \) of coinvariants. The second part of the lemma says that the action is given by \( \Delta : \text{Aut}_H(X) \rightarrow \mathbb{R}_+ \).
Assume that an \( l \)-group \( H \) acts freely on an \( l \)-space \( X \) and denote by \( S_0(X) \subset S(X) \) be the span of \( \{ f - f_h \}, f \in S(X), h \in H \) and by \( S(X)_H := S(X)/S_0(X) \) be the space of \( H \)-coinvaraints of \( S(X) \). Assume also that a group \( G \) acts on \( S(X) \) and this action of \( G \) commutes with the natural action of \( H \) on \( S(X) \).

**Corollary 0.40.**  (1) The space \( S(X)_H \) is canonically isomorphic to the space \( M(H\setminus X) \) of locally constant functions \( r \) on \( X \) such that \( r(hx) \equiv \Delta(h)r(x), h \in H, x \in X \) and such \( \text{supp}(r) \subset HC \) for some compact \( C \subset X \).

(2) For any \( H \)-invariant linear functional \( \lambda \) on \( S(X) \) there exists unique linear functional \( \bar{\lambda} \) on \( S(X)_H \) such that
\[
\bar{\lambda}(\bar{f}) \equiv \lambda(f), f \in S(X)
\]
where \( \bar{f} \) is the image of \( f \) in \( \bar{S}(X) \).

(3) The group \( G \) acts naturally on \( \bar{S}(X) \).

(4) \( \lambda \) is a \( G \)-invariant functional on \( S(X) \) then \( \bar{\lambda} \) is a \( G \)-invariant functional on \( S(X)_H \).

**Definition 0.41.** Let \( H \) be a closed subgroup of an unimodular group \( G \) and \( \Delta_H \) be the modular character of \( H \). Then the restriction of the functional \( \int \) to the subspace \( S_0(X) \subset S(X) \) vanishes and therefore the functional \( \int_f \) defines a \( G \)-invariant linear functional \( \int \) on the the space \( M(H\setminus G) = S(X)_H \).

**Lemma 0.42.** Assume that \( K \) is an open compact subgroup of \( G \) such that \( G = HK \). Then any \( K \)-invariant linear functional on \( M(H\setminus G) \) is proportional to \( \int \).

**Proof.** Since the linear functional \( \int \) on \( M(H\setminus G) \) is \( K \)-invariant it is sufficient to show that all \( K \)-invariant linear functionals on \( M(H\setminus G) \) are proportional.

Since \( G = HK \) the restriction map defines an isomorphism of \( K \)-vector spaces \( M(H\setminus G) \) and \( M(H\cap K\setminus K) \) and the claim follows from the uniqueness of the Haar measure. \( \square \)

**Problem 0.43.** (1) Show that in the case when \( G = SL(2, F) \) and \( H \subset G \) is the subgroup of upper triangular matrices. In this case we can identify the space \( M(H\setminus G) \) with the space \( V \) of locally constant functions \( f \) on \( F^2 - \{0\} \) such that \( f(cx, cy) = 1/||c||f(x, y) \) and that the functional \( \int \) on \( V \) is given by
\[
\int(f) = \int_Z f(x, y)dx\,dy
\]
where \( Z = \{(x, y) | \text{ max}(||x||, ||y||) = 1\} \).

(2) Explain why this result follows from Lemma 0.42 if we take \( K = SL(2, \mathcal{O}) \).
2. L-GROUPS

**Definition 0.44.** Let $G$ be a unimodular $l$-group with a Haar measure $dg$.

1. We denote by $\mathbb{C}(G)$ the space of $\mathbb{C}$-valued locally constant functions $\phi$ on $G$.

2. For any $f \in \mathcal{S}(G)$ we consider $f dg$ as a linear functional on the space $\mathbb{C}(G)$ given by
   
   $$fdg(\phi) := \int_G f\phi, \phi \in \mathbb{C}(G)$$

3. We denote by $\mathcal{H}(G)$ the space of linear functionals $h$ the space $\mathbb{C}(G)$ on $G$ of the form $h = fdg, f \in \mathcal{S}(G)$. and say that elements of $\mathcal{H}(G)$ are locally constant measures on $G$ with compact support.

4. Since $G$ is a unimodular $l$-group the left and right regular representations $l, r : G \rightarrow \text{Aut}(\mathcal{S}(G))$ define the left and right regular representations $G \rightarrow \text{Aut}(\mathcal{H}(G))$ which we also denote by $l$ and $r$.

5. For any $h' = f' dg', h'' = f'' dg'' \in \mathcal{H}(G')$ we define
   
   $$h' \square h'' := f'(g')f''(g'')dg'dg'' \in \mathcal{H}(G' \times G'')$$

6. We denote by $(h', h'') \rightarrow h' \ast h''$ the convolution
   
   $$h' \ast h'' := m_*(h' \square h'')$$

   where $m : G \times G \rightarrow G$ is the product map. In other words
   
   $$h' \ast h''(\phi) := h' \square h''(m^*(\phi)), \phi \in \mathbb{C}(G)$$

7. The algebra $\mathcal{H}(G)$ is called the **Hecke algebra** of $G$.

8. For any compact open subgroup $K \subset G$ we denote by $e_K \subset \mathcal{H}(G)$ the idempotent given by the Haar measure of $K$ and write $\mathcal{H}_K(G) = e_K \mathcal{H}(G)e_K$.

9. For any compact not necessarily open subgroup $K$ of $G$ we denote by $e_K \in \hat{K}$ the endomorphism of $\mathcal{H}$ defined by
   
   $$e_K(h) = m_*(e_K \square h)$$

   In other words if $h = fdg$ then $e_K(h) = f' dg$ where $f'(x) = \int_K f(kx)$.

10. For any $x \in G$ we denote by $\mathcal{E}_x \in \hat{K}$ the endomorphism of $\mathcal{H}$ defined by $\mathcal{E}_x(h)(g) := l(x)(h), h \in \mathcal{H}(G)$ and say that $\mathcal{E}_x$ is the **delta function** at $x$.

11. We denote by $\int$ the linear functional on $\mathcal{H}(G)$ given by $\int h := h(1)$.

**Problem 0.45.**

1. If $h' = f' dg, h'' = f'' dg$ then $h' \ast h'' = f' \ast f'' dg$

   where
   
   $$f' \ast f''(g) := \int_{g' \in G} f'(g')f''(g'^{-1}g)dg$$

2. $\mathcal{H}(G)$ is an idempotent algebra.

3. $\int f' \ast f'' = \int f' \int f''$. 
(4) For any $g \in G$ and any compact open subgroup $K \subset G$ there exists unique $a(g) \in \mathcal{H}_K(G)$ such that $\text{supp} \ a(g) \subset KgK$ and $\int a(g) = 1$.

(5) The element $a(g) \in \mathcal{H}_K(G)$ depends only on the double coset $KgK \in K \backslash G \backslash K$ and the set $\{a(g)\}, gK \in K \backslash G \backslash K$ is a basis of $\mathcal{H}_K(G)$.

(6) For any $n \in \mathbb{N}$ and any $g \in G$ we have $a(ng) = a(n)a(g), a(gn) = a(g)a(n)$.

(7) Let $K, K', K'' \subset G$ be compact subgroups such that $K'K'' = K$. The $e_{K'}e_{K''} = e_K$.

(8) For any $x \in G$ and any compact subgroup $K \subset G$ we have $e_xKe^{-1} = \mathcal{E}_x \in \mathcal{E}_x: e^{-1}$.

(9) $\mathcal{H}(G)$ has countable dimension.

2.2. Representations of $l$-groups.

Definition 0.46. Let $G$ be a unimodular $l$-group with a Haar measure $dg$.

(1) A representation $\pi : G \to \text{Aut}(V)$ of the group $G$ on a complex vector space $V$ is smooth if for any vector $v \in V$ the stabilizer $St_v \subset G$ of $v$ in $G$ is open.

(2) We denote by $\mathcal{M}(G)$ the category of smooth representations of $G$.

If $(\pi, V)$ is a smooth representation of $G$, we can give $V$ the structure of an $\mathcal{H}(G)$-module as follows. Given $E \in \mathcal{H}(G)$ and $v \in V$ we choose an open compact subgroup $K$ such that $E$ is right $K$-invariant and $Kv = v$. Since $E$ is right $K$-invariant and has compact support it is a finite sum of left shifts of the $\chi_K$

$$E = \sum_{i \in I} c_ig_i \chi_K, c_i \in \mathbb{C}, g_i \in G.$$ 

Now we define

$$E(v) := \sum_{i \in I} c_ig_iv.$$ 

In other words we constructed a functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$.

Proposition 0.47. Let $G$ be an $l$-group. The functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$ defines an equivalence of categories

$$\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H}(G))$$

between smooth representations of $G$ and non-degenerate $\mathcal{H}(G)$-modules.

Proof. Let $M$ be a smooth $\mathcal{H}$-module and $g \in G$. We want to define a representation $\pi : G \to \text{Aut}_\mathbb{C}(M)$. To define $\pi(g), g \in G$ consider the automorphism $\mathcal{E}_g$ of $\mathcal{H}(G)$ given by the left shift by $g$. The automorphism $\mathcal{E}_g$ commutes with the action of $\mathcal{H}^{op}(G)$ and therefore belongs to $\hat{\mathcal{H}}(G)$. Since by Lemma 0.16 the algebra $\hat{\mathcal{H}}(G)$ we can define $\pi(g) := \mathcal{E}_g M$.

Problem 0.48. Show that

(1) The assignment $g \to \pi(g)$ defines a representation of $G$ on $M$. 

(2) The functor \( M(G) \to M(H(G)) \) defines an equivalence of categories.

We will identify categories \( M(G) \) and \( M(H(G)) \) and write \( \text{Irr}(G) := \text{Irr}_{H(G)} \).

**Definition 0.49.** Let \( K \) be a compact subgroup of \( G \).

1. We denote by \( e_K \) an element of \( \text{End}_{H(G)}(G) \) given by the left convolution with the Haar measure on \( K \).
2. For any smooth \( G \)-module \( \pi : G \to \text{Aut}(V) \) we denote by \( \pi(e_K) \in \text{End}(V) \) the endomorphism defined as in Lemma 0.16.

**Problem 0.50.**

1. For any smooth \( G \)-module \( V \) the associated \( H(G) \)-module is non-degenerate.
2. Let \( C_l(G) \) be the space of \( C \)-valued functions on \( G \) and \( X(M) \) which are left-sided invariant under some open compact subgroup of \( G \). Construct an isomorphism \( C_l(G) = \hat{H}(G) \) where we consider \( H(G) \) as the left regular representation of \( G \).
3. Let \( V_1, V_2 \) be smooth representations of \( G \) and \( <,> : V_1 \times V_2 \to \mathbb{C} \) be a \( G \)-invariant pairing. There exists unique \( G \times G \)-equivariant morphism \( \kappa_{<,>} : V_1 \otimes V_2 \to \mathbb{C}(G) \) such that

   \[ \kappa_{<,>} : (v_1 \otimes v_2)(e) = <v_1, v_2> \]

   and the map \( <,> \to \kappa_{<,>} : \) defines a bijection between \( G \)-invariant pairings \( V_1 \times V_2 \to \mathbb{C} \) and \( G \times G \)-equivariant morphisms \( V_1 \otimes V_2 \to \mathbb{C}(G) \).
4. For any smooth representation \( (\pi, V) \) of \( G \) and any \( v \in V \) the map \( h \to \pi(h)v \) defines an element of the space \( \psi_v \in \text{Hom}_G(S(G), V) \). Show that for admissible representations \( (\pi, V) \) of \( G \) the map \( \psi_V : V \to \text{Hom}_G(S(G), V), v \to \psi_v \) is a bijection.
5. Consider the case \( G = \mathbb{G}_a(F) \) where \( F \) is a local non-archimedian field. Is the map \( \psi_{S(G)} \) a bijection?

The next lemma is a statement that our Hecke algebra resembles a semisimple algebra in a crucial sense.

**Lemma 0.51.** [Separation Lemma]. Suppose that \( G \) be an unimodular \( l \)-group countable at infinity. Then for any non-zero \( h \in H(G) \) there exists an irreducible representation \( \rho \) of \( G \) such that \( \rho(h) \neq 0 \).

**Remark 0.52.** The result is true for all locally compact groups and follows immediately from the theorem of Gelfand-Milman but in the case \( l \)-groups one can give a proof which does not use Functional Analysis.

**Proof.** Since \( G \) is unimodular we choose a Haar measure \( dg \) and identify \( H(G) \) with \( S(G) \), \( f \to fdg \). Consider the map \( H(G) \to H(G), h \mapsto h^+ \)
given by \( f(g)dg \rightarrow \bar{f}(g^{-1})dg \) where \( \bar{f}(g) \) is the complex conjugation of \( f(g) \) and define \( u = h \ast h^+ \). Then \( u = \psi dg \) where
\[
\psi(r) = \int_{g \in G} f(g)\bar{f}(r^{-1}g)dg
\]
Setting \( r = e \), it is obvious that \( \psi(e) \neq 0 \).

We have shown that \( h \neq 0 \) implies \( u \neq 0 \). So it is enough to find a representation \( \rho \) so that \( \rho(u) = \rho(h)\rho(h^+) \neq 0 \). Note that \( u^+ = u \). Thus
\[
u^2 = uu^+ = (hh^+)(hh^+)^+ \neq 0
\]
and more generally that \( u^n \neq 0 \). Choose an idempotent \( e \) of \( H \) such that \( e \ast u \ast e = u \). As follows from Lemma 0.19 it is sufficient to prove that there exists an irreducible representation \( \rho \) of the unital algebra \( eHe \) such that \( \rho(u) \neq 0 \). Since \( u \) is not nilpotent the Separation Lemma follows from Lemma 0.5.

**Remark 0.53.** The reduction to the case of a non-nilpotent element \( h \in H(G) \) is not purely algebraic since we use the notion of positivity specific for \( \mathbb{R} \subset \mathbb{C} \).

**Problem 0.54.** Prove the Separation Lemma without the assumption that \( G \) is unimodular.

**Definition 0.55.** Let \( G \) be a unimodular \( l \)-group. A smooth representation \( \pi \) of \( G \) is **compact** if for any \( v \in V \) and any open compact subgroup \( K \subset G \) there exists a compact \( C \) of \( G \) such that \( \pi(e_K)\pi(g)v = 0 \) for \( g \notin C \).

**Proposition 0.56.** If smooth irreducible representation \( \pi : G \rightarrow \text{Aut}(V) \) is compact then the corresponding representation \( \pi : H(G) \rightarrow \text{End}(V) \) of the Hecke algebra \( H(G) \) is also compact.

**Proof.** We fix a Haar measure \( dg \) on \( G \) and will identify the algebra \( H \) with the space \( S(G) \) of functions on \( G \).

Assume that the representation \( \pi \) of \( G \) is compact and fix an idempotent \( e \in H(G) \). We want to show that the representation \( \pi_e : H_e \rightarrow \text{End}(V_e), H_e = eHe, V_e = eV \) is compact. It is easy to see that it is sufficient to analyze the case when \( e = e_K \) is a Haar measure of an open compact subgroup \( K \subset G \). In this case we can identify the algebra \( H_e \) with the space of functions on \( K \backslash G/K \) with finite support.

We first check that the space \( V_e \) is finite-dimensional. Fix \( v \in V \). Since the representation \( \pi \) is irreducible the space \( V \) is spanned by \( \pi(g)v \). Thus the space \( V_e \) is spanned by vectors \( \pi(e_K)\pi(g)v \). But by the definition of compact representation, there exists a finite subset \( R \) of \( K \backslash G \) such that \( \pi(e_K)\pi(g)v = 0 \) for \( Kg \in K \backslash G - R \). So the the space \( V_e \) is finite-dimensional.

Since the representation \( \pi \) of \( G \) is compact the function \( \kappa(v, \lambda), v \in V_e, \lambda \in V_e^\vee \) on \( G \) defined by \( \kappa(v, \lambda)(g) := \lambda(\pi(e_K)\pi(g)(v)) \) has finite support. So we can define a bilinear form \( \kappa : V_e \times V_e^\vee \rightarrow H_e, (v, \lambda) \rightarrow \kappa(v, \lambda) \). This bilinear form defines a linear map \( \kappa : V_e \otimes V_e^\vee \rightarrow H_e \) and, since the space
$V_e$ is finite-dimensional, a linear map $\kappa : \text{End}(V_e) \to \mathcal{H}_e$. It is clear that the map $\kappa : \text{End}(V_e) \to \mathcal{H}_e$ is a morphism of $\mathcal{H}_e \otimes \mathcal{H}_e^{op}$-modules.

**Lemma 0.57.**

1. The composition $\pi_e \circ \kappa : \text{End}(V_e) \to \text{End}(V_e)$ is not equal to 0.

2. There exists a non-zero constant $d_\pi \in \mathbb{C}$ such that $d_\pi (\pi_e \circ \kappa) = \text{Id}_{V_e}$.

**Proof.** Let $\tilde{V}_e : = \text{Im}(\kappa)$ and $\rho : \mathcal{H}_e \to \text{End}(W)$ be a irreducible representation of $\mathcal{H}_e$ which is not equivalent to $\pi_e$. Then the composition $\rho \circ \kappa : \text{End}(V_e) \to \text{End}(W)$ is a morphism of irreducible $\mathcal{H}_e \otimes \mathcal{H}_e^{op}$-modules. Since $\rho$ is not equivalent to $\pi_e$ we see that $\rho \circ \kappa = 0$. We see that the restriction of $\rho$ on $\tilde{V}_e$ vanishes for all irreducible representation of $\mathcal{H}_e$ not equivalent to $\pi_e$. The Lemma follows now from 0.51.

Since the composition $\pi_e \circ \kappa : \text{End}(V_e) \to \text{End}(V_e)$ is a non-zero endomorphism of an irreducible $\mathcal{H}_e \otimes \mathcal{H}_e^{op}$-module there exists a non-zero constant $d_\pi \in \mathbb{C}$ such that $d_\pi \pi_e \circ \kappa = \text{Id}_{V_e}$.\hfill $\square$

**Problem 0.58.**

1. Show that the constant $d_\pi$ does not depend on a choice of an idempotent $e \in \mathcal{H}(G)$ as long as it is defined [that is when $V_e \neq \{0\}$].

2. Formulate and prove the converse of the Theorem.

**Definition 0.59.** We say that $d_\pi$ is the Formal Dimension of a compact representation $\pi$.

**Remark 0.60.** The formal dimension depends on a choice of a Haar measure $dg$.

**Problem 0.61.** Assume that the group $G$ is compact and the Haar measure $dg$ is such that $\int_G dg = 1$. Then for any irreducible representation $(\pi, V)$ of $G$ we have $d_\pi = \text{dim}(V)$.

**Theorem 0.62.** Let $(\pi, V)$ be an irreducible compact representation, $K \subset G$ an open compact subgroup such that $V_{eK} \neq \{0\}$ and

$$\mathcal{E}_{W,K} = d_\pi \kappa(\text{Id}_{V_{eK}}) \subset \mathcal{H}_{eK}$$

Then

1. $\mathcal{E}_{W,K} \in \mathcal{H}_{eK}$ is an idempotent.

2. For any smooth representation $(\rho, W)$ of $G$ we have a direct sum decomposition $W = W_0 \oplus W_1$ where $W_0$ is a multiple of $V$ and $W_1$ does not have subquotients isomorphic to $V$.

**Proof.** The first part follows immediate from the definition of the formal dimension. To construct a decomposition we define

$$W_0 : = \cup_{K'} \text{Im}(\rho(\mathcal{E}_{W,K'})), W_1 : = \cap_k \text{Ker}(\rho(\mathcal{E}_{W,K}))$$

where $K'$ runs through the set of open comact subgroups of $G$. As follows from the complete reducibility of the restriction of $\eta$ on $K$ the subspaces $W_0, W_1$ are $G$-invariant and $W = W_0 \oplus W_1$.\hfill $\square$
As follows from Lemma 0.57 any subquotient of $W$ isomorphic to $(\rho, W)$ is not killed by $\mathcal{E}_{W,K}$. Hence $W_1$ cannot have any such subquotients isomorphic to $V$.

To show that $W_0$ is a direct sum of copies of $V$ note that by the Lemma 0.57, $\rho(\mathcal{E}_{W,K}) = 0$ for any irreducible not equivalent to $\rho$. Therefore, all irreducible subquotients of $V_0$ are isomorphic to $(\pi, V)$ and the claim follows from 0.10. □

**Definition 0.63.**
(1) We denote by $G^0 \subset G$ the subgroup generated by $\{K\}$ when $K$ runs through the set of compact subgroups of $G$.
(2) We write $\Lambda(G) := G/G^0$.
(3) We denote the center of $G$ by $Z(G)$.
(4) A smooth representation $(\pi, V)$ of a group $G$ is compact modulo center if for any $v \in V$ and any open compact subgroup $K \subset G$ there exists a compact $C$ of $G$ such that $\pi(e_K)\pi(g)v = 0$ for $g \notin CZ(G)$.

**Problem 0.64.**
(1) Assume that $Z(G)G^0$ is a subgroup of finite index in $G$. Then an irreducible representation $\pi$ of $G$ is compact modulo center iff the restriction of $\pi$ to $G^0$ is compact.
(2) Let $G = \text{GL}(n)$. Show that
   (a) $G^0 = \{g \in G | \det g \in \mathcal{O}^*\}$.
   (b) $G^0$ is an open, normal subgroup of $G$ with $\Lambda(G) = \mathbb{Z} = F^*/\mathcal{O}^*$.
   (c) $Z(G)G^0$ is an open subgroup of finite index in $G$.

**Remark 0.65.** Analogous statements are true for an arbitrary reductive group $G$. For example, $\Lambda(G) = \mathbb{Z}^k$ for $G = \text{GL}(n_1) \times \cdots \times \text{GL}(n_k)$.

**Example 1.** If $G$ is a compact group then every smooth $G$-module $M$ is completely reducible, that is $M = \bigoplus W_{\alpha}$ where the $W_{\alpha}$ are irreducible. Thus, the representation theory is entirely controlled by the the knowledge of irreducible representations and in a simple way.

**Example 2.** $G = F^*$ (This is “almost” compact.) Let $\pi$ be a generator for the maximal ideal in the ring of $\mathcal{O} \subset F$. Then we have a decomposition

$$F^* = \mathbb{Z}\pi \oplus \mathcal{O}^*.$$  

Here $\mathcal{O}^*$ is compact and $\mathcal{M}(\mathbb{Z}) = \mathcal{M}(\mathbb{C}[t,t^{-1}])$, the category of sheaves on $\mathbb{C}^*$. Thus,

$$\text{Irr}(G) = \text{Irr}(\mathbb{Z}) \times (\mathcal{O}^*)^\vee$$  

where $(\mathcal{O}^*)^\vee$ is the discrete set of characters of the compact group $\mathcal{O}^*$.

The point here is that the structure of the representations is half discreet and half continuous. Specifically, it is a discrete sum of the category of sheaves on some space. We will see that this is a typical situation.
3. The Induction and Jacquet Functors.

Remark. The way to make an advance in representation theory is to find a way to construct representations. Practically our only tool is the induction.

3.1. Induction.

Definition 0.66. Let $H$ be a closed subgroup of $G$.

(1) We denote by $Res' = Res'_G : \mathcal{M}(G) \to \mathcal{M}(H)$ the restriction functor.

(2) For any smooth representation $(\rho, V)$ of $M$ we denote by $Ind'(V)$ the functions $f : G \to V$ such that

\[ \{ f : G \to V | f(hg) = \rho(h)f(g) \} \]

and

\[ \text{and} \]

\[ \text{(b) There exists an open subgroup } K \text{ of } G \text{ such that } f(kg) \equiv f(g), g \in G, k \in K. \]

(3) We define the representation $Ind'(\rho)$ of $G$ on $Ind'(V)$ by right shifts

\[ Ind'(\rho)(g)f(x) = f(xg) \]

(4) We consider $Ind'(V)$ as a functor from $\mathcal{M}(H)$ to $\mathcal{M}(G)$.

(5) We denote by $ind' = ind'_G(V)$ the subfunctor of $Ind'_G$ given by

\[ ind'_G(V) = \{ f \in Ind'(V) | f \text{ has compact support modulo } H \}. \]

Lemma 0.67. (1) The functor $Ind'$ is the right adjoint of $Res'$.

(2) If $H$ is open then the functor $ind'$ is the left adjoint of $Res'$.

Proof. (1) Given $(\rho, V)$ and $(\pi, W) \in \mathcal{M}(G)$ we define a map $\kappa : Hom_G(W, Ind'(V)) \to Hom_M(Res'(W), V)$ by

\[ \kappa(\phi)(w) := \phi(e_H, \rho)(w), \phi \in Hom_G(W, Ind'(V)), w \in W \]

Conversely we define a map

\[ \kappa' : Hom_M(Res'(W), V) \to Hom_G(W, Ind'(V)) \]

by

\[ \kappa'(\psi)(w)(g) := \psi(\pi(g)(w)), \psi \in Hom_M(Res'(W), V), w \in W \]

(2) Consider a vector $e_{H,v} \in ind'(V)$ given by $e_{H,v}(g) = \rho(g)v$ if $g \in H$ and $e_{H,v}(g) = 0$ if $g \in G - H$ and define a map

\[ \theta : Hom_G(\text{ind}'(V), W) \to Hom_M(V, \text{Res}'(W)) \]

by $\theta(\phi)(v) := \phi(e_{H,v})$.

Problem 0.68. (1) Show that $\kappa$ and $\kappa'$ are the inverse maps.

(2) Show that $\theta : Hom_G(\text{ind}'(V), W) \to Hom_M(V, \text{Res}'(W))$ is a bijection.
As we will see it is better to replace the usual induction by the unitary induction. Assume that the group \( G \) is unimodular.

**Definition 0.69.** Let \( H \) be a closed subgroup of \( G \).

1. We define the functor \( \text{Ind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G) \) of the unitary induction by
   \[
   \text{Ind}(\rho) := \text{Ind}_H^G(\Delta_H^{1/2} \otimes \rho)
   \]
2. We denote by \( \text{ind}_H^G \) the subfunctor of \( \text{Ind}_H^G \) of functions compact modulo \( H \).
3. We define the functor \( \text{Res}_H^G : \mathcal{M}(H) \to \mathcal{M}(G) \) of the unitary restriction by
   \[
   \text{Res}_H^G(\pi) := \text{Res}_H^G(\pi) \otimes \Delta_H^{-1/2}
   \]

**Problem 0.70.** (1) Show that the functor \( \text{Res}_H^G \) is the left adjoint to \( \text{Ind}_H^G \).

(2) Let \( (\rho, W) \) be smooth representation of the group \( H \) and \( <, > \) be an \( H \)-invariant bilinear Hermitian form on \( W \). Then for any \( f', f'' \in \text{ind}_H^G(W) \) the function \( <f'(g), f''(g)> \) on \( G \) belongs to the space \( \mathcal{M}(H \setminus G) \) [see 0.40] and the bilinear form
   \[
   [f', f''] := \int <f'(g), f''(g)>
   \]
   defines a \( G \)-invariant bilinear Hermitian form on the space \( \text{ind}_H^G(W) \).

**Proposition 0.71.** (1) Both functors \( \text{Ind}_H^G \) and \( \text{ind}_H^G \) are exact.

(2) If \( H \setminus G \) is compact, \( \text{Ind} = \text{ind} \).

(3) If \( H \setminus G \) is compact, induction maps admissible representations to admissible representations.

**Proof.** The parts (1) and (2) are obvious.

(3) Let \( V \) be an admissible representation of \( H \) and fix \( K \subset G \) a compact open subgroup. Let \( \{Hg_iK\} \) be a system of coset representatives for \( H \setminus G/K \). By our assumption, this is a finite set. It is clear that an element, \( f \), of \( L(V)^K \) is determined by its values on the \( g_i \). Moreover, the image of \( g_i \) must lie in the subset of \( V \) fixed by \( H \cap g_iKg_i^{-1} \) which is finite dimensional since we are assuming that \( V \) is admissible.

3.2. Jacquet Functor.

**Definition 0.72.** For any smooth representation \( (\rho, V) \) of \( G \) we denote by \( V_G \) the space of coinvariants \( V/V(G) \) where \( V(G) \) is the subspace spanned by vectors \( \pi(g)v - v, v \in V, g \in G \) and denote by \( J : V \to V_G \) the natural projection.

**Remark 0.73.** When \( G \) is a finite group, it is often useful to consider the functor of invariants \( V^G := \text{Hom}_G(\mathbb{C}_G, V) \). It turns out that for non-compact \( l \)-groups this notion is not useful. However, the functor \( J_G \) of coinvariants is useful since it is often exact.
4. Unitary Structure

**Proposition 0.74.**

1. The functor $J_G$ from the category $\mathcal{M}(G)$ to the category of vector spaces is right exact.
2. If $G$ is compact then $V(G) = \ker e_G$ and the functor $J_G$ is exact.
3. If $G = \bigcup_i U_i$ is the union of an increasing family of compact groups $G_i$, then the functor $J_G$ is exact and $V(G) = \{ v \in V | \int_{g \in G_i} \rho(g)v \, dg = 0 \text{ for some } i > 0 \}$. 

**Proof.** (1) is obvious.

(2) We have an exact sequence

$$0 \to V(G) \to V \to V_G \to 0.$$ 

When $G$ is compact, this implies that the composition $V_G \to V \to V_G$ is a bijection. But the functor $V \to V_G$ is left exact.

(3) If $G = \bigcup_i U_i$ then $J_G(V) = \lim U_i(V)$. But the direct image of exact functors is exact and the first part of (3) follows from (2).

Since $J_G(V) = \lim U_i(V)$ the second part of (3) follows from the equality $V(G_i) = \{ v \in V | \int_{g \in G_i} \rho(g)v \, dg = 0 i > 0 \}$ which is an immediate consequence of the complete reducibility of smooth representations of compact groups.

**Definition 0.75.** Let $P = M \ltimes U$ be an $l$-group such that both groups $M$ and $U$ are unimodular and $\rho : P \to \text{Aut}(V)$ be a smooth representation. Since $M$ normalizes $U$ we have a representation $\tilde{\rho} : M \to \text{Aut}(V_U)$ such that

$$\tilde{\rho}(m)(v) = J(\rho(m)v), v \in V$$

where $J : V \to V_U$ is the natural projection. If $P$ is a subgroup of a unimodular group $G$ we define the Jacquet functor corresponding to $P = M \ltimes U \subset G$ as the composition $r_{M,U} := \tilde{r}_{M,U} \circ \text{Res}_G^P : \mathcal{M}(G) \to \mathcal{M}(M)$.

4. Unitary Structure

Let $G$ be an $l$-group and $(\rho, V)$ a smooth representation of $G$.

**Definition 0.76.**

1. A *unitary structure* on a $G$-module $(\pi, V)$ is a positive definite, $G$-invariant Hermitian scalar product $Q : V \otimes V \to \mathbb{C}$.

2. Let $P = MU$ be a parabolic subgroup of $G$ and $(\rho, W, \langle, \rangle)$ be a unitary representation of $M$. We define the unitary structure $Q$ on $(\pi, V) = (i_M(\rho), i_M(W))$ by

$$Q(f_1, f_2) = \int_{x \in P \backslash G} \langle f_1(x), f_2(x) \rangle$$

where the linear functional $\int$ is as in Definition 0.40.

**Remark 0.77.** We do not assume that $V$ is complete with respect to this structure.

The essential uniqueness of an invariant scalar product for irreducible representations follows from the following version of the Schur’s lemma.
**Lemma 0.78.** [Schur’s Lemma] Let $G$ be a reductive $F$-group and $V$ be an irreducible $G$-module. Then any two $G$-invariant unitary structures on $V$ are proportional.

**Proof.** Let $V^+$ be anti-linear dual of $V$, that is the space of anti-linear functionals. A $G$-invariant Hermitian scalar product $Q : V \otimes V \to \mathbb{C}$ defines a $G$-equivariant semi-linear map $V \to V^+$. As $V$ is smooth, we obtain a semi-linear map $\kappa : V \to (V^+)_\text{sm}$. Since $V$ is admissible we see that $V^\text{sm}$ is also admissible and irreducible. Therefore the $G$-equivariant linear map $\kappa : V \to (V^+)_\text{sm}$ is a bijection. It follows now from the Schur’s lemma that any two non-zero $G$-equivariant semi-linear maps $V \to (V^+)_\text{sm}$ are proportional.

**Remark 0.79.** There are not many ways of constructing unitary representations. The only general procedure is to find a space $X$ with a $G$-action. Then $G$ acts on $C^\infty(X)$ and we can find $V \subset L^2(X, d\mu)$. However, it is not clear how to find such $X$. The two natural choices are $X$ = point which gives the trivial representation, and $X = G$ which is what we did above.

It is somehow more difficult to classify unitary representations than general ones.

**Lemma 0.80.** Any smooth admissible unitary representation $(\pi : G \text{Aut}(V), Q)$ of $G$ is completely reducible. [That is, $V = \bigoplus V_i$ where the $V_i$ are irreducible unitary subrepresentations.]

**Remark 0.81.** The assumption of the admissibility is important.

**Proof.** We want to show that for any $G$-invariant subspace $W \subset V$ we can find a $G$-invariant complement. Consider the orthogonal complement,

$$W^\perp := \{w^\perp \in V | Q(w, w^\perp) = 0, \forall w \in W\}.$$ 

Since the form $Q$ is $G$-invariant the subspace $W^\perp \subset V$ is also $G$-invariant. Since $Q$ is positive definite we have $W \cap W^\perp = 0$. It remains to check that $W + W^\perp = V$. For this it is enough to check that for any compact open subgroup $K \subset G$ we have $W^K + (W^\perp)^K = V^K$. Since [by the admissibility] the space $V^K$ is finitely-dimensional we have $V^K = W^K \oplus (W^K)^\perp \cap V^K$. So it is sufficient to show that $(W^K)^\perp \cap V^K \subset W^\perp$. Since the group $K$ is compact we have $W = W^K \oplus L$ where $L := \{l \in V | \int_K \pi(k)ldk = 0\}$. Then $<l, v> = 0$ for all $v \in V^K, l \in L$ and we see that $<w, v> = 0$ for all $v \in (W^K)^\perp \cap V^K, w \in W$.

**Corollary 0.82.** Let $V$ be an admissible unitary representation $V$ of $G$ such that $\text{End}_G(V) = \mathbb{C}$. Then $V$ is irreducible.

This Corollary provides a method for establishing the irreducibility of some representations.

**Definition 0.83.** (1) For any $v \in V, \hat{\lambda} \in \hat{V}$ we define the matrix coefficient $m_{\hat{\lambda}, v}(g)$ as a function on $G$ given by $m_{\hat{\lambda}, v}(g) = \hat{\lambda}(\pi(g)v), v \in V, \hat{\lambda} \in \hat{V}$.
Assume now that $(\pi, V)$ is an irreducible representation of $G$ such that the restriction to the center $Z(G)$ is equal to $\chi Id_V$ where $\chi : Z(G) \to \mathbb{C}^*$ is a unitary character. We say that $V$ is square integrable modulo center if

$$\int_{G/Z} |m_{\xi, \xi}(g)|^2 dg < \infty.$$ 

**Problem 0.84.** Let $(\pi, V)$ be an irreducible representation of $G$ which is essentially square integrable modulo center. Then

1. $(\pi, V)$ admits a unitary structure.
2. $|m_{v, \xi}(g)|^2 \in L^1(G/Z(G))$ for all $v \in V, \lambda \in \tilde{V}$.

## 5. Geometry of general linear groups

### 5.1. Flags

Let $E$ be a field and $V$ a vector space over $E$ of dimension $n, G := \text{Aut}(V)$.

**Definition 0.85.**

1. A flag in $V$ is a strictly increasing sequence $F = \{\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_k = V\}$.
2. A flag $F$ is complete if $k = n$.
3. Given two complete flags $F = \{\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_n = V\}$ and $F' = \{\{0\} \subset V'_1 \subset V'_2 \subset \ldots \subset V'_n = V\}$ we define the relative position $w = w(F', F)$ as a function from $[1, n]$ to $[1, n]$ such that $w(i) := \min\{j|V_i \subset V_{i-1} + V'_j\}, 1 \leq j \leq n$.
4. We denote by $B$ the set of complete flags in $V$ and denote by $(g, F) \to gF$ the natural action of the group $G$ on $B$.
5. For any flag $F = \{\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_k = V\}$ we denote by $P_F \subset G$ the stabilizer of $F$ and by $U_F \subset P_F$ the subgroup of transformations acting trivially on quotients $V_i := V_i/V_{i-1}$.

**Lemma 0.86.** Let $F, F'$ be complete flags and $w := w(F, F')$.

1. The function $w$ from $[1, n]$ to $[1, n]$ is a permutation. In other words $w$ is a bijection from $\{1, 2, \ldots, n\}$ to itself.
2. Let $L_i \subset V_i$, $1 \leq i \leq n$ be one-dimensional subspaces not contained in $V_{i-1} + V'_{j-1}, j := w(F, F')(i)$. Then for all $1 \leq i \leq n$ we have $V_i = \oplus_{k=1}^{i} L_k, V'_j = \oplus_{k=1}^{j} L_{w^{-1}}(k)$.
3. For any pair $(F, F') \in B^2, g \in G$ we have $w(gF, gF') = w(F, F')$.
4. For any pair $(\tilde{F}, \tilde{F}') \in B^2$ such that $w(F, F') = w(\tilde{F}, \tilde{F}')$ there exists $g \in G$ such that $(gF, gF') = (\tilde{F}, \tilde{F}')$.

**Proof.**

1. Assume that $w(F, F')(a) = w(F', F)(b) = j$ for some pair $1 \leq a < b \leq n$. By the definition we can find $v_a \in V_a - V_{a-1}, v_b \in V_b - V_{b-1}$ such that $v_a, v_b \in V'_j - V'_{j-1}$. Let $\tilde{v}_a, \tilde{v}_b$ be images of $v_a, v_b$ in $V'_j := V'_j/V'_{j-1}$.
Since $\tilde{v}_a \neq 0$ and $dim(\tilde{V}_j) = 1$ there exists $\lambda \in \mathbb{C}$ such that $\tilde{v}_b = \lambda \tilde{v}_a$. In other words $v_b - \lambda v_a \in V'_{j-1}$. But this contradicts the assumption that $w(\mathcal{F}, \mathcal{F}')(b) = j$.

(2) We prove the equality $V_i = \oplus_{k=1}^i L_k$ be the induction in $i$. The claim is obviously true for $i = 1$. By inductive assumptions the sum $\oplus_{k=1}^i L_k$ is contained in $V_{i-1} + L_i \subset V_i$ and properly contains $V_{i-1}$. Since $dim(V_i) = dim(V_{i-1}) + 1$ we see that $V_i = \oplus_{k=1}^i L_k$.

The proof of the equality $V_j' = \oplus_{k=1}^j L_{v^{-1}}(k)$ is completely analogous.

(3) Follows immediately from the definition.

(4) Follows from the existence of $g \in G$ such that $gL_i = \tilde{L}_i$, $1 \leq i \leq n$ where $\tilde{L}_i$ are lines corresponding to the pair $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}')$.

**Problem 0.87**

Show that $N_G(P) = P$ for any parabolic subgroup $P$ of $G$ where $N_G(P)$ is the normalizer of $P$ in $G$.

**Definition 0.88.**

1. For any $w \in W$ we define
   
   $$X(w) := \{(\mathcal{F}, \mathcal{F}') \in \mathcal{B} \times \mathcal{B} | w(\mathcal{F}, \mathcal{F}') = w\}.$$ 

2. A splitting of $V$ is a choice $\mathcal{SP}$ of a direct sum decomposition $V = \oplus_{i=1}^k W_i$.

3. With any splitting $\mathcal{SP}$ of $V, V = \oplus_{i=1}^k W_i$ we associate a flags

   $$\mathcal{F}(\mathcal{SP}) := \{\{0\} \subset V_1 \subset V_2 \subset ... \subset V_k = V\}$$

   and

   $$\tilde{\mathcal{F}}(\mathcal{SP}) := \{\{0\} \subset \tilde{V}_1 \subset \tilde{V}_2 \subset ... \subset \tilde{V}_k = V\}$$

   where $V_i := \oplus_{j=1}^i W_i$ and $\tilde{V}_i := \oplus_{j=k-i+1}^k W_i$.

4. A subgroup $P$ of $G$ is a parabolic if it is equal to the stabilizer $P_\mathcal{F} \subset G$ of a flag $\mathcal{F}$.

5. A subgroup $B$ of $G$ is a Borel if it is equal to the stabilizer of a complete flag $\mathcal{F}$.

6. We say that $U_\mathcal{F}$ is the unipotent radical of $P_\mathcal{F}$

7. A subgroup $M$ of $G$ is Levi if it is a stabilizer of some splitting $\mathcal{SP}$.

8. Two parabolic subgroups $P, Q$ of $G$ are opposite if the intersection $P \cap Q$ is a Levi subgroup in both $P$ and $Q$.

We choose a basis $e_1, e_2, ..., e_n$ of $V$ and identify the group $G$ with $GL(n, E)$ and denote by $T \subset G$ the subgroup of diagonal matrices.

**Definition 0.89.**

1. We denote by $\mathcal{F}_0 = \{\{0\} \subset V_1 \subset V_2 \subset ... \subset V_n = V\}$ the complete flag in $V$ such that $V_i$ is the span of $e_1, e_2, ..., e_i$.

2. We denote by $\mathcal{F}_0 = \{\{0\} \subset L_1 \subset L_2 \subset ... \subset L_n = V\}$ the complete flag in $V$ such that $L_i$ is the span of $e_n, e_{n-1}, ..., e_{n-k+1}$. 
(3) We denote by $B$ the stabilizer of the flag $\mathcal{F}_0^-$ and by $B^-$ the stabilizer of the flag $\mathcal{F}_0^+$. It is clear that $B$ is the group of upper-triangular matrices and $B^-$ is the group of lower-triangular matrices.

(4) A partition of $n$ is a decomposition of the interval $[1,n] \subset \mathbb{Z}$ in a disjoint union of subsets $S_1, S_2, ..., S_r$, some of which could be empty.

(5) A partition $[1,n] = S_1 \cup S_2 \cup ... \cup S_r$ is standard if $S_i = [n_1 + ... + n_{i-1} + 1, n_1 + ... + n_i]$ for some $n_i \in \mathbb{N}, 1 \leq i \leq r, n = n_1 + ... + n_r$.

(6) For a partition $\theta = (S_1, S_2, ..., S_r)$ of $n$ we denote by $SP_\theta$ the splitting of $V = \oplus_{i=1}^k W_i$ where $W_i$ is the span of $e_j, j \in S_i$.

(7) We denote by $\mathcal{F}_\theta$ the corresponding flag and by $P_\theta$ the stabilizer of $\mathcal{F}_\theta$.

(8) A standard parabolic subgroup of $G$ is a subgroup of the form $P_\theta$ for some standard partition $\theta$ of $n$.

(9) For any partition $\theta = (S_1, S_2, ..., S_r)$ of $n$ we denote by $M_\theta$ the stabilizer of the splitting $V = \oplus_{i=1}^k W_i$ where $W_i$ is the span of $e_j, j \in S_i$. It is clear that the group $M_\theta = G_1(\theta) \times ... \times G_r(\theta)$ where $G_i(\theta)$ is isomorphic to $GL(n_i,E)$, $n_i := |S_i|$.

(10) For any permutation $w$ we denote by $B_w \subset B$ the subspace of flags $\mathcal{F}$ such that $w(\mathcal{F}_0^-, \mathcal{F}) = w$.

(11) A standard Levi subgroup of $G$ is a subgroup of the form $M_\theta$ for a standard partition $\theta$ of $n$.

(12) We write $M < G$ if $M$ is a standard Levi subgroup of $G$.

(13) We denote by $B \subset G$ the subgroup of upper-triangular matrices.

(14) We denote by $\delta : B \to E^*$ the character which maps $b \in B$ into the product $\prod_{1 \leq i < j \leq n} b_{ii}b_{jj}^{-1}$ where $b_{ii}, 1 \leq i \leq n$ are the diagonal entries of $b$.

(15) We say that a subgroup $P$ of a product $GL(m_1,E) \times ... \times GL(m_t,E)$ is parabolic if it has a form $P = P_1 \times ... \times P_t$ where all the subgroups $P_i \subset GL(m_i,E), 1 \leq i \leq t$ are parabolic.

(16) For any $w \in W$ we define $U_w := U \cap wB^-w^{-1} \subset U$.

**Problem 0.90.**

(1) For any $w \in W$ the group $U_w$ acts simply transitively on the set $B_w$.

(2) Any parabolic subgroup of $G$ is conjugated to unique standard parabolic subgroup.

(3) Any Levi subgroup of $G$ is conjugated to a standard Levi subgroup $M_\theta < G$ but such $\theta$ is not necessarily unique.

(4) Any subgroup of $G$ containing $B$ is a standard parabolic subgroup of $G$.

(5) For any splitting $SP V = \oplus_{i=1}^k W_i$ we have

$$P_{\mathcal{F}(SP)} = M_{SP} \rtimes U_{\mathcal{F}(SP)}.$$
(6) Let $B, B'$ be opposite Borel subgroups of $GL(n, E)$. Then there exists $g \in GL(n, E)$ such that $gBg^{-1}$ is the subgroup of upper-triangular matrices and $gB'g^{-1}$ is the subgroup of lower-triangular matrices.

(7) Let $P = MU, Q = NV$ be parabolic subgroups of $G$ such that $Q \subset P$ and $N \subset M$. Then $Q_{M} := M \cap Q$ is a parabolic subgroup of $M$ and $Q_{M} = N \times V_{M}, V_{M} := V \cap M$.

Definition 0.91. Let $\theta = (S_{1}, ..., S_{r}), \tau = (R_{1}, ..., R_{t})$ be two partitions of $n$ and fix $i, 1 \leq i \leq r$.

1. We denote by $\tau_{i}$ the partition $\{S_{i} \cap R_{j}\}, 1 \leq j \leq t$ of $S_{i}$ and denote by $P_{\tau_{i}} \subset G_{i}(\theta)$ the corresponding parabolic subgroup.
2. We write $P_{\tau}(\theta) := \prod_{i=1}^{r} P_{\tau_{i}} \subset M_{\theta}$ and write $P_{\tau}(\theta) = M_{\tau}(\theta)U_{\tau}(\theta)$.
3. Completely analogously we define partitions $\theta_{j}$ of $R_{j}$ and a subgroup $P_{\theta}(\tau) = M_{\theta}(\tau)U_{\theta}(\tau) \subset M_{\tau}$.

Problem 0.92. Show that $M_{\tau}(\theta) = M_{\theta}(\tau) = M_{\theta \wedge \tau}$ where $\theta \wedge \tau$ the partition
\[
\{S_{i} \cap R_{j}\}, 1 \leq i \leq r, 1 \leq j \leq t
\]
of $n$.

Remark 0.93. The claim make sense since Levi subgroups $M_{S_{1}, S_{2}, ..., S_{r}} \subset G$ do not depend on the order of the subsets $S_{1}, S_{2}, ..., S_{r}$.

5.2. Symmetric group.

Definition 0.94. (1) Let $W := S_{n}$ be the symmetric group on $n$ letters.
(2) We consider an imbedding $w \to w_{i,j} := \delta_{w(i),j}$ of $S_{n}$ into the subgroup of permutation $n \times n$-matrices and identify the symmetric group $W := S_{n}$ with it image in $GL(n, E)$.
(3) for any diagonal matrix $t \in T, w \in W$ we define $t^{w} := wtw^{-1}$.
(4) We denote by $w_{0} \in S_{n}$ the permutation $i \leftrightarrow n - i + 1$.
(5) For any $i, 1 \leq i \leq n$ we denote by $s_{i} \in S_{n}, 1 \leq i < n$ the permutation $i \leftrightarrow i + 1$. We say that elements $s_{i} \in W, 1 \leq i < n$ are simple reflections.
(6) For any $w \in W$ we define
\[
J_{w} := \{(i, j), 1 \leq i < j \leq n w(i) > w(j)\}
\]
and write $l(w) := |J_{w}|$.
(7) For any partition $\theta = (S_{1}, ..., S_{r})$ of $n$ we denote by $W_{\theta} \subset S_{n}$ the stabilizer of the partition $\theta$. It is clear that $W_{\theta} = \prod_{i=1}^{r} W_{i}(\theta)$ where $W_{i}(\theta)$ is the group of permutations of the set $S_{i}$.
(8) We now define the Bruhat order on the symmetric group $S_{n}$. For any permutations $w : i \to a_{i}, w' : i \to a'_{i}, 1 \leq i \leq n$ we say that $w'$ is a reduction of $w$ if the sequence $\{a'_{i}\}$ can be obtained from the
sequence \( \{a_i\} \) by interchanging \( a_i \leftrightarrow a_j \) for some \( i, 1 \leq i < n, j > i \) \( a_i > a_j \). Define \( w' < w \) if \( w' \) can be obtained from \( w \) by a sequence of reductions.

(9) For any \( w \in W \) we define \( W_{\leq w} := \{w' \in W| w' \leq w\} \).

**Problem 0.95.**

1. The group \( S_n \) is generated by the set of simple reflections \( \{s_i\}, 1 \leq i < n \).
2. For any \( w \in S_n \) and any decomposition \( w = s_{i_1}...s_{i_t} \) in a product of simple reflections we have \( l \geq l(w) \).
3. For any \( w \in S_n \) there exists a decomposition \( w = s_{i_1}...s_{i_t} \) in a product of simple reflections such that \( l = l(w) \). In this case we say that \( w = s_{i_1}...s_{i_t} \) is a reduced decomposition of \( w \).
4. Let \( w = s_{i_1}...s_{i_t} \in W \) be a reduced decomposition and \( w' \) is obtained by omitting some factors in this decomposition. Then \( w' < w \).
5. Conversely let \( w', w \in W \) be such that \( w' < w \) and \( w = s_{i_1}...s_{i_t} \) be a reduced decomposition. Then we can obtain \( w' \) omitting some factors in this decomposition.
6. The generators \( \{s_i\}, 1 \leq i < n \) satisfy the relations:
   \[
   s_i^2 = e; \quad s_i s_j = s_j s_i, |i - j| > 1, 1 \leq i, j < n \quad \text{and} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \leq i < n - 1.
   \]
7. * The group \( S_n \) is defined by this set of relations.
8. For any \( t \in T, w \in W \) we have
   \[
   \delta(t)/\delta(tw) = \prod_{(i,j) \in J_w} a_i a_j^{-1}
   \]
   where \( a_i = t_{ii} \).
9. If \( w' < w \) then \( w' w_0 > w w_0 \).

**Definition 0.96.**

1. For any \( w \in W \) we define
   \[
   X(w) := \{(F, F') \in B \times B|w(F, F') = w\}.
   \]
   So \( X(e) = \Delta_B \) where \( \Delta_B \subset B \times B \) is the diagonal and
   \[
   \{F_0\} \times B_w = \{F_0\} \times B \cap X_w.
   \]
2. For any \( i, 1 \leq i < n \) we define \( \tilde{X}(s_i) = X(s_i) \cup X(e) \).
3. For any \( w_1, w_2, ..., w_r \in W \) we define
   \[
   Z(w_1, w_2, ..., w_r) = \{(F_1, F_2, ..., F_{r+1}) \in B^r| (F_1, F_2) \in X(w_1), (F_2, F_3) \in X(w_2), ..., (F_r, F_{r+1}) \in X(w_r)\}
   \]
4. For any \( l \geq 2 \) we denote by \( q = q_{l+1} : B^l \to B^2 \) the projection to the first and the last factors.
5. For any simple reflection \( s_i \in W \) we define \( \tilde{X}(s_i) := X(s_i) \cup \Delta_B \).
6. For any reduced decomposition \( w = s_{i_1}...s_{i_t} \) we define
   \[
   \tilde{Z}(s_{i_1}, s_{i_2}, ..., s_{i_t}) = \{(F_1, F_2, ..., F_{t+1}) \in B^t| (F_1, F_2) \in \tilde{X}(s_1), (F_2, F_3) \in \tilde{X}(s_2), ..., (F_t, F_{t+1}) \in \tilde{X}(s_t)\}.
   \]
PROBLEM 0.97. For any \( w \in W \) and a simple reflection \( s_i \) we have

1. \( Z(w, s_i) = X(ws_i) \) if \( l(ws_i) > l(w) \).
2. \( Z(w, s_i) = X(ws_i) \cup X(w) \) if \( l(ws_i) < l(w) \).
3. If \( l(ws_i) > l(w) \) then \( W_{\leq ws_i} = W_{\leq w} \cup W_{\leq w}s_i \).

LEMMA 0.98. Show that for any reduced decomposition \( w = s_{i_1}...s_{i_t} \), the restriction of \( q \) on \( Z(i_1, ..., i_t) \) defines a bijection between \( Z(i_1, ..., i_t) \) and \( X(w) \).

PROOF. I’ll explain the proof in the case when \( G = GL(3) \), \( w = w_0 \in S_3 \) is the longest elements and \( w_0 = s_1s_2s_1 \) a reduced decomposition. The general case is quite similar.

In our case
\[
X(w_0) = \{ F^0 = (L_1^0 \subset L_2^0), F' = (L_1' \subset L_2') | L_1 \not\subset L_2', L_1' \not\subset L_2 \} \]

and
\[
Z(s_1, s_2, s_1) = \{ F^0 = (L_1^0 \subset L_2^0), F^1 = (L_1^1 \subset L_2^1), F^2 = (L_1^2 \subset L_2^2), F' = (L_1' \subset L_2') \}
\]

such that
\[
L_1^0 = L_1', L_2^0 = L_2', L_1^1 = L_1', L_2^1 \neq L_1', L_1^2 \neq L_1', L_2^2 \neq L_2'.
\]

It is easy to check that these conditions imply that \( L_1 \not\subset L_2', L_1' \not\subset L_2 \). So \( q(Z(s_1, s_2, s_1)) \subset X(w_0) \).

Conversely for any \( (F^0 = (L_1^0 \subset L_2^0), F' = (L_1' \subset L_2')) \in X(w_0) \) we can define \( F^1 = (L_1^0 \subset L_1') := L_1^0 \oplus L_1' \) and \( F^2 = (L_1^2 \subset L_2') \). Then the sequence \((F^0, F^1, F^2, F')\) belongs to \( Z(s_1, s_2, s_1) \) and it is easy to check the uniqueness of such a sequence for any pair \((F^0, F') \in X(w_0)\).

PROPOSITION 0.99. For any reduced decomposition \( w = s_{i_1}...s_{i_t} \), we have

\[
q(Z(s_{i_1}, s_{i_2}, ..., s_{i_t})) = \cup_{w' \leq w} X(w')
\]

PROOF. The proof is by the induction in \( l \). For \( l = 0 \) the claim is clear. So assume that \( w = \bar{w}s_{i_l}l(\bar{w}) = l - 1 \) and that the result is known for the reduced decomposition \( \bar{w} = s_{i_1}...s_{i_{l-1}} \). By the definition

\[
Z(s_{i_1}, s_{i_2}, ..., s_{i_t}) = \{(z', \mathcal{F}) \in \bar{Z}(s_{i_1}, s_{i_2}, ..., s_{i_{l-1}}) \times \mathcal{B} \} \subseteq X(s_l)
\]

As follows from the inductive assumption we have

\[
q(Z(s_{i_1}, s_{i_2}, ..., s_{i_l})) = \cup_{w' \leq \bar{w}} Y_{w'} \cup \cup_{w' \leq \bar{w}} Z_{w'}
\]

where

\[
Y_{w'} = \{(x, \mathcal{F}) \subset X(\bar{w'}) \times \mathcal{B} | q_l(x, \mathcal{F}) \in X(s_l) \}
\]

and

\[
Z_{w'} = \{(x, \mathcal{F}) \subset X(\bar{w'}) \times \mathcal{B} | q_l(x) = \mathcal{F} \}
\]

Proposition follows now from the previous Problem.
Problem 0.100.  
(1) For any standard partition $\theta$ of $n$ and any $w \in S_n$ there exists unique $\tilde{w} \in wW_\theta$ such that $\tilde{w}(i) < \tilde{w}(j)$ for any pair $(i,j), i < j$. Moreover $l(\tilde{w}) < l(w')$ for all $w' \in wW_\theta, w' \neq \tilde{w}$.

(2) For any two standard partitions $\theta, \theta'$ of $n$ and any $w \in S_n$ there exists unique $\tilde{w} \in W_{\theta}wW_{\theta'}$ such that $\tilde{w}(i) < \tilde{w}(j)$ for any pair $(i,j), i < j$ which belong to the same part of the partition $\theta$ and $\tilde{w}^{-1}(i) < \tilde{w}^{-1}(j)$ for any pair $(i,j), i < j$ which belong to the same part of the partition $\theta'$. Moreover $l(\tilde{w}) < l(w')$ for all $w' \in W_{\theta'}wW_{\theta}, w' \neq \tilde{w}$.

(3) We denote by $W_{\theta,\theta'} \subset W$ the set of shortest elements in two-sided classes $W_{\theta}wW_{\theta'}$.

(4) For any standard Levi subgroup $M < G$ we define $W_M = N_M(T)/T$.

Definition 0.101. Let $M, M'$ be a pair of standard Levi subgroups of $G$ corresponding to standard partitions $\theta, \theta'$ of $n$. We define

(1) $W(M, *) := \{ w \in W | w(M) \text{ is a standard Levi subgroup} \}$. It is clear that the subset $W(M, *)$ of $W$ is right $W_M$-invariant.

(2) $l_G(M) := |(M, *)/W_M|$.

(3) If $M' < G$ is another standard Levi subgroups we write $W(M, M') := \{ w \in W | w(M) = M' \}$ where $w(M) := w^{-1}Mw$ and say that standard Levi subgroups $M, M'$ are associated if $W(M, M') \neq \emptyset$. In this case we write $M \sim M'$.

(4) We write $W^{M,N} := W^{\theta,\theta'} \subset W$.

Example 0.102. Let $G = GL(3, F)$.

(1) $G$ has four standard parabolic subgroups $B, P, Q$ and $G$ where

\[
    P = \begin{bmatrix}
    * & * & * \\
    * & * & * \\
    0 & 0 & *
    \end{bmatrix}
\]

\[
    Q = \begin{bmatrix}
    * & * & x \\
    0 & * & * \\
    0 & * & *
    \end{bmatrix}
\]

(2) $G$ has four standard Levi subgroups $T, M_P, M_Q, G$ where $T$ is the subgroup of diagonal matrices,

\[
    M_P = \begin{bmatrix}
    * & * & 0 \\
    * & * & 0 \\
    0 & 0 & *
    \end{bmatrix}
\]

\[
    M_Q = \begin{bmatrix}
    * & 0 & 0 \\
    0 & * & * \\
    0 & * & *
    \end{bmatrix}
\]

(3) The Levi subgroups $M_P$ and $M_Q$ are associated.
(4) $W_G = S_n$, $W_{M_p} = \{e, s_1\}, W_{M_Q} = \{e, s_1\}$ where $S_n$ is the symmetric group on $n$ letters and $s_i \in S_n, 1 \leq i < n$ are the permutations $i \leftrightarrow i + 1$.

(5) $W(M_p, M_Q) = W_{M_p} s_2 s_1$.

(6) $W(M_0, \ast) = S_3, W(M_p, \ast) = W_{M_p} \cup W_{M_p} s_2 s_1$.

(7) $l_G(M_0) = 6, l_G(M_p) = l_G(M_Q) = 2$.

**Problem 0.103.**

(1) The imbedding $W \hookrightarrow GL(n, E)$ induces a bijection $W \to B \backslash GL(n, F)/B$.

(2) For any pair $\theta, \theta'$ of standard partitions of $n$ the imbedding $W \hookrightarrow GL(n, E)$ induces a bijection

$$W_{\theta'} \backslash W/W_\theta \to P_{\theta'} \backslash G/P_{\theta}.$$  

(3) For any two standard parabolic subgroups $P, Q$ of $G$ s $\theta, \theta'$ of $n$ the set $W^{\theta, \theta'} \subset W$ is equal to the set of elements $w \in W$ such that $w(M_0 \cap B) \subset B$ and $w^{-1}(M_0 \cap B) \subset B$ where as before $w(X) := w^{-1}Xw$ for any subset $X$ of $G$.

(4) Show that $W^{M_p, M_Q} = \{e, s_2 s_1\}$ where we use notation from the previous example.

Consider now the case when $E$ is a local field. Then $B$ is a compact topological space.

**Definition 0.104.** For any $w \in W = S_n$ we denote by $\bar{X}(w)$ the closure of the $X(w)$ in $B \times B$.

**Problem 0.105.**

(1) The closure $\bar{X}(s_i)$ of $X(s_i)$ in $B^2$ is equal to the union $X(s_i) \cup X(e)$ for all $i, 1 \leq i < n$.

(2) Let $q_1, q_2 : X(s_i) \to B$ be the restriction of natural projections $p_1, p_2 : B \times B \to B$. Then $q_1, q_2$ are fibrations with fibers $\mathbb{P}^1$.

(3) The set $Z(i_1, \ldots, i_l)$ is compact.

(4) For any $w \in W$ and any reduced decomposition $w = s_{i_1} \ldots s_{i_l}$ we have $\bar{X}(w) = q(Z(i_1, \ldots, i_l))$.

(5) $\bar{X}(w) = \cup_{w' \leq w} X(w')$.

**5.3. Mackey theory.** For a finite group $G$ we denote by $\mathcal{M}(G)$ the category of representations of $G$.

**Definition 0.106.** Let $G$ be a finite group and $P, Q$ subgroups of $G$.

(1) For any $w \in G$ we define $w(P) := w^{-1}Pw$ we denote by $R_w(P, Q)$ [or simply $R_w$] the functor $\mathcal{M}(P) \to \mathcal{M}(Q)$ given by the composition

$$\text{Ind}^Q_{w(P) \cap Q} \circ \sigma \circ \text{Res}^P_{P \cap w^{-1}(Q)}$$

where Ind is the functor of induction, Res is the functor of restriction and $\sigma$ is the isomorphism $\mathcal{M}(w(P) \cap Q) \to \mathcal{M}(P \cap w^{-1}(Q))$ induced by the isomorphism $Ad(w)$.  

We denote by \( \alpha_w(P, Q) \) [or simply \( \alpha_w \)] the functorial morphism \( R_w \rightarrow \text{res}^G_Q \circ \text{ind}_P^G \) such that for any representation \( (\rho, V) \) of \( P \) and \( f \in T_w(V) \) we have
\[
\alpha_w(P, Q)(f)(g) = 0 \text{ if } g \notin PwQ \text{ and } \\
\alpha(P, Q)(f)(pwq) = \rho(p)f(q)
\]
where \( f \in R_w(V) = \text{ind}_P^G(V) \) is given by a function \( f : G \rightarrow V \), such that \( f(pg) \equiv \rho(p)f(g), g \in G, p \in P. \)

**Lemma 0.107.** [Mackey] Let \( w_i \in G, 1 \leq i \leq r \) be representatives of double cosets \( P \backslash G/Q. \) Then the morphism
\[
\oplus_{i=1}^r \alpha_{w_i} : \oplus_{i=1}^r R_{w_i} \rightarrow \text{res}^Q_G \circ \text{ind}_P^G
\]
is an isomorphism.

**Proof.** By the definition the space of the representation \( \text{res}^Q_G \circ \text{ind}_P^G(V) \) consists of functions \( f : G \rightarrow V \), such that \( f(pg) \equiv \rho(p)f(g), g \in G, p \in P, \)
and the group \( Q \) acts by right shifts. The decomposition \( G = \cup_{i=1}^r Pw_iQ \)
induces a direct sum decomposition
\[
\text{res}^Q_G \circ \text{ind}_P^G(V) = \oplus_{i=1}^r R_{w_i}^r(V)
\]
of \( Q \)-submodules where
\[
R_{w_i}^r(V) := \{ f \in \text{res}^Q_G \circ \text{ind}_P^G(V) | \supp(f) \subset Pw_iQ \}
\]
Now we observe that the map \( f \rightarrow \hat{f}, h(q) := f(w_iq) \) defines an equivalence between \( Q \)-modules \( R_{w_i}^r(V) \) and \( R_{w_i}(V). \)

We will need a variant

**Example 0.108.** Let \( k = \mathbb{F}_q \) be a finite field, \( G = GL(n, k) \) and \( P = Q = B \) and \( U_0 \subset B \) be the subgroup of unipotent upper triangular matrices. For any representation \( \pi \) of \( T \) we denote by \( \text{Inf}_B^P(\pi) \in \text{Ob}(\mathcal{M}(B)) \) the composition \( \pi \circ p_T \) where \( p_T : B \rightarrow T \) is the projection \( B \rightarrow B/U_0 = T. \)
Since [see Problem 0.103] \( G = \cup_{w \in S_n} BwB \) we have an isomorphism
\[
\text{res}_G^B \circ \text{ind}_B^G(\pi) = \oplus_{w \in S_n} \text{ind}_{Bw(B)}^B(\pi^w)
\]
where \( \pi^w(b) := \pi(wbw^{-1}). \)

Let \( c_T^B : \mathcal{M}(B) \rightarrow \mathcal{M}(T) \) be the functor of \( U_0 \)-invariants.

**Problem 0.109.** For any representation \( \rho \) of \( T \) and any subgroup \( H \) of \( B \) containing \( T \) we have
\[
c_B^T \circ \text{ind}_H^B \circ \text{Inf}_T^H(\rho) = \rho
\]
We see that
\[
c_B^T(\text{ind}_{Bw(B)}^B(\pi^w)) = \pi^w
\]
and therefore
\[
r_T^G \circ i_T(\pi) = \oplus_{w \in S_n} \pi^w
\]
where \( r_T := c_T^B \circ \text{res}_G^B, i_T := \text{ind}_B^T \circ p. \)
Definition 0.110. Let $\theta$ be a partition of $n$.

1. We denote by $i_{\theta}$ the functor from the category $\mathcal{M}(M_{\theta})$ to $\mathcal{M}(G)$ which associates with a representation $V$ of the group $M_{\theta}$ the representation $i_{\theta}(V)$ where we extend the action of the group $M_{\theta}$ on $V$ to the action of $P_{\theta}$ through the homomorphism $P_{\theta} \to M_{\theta}$.
2. We denote by $r_{\theta}$ the functor from the category $\mathcal{M}(G)$ to $\mathcal{M}(M_{\theta})$ which associates with a representation $V$ of the group $G$ the action of the group $M_{\theta}$ on the subspace $V^{U_{\theta}}$ of $U_{\theta}$-invariants.

Let $\theta'$ be another partition of $n$. As we know the partition $\theta' \wedge \theta$ defines a parabolic subgroup $P_{\theta}(\theta)$ of the group $M_{\theta}$.

3. We define the functors $r_{\theta'}(\theta)$ from the category $\mathcal{M}(M_{\theta})$ to $\mathcal{M}(\theta' \wedge \theta)$ and $i_{\theta}(\theta)$ from the category $\mathcal{M}(\theta' \wedge \theta)$ to $\mathcal{M}(M_{\theta}')$ in the way we defined the functors $i_{\theta}$ and $r_{\theta}$.

4. For any representation $(\pi, V)$ of the group $M_{\theta}$ we denote by $\Psi_{\theta}'(V) \subset i_{\theta}(V)$ the subspace of functions $f : G \to V$ such that $f \in i_{\theta}(V)$ and supp$(f) \subset P_{\theta}P_{\theta'}$.

5. The group $P_{\theta}'$ acts on the space $\Psi_{\theta}'(V)$ by right translations and we denote by $\Psi_{\theta}'(V)$ the representation of the group $M_{\theta}$ on the subspace of $U_{\theta}$-invariants in $\Psi_{\theta}'(V)$. By the construction $\Psi_{\theta}'(V)$ is a subrepresentation of $i_{\theta}(V)$.

Problem 0.111. For any $f \in \Psi_{\theta}'(V)$ we denote by $a(f)$ the restriction of $f$ to $M_{\theta} \subset P_{\theta}P_{\theta'}$. Show that:

1. $a(f)(qm') = \pi(q)f(m')$ for all $m' \in M_{\theta'}$, $q \in M_{\theta} \cap P_{\theta} = P_{\theta}(\theta')$. In other words $a$ is a morphism from the representation $\Psi_{\theta}'(V)$ of the group $M_{\theta}$ to $i_{\theta}(\theta) \circ r_{\theta}(\theta)(V)$.
2. The morphism $a : \Psi_{\theta}'(V) \to i_{\theta}(\theta) \circ r_{\theta'}(\theta)(V)$ is an isomorphism.

Proposition 0.112. We denote by $X = X(\theta, \theta')$ the set of double cosets $W_{\theta} \setminus W_{\theta'} / W_{\theta'}$ and choose representatives $w_{x} \in W$ for $x \in X$. Then we have a functorial isomorphism

$$r_{\theta'} \circ i_{\theta} = \oplus_{x \in X} i_{\theta'}(\theta^{w_{x}}) \circ \sigma_{x} \circ r_{\theta'}w_{x}^{-1}(\theta)$$

where $\sigma_{w}$ is the isomorphism between categories $\mathcal{M}(M(\theta^{w} \wedge \theta'))$ and $\mathcal{M}(M(\theta \wedge \theta^{w^{-1}}))$ defined by the conjugation by $w \in W$.

Proof. The proof is completely analogous to the proof of the Mackey’s lemma. \hfill \Box

5.4. Mackey theory for local fields.

6. Representation of $GL(n, F)$. 

Let $F$ be a local non-archimedian field, $\mathcal{O} \subset F$ the ring of integers, $\mathcal{P} \subset \mathcal{O}$ the maximal ideal, $t$ a generator of $\mathcal{P}$ and $k = \mathcal{O}/\mathcal{P}$ the residue field $k = \mathbb{F}_{q}$. We denote by $\nu : F^{*} \to \mathbb{Z}$ the valuation such that $\nu(t) = 1$ and
define the norm by \(|x| = q^{-\nu(x)}\). For any \(x \in \mathcal{O}\) we denote by \(\bar{x}\) the image of \(x\) in \(k\).

Let \(G\) be a reductive \(F\)-group and \(G = G(F)\). Then \(G\) is an \(l\)-group. Here is the main result.

**Theorem 0.113.** For any compact open subgroup \(K \subset G\) the algebra \(\mathcal{H}_K(G)\) is finitely generated as a module over it center.

**Corollary 0.114.** All irreducible smooth representations of \(G\) are admissible.

**Proof.** Let \(\pi : G \to Aut(V)\) be a smooth irreducible representation. We want to show that for any compact open subgroup \(K \subset G\) the space \(V^K\) is finite-dimensional. As follows from Proposition 0.47 and Lemma 0.19 the space \(V^K\) is an irreducible \(\mathcal{H}_K(G)\)-module. By the Schur’s lemma the center \(Z_K\) of \(\mathcal{H}_K(G)\) acts by scalars. Since the algebra \(\mathcal{H}_K(G)\) is finitely generated as a module over \(Z_K\) we see that the space \(V^K\) is finite-dimensional. □

We concentrate on the case when \(G = GL(n, F)\). Although, arguments in the general case are not much different from the case when \(G = GL(n, F)\) they require a thorough knowledge of the structure theory of reductive groups over local fields.

### 6.1. Lattices.

**Definition 0.115.** (1) Let \(V\) be a \(F\)-vector space of dimension \(d < \infty\). A lattice of \(V\) is an \(O\)-submodule \(L\) of \(V\) such that \(L \otimes_O F = V\).

(2) A basis of a lattice \(L\) is a set of vectors \(l_1, \ldots, l_d \in L\) such that the map 
\[
O^d \to L, (c_1, \ldots, c_d) \mapsto c_1 l_1 + \cdots + c_d l_d
\]
is a bijection.

**Problem 0.116.** Let \(L \subset V\) be a lattice, \(\bar{L} := L/P L\). Then

(1) \(\text{dim}_k(\bar{L}) = d\).

(2) Let \(l_1, \ldots, l_d \in L\) be elements such that the set \(\bar{l}_1, \ldots, \bar{l}_d \in \bar{L}\) is a basis of the \(k\)-vector space \(\bar{L}\). Then the set \(l_1, \ldots, l_d\) is a basis of \(L\).

(3) Let \(M \subset L\) be an \(O\)-submodule and \(W \subset V\) be the \(F\)-subspace generated by \(M\). Then \(M\) is a lattice of \(W\).

(4) For any \(l \in L - P L\) there exists a subspace \(W\) of \(V\) such that \(l \notin W + P L\).

(5) Let \(M\) be another lattice an \(m \in M - P M\). Then there exists a subspace \(W\) of \(V\) of codimension 1 such that \(l \notin W + P L\) and \(m \notin W + P M\).

**Lemma 0.117.** Let \(L \subset V\) be a lattice.

(1) For any \(l \in L - P L\) and any subspace \(W\) of \(V\) of codimension 1 such that \(l \notin W + P L\) the map \(\kappa : O \oplus L \cap W \to L, (a, x) \mapsto av + x, a \in O, x \in L_W\) is a bijection.
(2) For any other lattice $M$ of $V$ there exists a basis $l_1, ..., l_d$ of $L$ and non-zero elements $c_1, ..., c_d \in F$ such that the set $c_1l_1, ..., c_dl_d$ is a basis of $M$.

**Proof.** (1) Since $l \notin W$ the map $\kappa$ is an imbedding. To show the surjectivity we observe that any $x \in L$ can be uniquely written in the form $x = y + al, a \in F; y \in W$. I claim that $a \in \mathcal{O}$.

Assume that $a \notin \mathcal{O}$. Then $a^{-1} \in \mathcal{P}$ and we have $l = a^{-1}x - a^{-1}y$. But such a decomposition contradicts the assumption that $l \notin W + \mathcal{P}L$.

Since $a \in \mathcal{O}$ we see that $y = x - al \in L$. So $y \in L \cap W$.

We prove (2) by the induction in $d = \dim_F(V)$. Choose $c_1 \in F$ such that $c_1L$ contains $\mathcal{P}M$ but does not contain $M$. Choose now a vector $l \in L$ such that $c_1l \notin \mathcal{P}c_1M$. As was shown in the last problem there exists a subspace $W$ of $V$ of codimension 1 such that $c_1l \notin W + \mathcal{P}M$ and $l \notin W + \mathcal{P}L$. By the inductive assumptions we can find a basis $l_2, ..., l_d$ of $L \cap W$ and elements $c_2', ..., c_d' \in F$ such that the set $c_2'l_2, ..., c_d'l_d$ is a basis of $M \cap W$. But it follows now from the part (1) that the set $c_1l_1, ..., c_dl_d, c_i = c_1^{-1}c_i', 1 < i \leq d$ is a basis of $M$.

**Problem 0.118.** For any lattice $L$ and a complete flag $W_1 \subset W_2 \subset ... \subset W_d = V$ of $V$ there exists a basis $l_1, ..., l_d$ of $L$ such that $l_i \in L \cap W_i, 1 \leq i \leq d$.

**6.2. The Geometry of GL(n, F).** Let $G = GL(n, F)$.

To describe the geometry of $G$ we introduce a number of definitions. These definitions can be extended to the case of an arbitrary reductive group.

**Definition 0.119.**

1. Let $K_0 := GL(n, \mathcal{O}) \subset GL(n, F)$.
2. We denote by $T$ the diagonal group, by $B \subset G$ the subgroup of upper-triangular matrices, by $U \subset B_0$ the subgroup of unipotent matrices and by $\bar{U} \subset G$ the subgroup of lower-triangular matrices. Then $B = TU$ and $T_0 := T \cap K_0$ is the maximal compact subgroup of $T$.
3. We denote by $\Lambda$ the quotient $\Lambda := T/T_0$. The map from $\mathbb{Z}^n$ to $\Lambda$ which associates with $(e_1, ..., e_n) \in \mathbb{Z}^n$ the class of the diagonal matrix with elements $a_{ii} = t^{e_i}, 1 \leq i \leq n$ provides an identification of the group $\Lambda$ with $\mathbb{Z}^n$.
4. We define

$$\Lambda^+ = \{(e_1, ..., e_n)|e_1 \geq e_2 \geq \cdots \geq e_n\}.$$  

The subset $\Lambda^+ \subset \Lambda$ is called the **Weyl chamber**.

5. We define

$$\Lambda^{++} = \{(e_1, ..., e_n)|e_1 > e_2 > \cdots > e_n\}$$
(6) We fix an imbedding $\Lambda \hookrightarrow T$ by
\[ \lambda = (l_1, \ldots, l_n) \mapsto \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix}. \]

(7) For any $r > 0$ we denote by $K_r \subset K_0$ the kernel of the natural projection $p_r : \text{GL}(n, \mathcal{O}) \to \text{GL}(n, \mathcal{O}/\mathbb{P}^r)$. We say that $K_r, r > 0$ are congruence subgroups of $G$.

(8) For any congruence subgroup $K$ of $G$ we denote by $\mathcal{H}_K(K_0) \subset \mathcal{H}_K(G)$ the subset of measures supported on $K_0$.

(9) We fix a Haar measure $dg$ on $\text{GL}(n, F)$ such that $\int_{K_0} dg = 1$.

**Remark 0.120.** To simplify notations we denote the subgroup of upper-triangular matrices by $B$ and not by $B_0$ as in the definition 0.86.

**Problem 0.121.** Show that

1. $G = K_0 \Lambda^+ K_0$. [The Cartan decomposition].
2. $G = K_0 B = B K_0$. [The Iwasawa decomposition].
3. For any congruence subgroup $K \subset G$ and any $k', k'' \in K_0, g \in G$ we have $a(k')a(g)a(k'') = a(k'gk'')$ where $a(k) \in \mathcal{H}_K$ are as in 0.45.
4. For any set $\{x_1, \ldots, x_r\}$ of representatives for $K \setminus K_0 = K_0/K \mathcal{H}_0 = \mathcal{H}_K(K_0) \subset \mathcal{H}_K(G)$ is equal to the span of $a(x_i)$.
5. The shifts of congruence subgroups form a basis for the topology for $G$.

A hint. Use the Lemma 0.117 and the Problem 0.118 to prove (1) and (2).

**Definition 0.122.** Let $\lambda = (m_1, \ldots, m_n)$ where the sequence $m_i$ decreasing but is not necessarily strictly decreasing
\[ m_1 = \ldots = m_{i_1} > m_{i_1+1} = \ldots = m_{i_1+i_2} > \ldots > m_{i_1+i_2+\ldots+i_r+1} = \ldots = m_n. \]

1. We denote by $P_{\lambda} \subset \text{GL}(n, F)$ the subgroup of block upper-triangular matrices with blocks of the size $n_j := m_{i_j} - m_{i_{j-1}}$, $1 \leq j \leq r$.
2. We denote by $\bar{P}_\lambda \subset \text{GL}(n, F)$ the subgroup of block lower-triangular matrices with blocks of the size $n_j, 1 \leq j \leq r$.
3. We denote by $P_\lambda \subset \text{GL}(n, F)$ the subgroup of block diagonal matrices with blocks of the size $n_j, 1 \leq j \leq r$.
4. We denote by $U_{\lambda}$ the unipotent radical of $P_{\lambda}$ and by $\bar{U}_{\lambda}$ the unipotent radical of $\bar{P}_\lambda$.
5. We denote by $\Lambda^+_M$ the subset of $\mu \in \Lambda^+$ such that $M_\mu = M_{\lambda}$.
6. For any congruence subgroup $K$ we define
\[ K_\lambda^+ = K \cap U_{\lambda}, K_\lambda^- = K \cap \bar{U}_{\lambda}, K_\lambda^0 = K \cap M_{\lambda}. \]

**Problem 0.123.** Show that

1. $M_{\lambda}$ is a standard Levi subgroup of $P_{\lambda}$ and of $P_{\lambda-1}$ for any $\lambda \in \Lambda^+$. Moreover $M_{\lambda}$ is the centralizer of $\lambda$ in $G$. 
(2) Any Levi subgroup is conjugated to a standard Levi subgroup $M_\lambda < G$ but such standard Levi subgroup is not necessarily unique.

(3) For each semisimple $g \in G$ the subgroups $P_g$ and $P_{g^{-1}}$ form a pair of opposite parabolic subgroups.

(4) For any $\lambda \in \Lambda^+$

(a) $K = K_\lambda^+ K_\lambda^0 K_\lambda^-$, $\lambda(K \cap P)\lambda^{-1} \subset K \cap P$ and $\lambda^{-1} K_\lambda^- \lambda \subset K_\lambda^-$.

(b) $(\text{Ad}\lambda^n)|_{K_\lambda^+} \to \{e\}$ as $n \to \infty$, and $(\text{Ad}\lambda^{-n})|_{K_\lambda^-} \to \{e\}$ as $n \to \infty$.

(c) $\bigcup_n \text{Ad}(\lambda^{-n}) (K_\lambda^+) = U_\lambda$.

Remark 0.124. This problem has a natural extension to the case of an arbitrary reductive group.

6.3. The structure of Hecke algebras. Let $K \subset K_0$ be a congruence subgroup. Since $K_0$ normalizes $K$ we have $Kx = xK$ for all $x \in K_0$.

Let $C$ be the span of $\{a(\lambda) | \lambda \in \Lambda^+\}$. The next proposition is key for our analysis of the structure of the algebra $H_K(G)$.

Proposition 0.125.

(1) $H_K(G) = H_0 C H_0$.

(2) $C$ is a commutative, finitely generated algebra.

Remark. This is saying that that $H_K(G)$ is somehow of finite type but it is neither generated over $C$ on the left nor on the right but rather “in the middle”. It is a question whether one can use this property directly to show that $H_K(G)$ is of finite over the center.

Proof. (1). By the Cartan decomposition, $G = \bigcup_{\lambda \in \Lambda^+} K_0 \lambda K_0$. Since $K_0 = \bigcup_{i=1}^r K x_i = \bigcup_{i=1}^r x_i K$ we have

$$G = \bigcup_{\lambda \in \Lambda^+, 1 \leq i, j \leq r} K x_i \lambda x_j K.$$  

This implies that the $a(x_i \lambda x_j)$ form a basis for $H_K(G)$. But we as we have seen $a(x_i \lambda x_j) = a(x_i) a(\lambda) a(x_j)$. This equality proves (1). \qed

To prove the part (2) of the Proposition we have to show that

$$a(\lambda) a(\nu) = a(\nu) a(\lambda)$$

for all $\lambda, \nu \in \Lambda^+$. Of course it is sufficient to show that $a(\lambda) a(\nu) = a(\lambda + \nu)$. In other words it is sufficient to show that

$$(K \lambda K) (K \nu K) = K \lambda \nu K$$

for all $\lambda, \nu \in \Lambda^+$. This equality is not trivial since the elements of $\Lambda^+$ do not normalize $K$. The idea is to decompose $K$ into parts that can be moved to the right and to the left.

We will use the following notation. For any congruence subgroup $K$ we define

$$K^+ = K \cap U, K^- = K \cap U$$

and $K^0 = K \cap T$.

Lemma 0.126.

(1) $K = K^+ K^0 K^-$.  

(2) $K = K^- K^0 K^+$.  

(3) If $\lambda \in \Lambda^+$ then $\lambda K^+ \lambda^{-1} \subset K^+$ and $\lambda^{-1} K^- \lambda \subset K^-$. 

(4) For any $\lambda \in \Lambda$ we have 
$$\int_{K_i \lambda K_i} dg / \int_{K_i} dg = \text{mod}^{-1}_U(\lambda).$$

**Proof.** (1) Since $K_j \subset K_i$, $j > i$ are normal subgroups and $i K_i = \{e\}$ it is sufficient to prove that for any $i > 0, k \in K_i$ there exist $k^+ \in K_i^+, k^0 \in K_i^0, k^- \in K_i^-$ such that $k \in k^+ k^0 k^- K_{i+1}$. To prove the existence of such a decomposition consider a map $\tilde{K}_i : K_i \rightarrow M_n(k)$ given by $x \rightarrow \frac{x - 1}{2}$ where as before $y \rightarrow \tilde{y}$ is the projection $O \rightarrow O/P = k$.

**Problem 0.127.** Show that the map $\tilde{K}_i : K_i \rightarrow M_n(k)$ defines a group isomorphism $\kappa_i : K_i K_{i+1} \rightarrow M_n(k)$.

To finish the proof of (1) it is sufficient now to observe that any matrix in $M_n(k)$ is a sum of an upper nilpotent, a lower nilpotent and a diagonal matrices.

The proof of (2) is completely analogous. The part (3) is immediate and (4) follows immediately from (1).

We can now finish the proof of the part (2) of the Proposition. We have reduced the problem to showing $K \lambda K \nu K = K \lambda \nu K$. It is clear that $K \lambda \nu K \subset K \lambda K \nu K$. On the other hand we have

$$K \lambda K \nu K = K \lambda K^+ K^0 K^- \nu K$$

$$= K(\lambda K^+ \lambda^{-1} \lambda K^0 \lambda^{-1} \lambda \nu \nu^{-1} K^- \nu)_K$$

$$\subset \lambda K^+ K^0 \lambda \nu K^- K = K \lambda \nu K$$

where the inclusions $\lambda K^+ \lambda^{-1} \subset K^+, \lambda K^0 \lambda^{-1} \subset K^0$ and $\nu^{-1} K^- \nu \subset K^-$ follow from the previous Lemma.

**Remark.** This decomposition is true only for congruence subgroups $K_r, r > 0$ but not for the group $K_0$.

### 6.4. Modules.

We have shown that $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$ with $C$ commutative and that $a(\lambda^n) = a(\lambda)^n$ for $\lambda \in \Lambda^+$. We use this information to study $\mathcal{H}_K$-modules. In this subsection we fix a congruence subgroup $K \subset G$ and $\lambda \in \Lambda^+$.

Let $(\pi, V)$ be a representation of $G$ and $\pi_K$ be the associated representation of $\mathcal{H}_K$ on $V^K$. I’ll use notations and results of the Problem 0.45.

**Lemma 0.128.** $\text{Ker} a(\lambda)|_V^K = \text{Ker} e_{\lambda^{-1} K_{\lambda}^+\lambda}|_V^K$.

**Proof.** By the definition $a(\lambda) = e_K * \mathcal{E}_\lambda * e_K$. Using the decomposition $K = K_\lambda^+ K_\lambda^0 K_\lambda^-$ we see that $e_{K_\lambda} = e_{K_\lambda^+} * e_{K_\lambda^0} * e_{K_\lambda^-}$ and therefore

$$a(\lambda) = \mathcal{E}_\lambda * e_{\lambda^{-1} K_{\lambda}^+\lambda} * e_{\lambda^{-1} K_{\lambda}^0 \lambda} * e_{\lambda^{-1} K_{\lambda}^- \lambda} e_K$$

Since $\lambda^{-1} K_{\lambda}^0 \lambda = K^0$ and $\lambda^{-1} K_{\lambda}^- \lambda \subset K^-$ we see that $a(\nu) = \mathcal{E}_\nu * e_{\nu^{-1} K_{\nu}^+} * e_K$.

Since $e_K$ acts as the identity on $V^K$ and $\mathcal{E}_\nu$, which acts on $V$ as $\pi(\nu)$, is invertible we see that
\[ \text{Ker} \ a(\lambda)|_{V^K} = \text{Ker} \ e_{\nu^{-1}K_{\lambda}^+}|_{V^K} \]

\[
\text{Proposition 0.129.} \quad \text{For any} \ \lambda \in \Lambda^+ \ \text{we have}
\]
\[ \cup_n \text{Ker} \ a(\lambda^n) \cap V^K = V(U_\lambda) \cap V^K \]

where as before \( V(U_\lambda) \subset V \) is the span of vectors \( v - \pi(u)v, v \in V, u \in U_\lambda \).

\[
\text{Proof.} \quad \text{Let} \ U_m := \lambda^{-m}K^+_{\lambda} \lambda^m. \ \text{As we know [see Problem 0.123]} \ U_1 \subset U_2 \subset \cdots \subset U_m \subset \cdots \ \text{and} \ U = \cup_m U_m. \ \text{So}
\]
\[ \text{Ker} \ e_{U_1} \subset \text{Ker} \ e_{U_2} \subset \cdots \]

and as follows from the Problem 0.74 we have \( V(U_\lambda) = \cup_m \text{Ker} \ e_{U_m} \). The Proposition follows now from Lemma 0.128 applied to \( \nu = \lambda^n \).

Let \((\rho, V)\) be a representation of \( G, \lambda \in \Lambda^+, P = P_\lambda = M_\lambda U_\lambda \) and \( K \) a congruence subgroup of \( G \). To simplify notations we will write in the formulation of the next problem \( M \) instead of \( M_\lambda \) and \( U \) instead of \( U_\lambda \). As before we denote by \( J : V \to V_U \) the projection onto coinvariants and by \( \rho_M : M \to \text{Aut}(V_U) \) the representation as in 0.75. We fix \( \mu \in \Lambda^+ \).

\[
\text{Proposition 0.130.} \quad \begin{align*}
(1) & \quad J(\rho(a_{\mu})(v)) = \Delta^{1/2}(\mu) r_M(\mu)J(v), \ v \in V^K \\
(2) & \quad \text{The image} \ J(V^K) \subset V_U \ \text{is} \ r_M(\mu)-\text{invariant.}
\end{align*}
\]

Assume now that the representation \((\rho, V)\) of \( G \) is admissible.

Then

\[
(3) \quad \text{The restriction of} \ r_M(\rho)(\mu) \ \text{on} \ J(V^K) \ \text{is invertible.}
\]

\[
(4) \quad J(V^K) = r_M(V)|_{V^K}. \]

\[
\text{Proof.} \quad (1) \ \text{follows from the definition the representation} \ r_M(\rho).
\]

(2) We have to show that \( r_M(\rho)(\mu)(J(v)) \in J(V^K) \) for any \( v \in V^K \). Let \( w := \rho(\mu)(v) \). Then by (1) we have \( r_M(\rho)(\mu)(J(v)) = cJ(w) \), \( c \in \mathbb{C} \). So it is sufficient to show that \( J(w) \in J(V^K) \).

Let \( \tilde{v} := \int_k \rho(k)wdk \). It is clear that \( \tilde{v} \in V^K \). On the other hand as follows from the Problem 0.123 \( xw = w \) for all \( x \in K^0K^- \) and therefore \( \tilde{v} := \int_{K^+} \rho(k^+)wdk^+ \). So

\[
J(w) = J(\int_{K^+} \rho(k^+)w)dk^+ = J(\tilde{v}) \in J(V^K).
\]

To prove (3) we observe that \( r_M(\rho)(\mu) \) is an invertible transformation of the space \( r_M(V) \). Therefore the restriction of \( r_M(\rho)(\mu) \) to \( J(V^K) \subset r_M(V) \) does not annihilate any non-zero vector. Since \( V \) is admissible the space \( J(V^K) \) is finitely dimensional and therefore the restriction of \( r_M(\rho)(\lambda) \) to \( J(V^K) \) is invertible.

The inclusion \( J(V^K) \subset (r_M(V))|_{V^K} \) follows from the definition. So for a proof of (4) it is sufficient to prove that \( (r_M(V))|_{V^K} \subset J(V^K) \). As follows
from (3) it is sufficient to show that for any $\xi \in r_M(V)^{K^0}$ there exist $\mu \in \Lambda^+$ such that $r_M(\rho(\mu))(\xi) \in V^K$.

To find such $\mu$ we fix a lift $v \in V^{K^0}$ of $\xi$. Since $V$ is smooth there exists a congruence subgroup $K'$ such that $v$ is $K'$-invariant. Then as follows from 0.123 the vector $v' := \rho(\lambda N)v$ is invariant with respect to $\lambda N K', \lambda^{-N}$ as well as respect to $K^0$.

Since, [see 0.123] \[ \cup_N \lambda^N K' \lambda^{-N} = U \] there exists $N > 0$ such that $K^- \subset \lambda^N K' \lambda^{-N}$. Set $w := \int_K \rho(k)v'$. It is clear that $w \in V^K$ and that $w$ is $K^-$-invariant. On the other hand since $\lambda k_0 = k_0 \lambda$ for all $k_0 \in K^0$ and and $v'$ is $K^0$-invariant we see that $w$ is $K^0 K^-$-invariant. Therefore

$$w := \int_K \rho(k)v' = \int_{K^+} \rho(k)v'.$$

On the other hand we have

$$J(\rho(k_+ v')) = J(v'), k_+ \in K^+.$$  

So $J(w) = J(v') = r_M(\rho)(\lambda N)\xi$ and we see that $r_M(\rho)(\lambda N)\xi \in J(V^K)$. \[ \square \]

**Remark 0.131.** One can show that the condition of the admissibility of $\rho$ is not necessary for the validity of the parts (3) and (4) of the Proposition.

**6.5. An Application.** We start with a result on representations of the group $SL(n, F)$. Define

$$\Lambda_1 := \{(m_1, \ldots, m_n) \in \Lambda | m_1 + \cdots + m_n = 0\}$$

and $\Lambda^+_1 := \Lambda^+ \cap \Lambda_1$. As follows from the Cartan decomposition for $GL_n(F)$ we have an analogous decomposition

$$SL(n, F) = \cup_{\lambda \in \Lambda_1} SL(n, O)\lambda SL(n, O)$$

**Definition 0.132.** A representation $(\pi, V)$ of a group $GL(n, F)$ or $SL(n, F)$ is *quasi-cuspidal* if $r_{M, U}(V) = \{0\}$ for all parabolic subgroups $P = MU \neq G$.

**Remark 0.133.** (1) As all unipotent subgroups of $GL(n, F)$ lie in $SL(n, F)$ a representation of $GL(n, F)$ is quasi-cuspidal if the restriction onto $SL(n, F)$ is quasi-cuspidal.

(2) Since any parabolic subgroup is conjugate to a standard one we can restate the definition by saying that $r_{M, U}(V) = \{0\}$ for unipotent radicals of standard parabolic subgroups $P \neq G$.

**Lemma 0.134.** Let $(\pi, V)$ be a quasi-cuspidal representation of $SL(n, F)$, $v \in V$ and $K$ be a congruence subgroup. Then $\rho(a(\lambda))(v) = 0$ for almost all $\lambda \in \Lambda^+_1$.
Proof. For any \( i, 1 \leq i < n \) we define
\[
\nu_i := (n - i, n - i, ..., n - i, -i, -i, ..., -i) \in \Lambda_1^+ \]
where \( n - i \) appears \( i \) times and \(-i\) appears \( n - i\) times. Let \( \Lambda_1^+(0) \subset \Lambda_1^+ \) be the subsemigroup generated by \( \nu_i, 1 \leq i < n \).

Problem 0.135.
(1) The set \( \nu_i, 1 \leq i < n \) generates the semigroup \( \Lambda_1^+ \).
(2) For any \( N > 0 \) there exists a finite subset \( S \subset \Lambda_1^+ \) such that \( S + NA_1^+ = \Lambda_1^+ \).

Since \( a(\lambda)a(\nu) = a(\nu)a(\lambda) \) for \( \lambda, \nu \in \Lambda_1^+ \) [see Proposition 0.125] for a proof of the Lemma it is sufficient to show that \( a(\nu_i)^Nv = 0 \) for any \( i, 1 \leq i < n - 1 \) and any \( v \in V \) if \( N >> 0 \). In other words we have to show that \( \cup_n Ker(a(\nu_i)^N) = V \).

Let \( P_i = M_iU_i \) be the parabolic subgroup corresponding to \( \nu_i \). Since \( (\pi, V) \) is quasi-cuspidal we have \( V = V(U_i) \) and it follows from Proposition 0.130 that \( \cup_n Ker(a(\nu_i)^N) = V \).

Theorem 0.136. A representation \( (\pi, V) \) of \( SL(n, F) \) is quasi-cuspidal if and only if it is compact.

Proof. a) Assume that a representation \( (\pi, V) \) of \( SL(n, F) \) is quasi-cuspidal. To prove that \( (\pi, V) \) is compact we have to show that for any congruence subgroup \( K \) and any \( v \in V \) the function
\[
SL(n, F) \to V, g \to \pi(e_K)\pi(g)v \]
has compact support. By changing \( K \) we can assume that our vector \( v \) is \( K \)-invariant. Then,
\[
\pi(e_K)\pi(g)v = \pi(e_K)\pi(g)\pi(e_K)v \]
As follows from the Cartan decomposition for \( SL_n(F) \) it is sufficient to show that the function
\[
\Lambda_1^+ \to V, \lambda \to \pi(e_K)\pi(k_0)v \]
has finite support for any \( k_0 \in K^0 \). By replacing \( v \) by \( \pi(k_0)v \) we see that it is sufficient to prove that the function
\[
\Lambda_1^+ \to V, \lambda \to \pi(e_K)v \]
on \( \Lambda_1^+ \) has finite support. But this is an immediate consequence of the previous Lemma.

b) Conversely, suppose that the representation \( (\pi, V) \) on \( SL(n, F) \) is compact. By reversing the reasoning given above, we see that the function \( \lambda \mapsto \pi(a(\lambda))v \) from \( \Lambda_1^+ \) to \( V \) has finite support for all \( v \in V \). Therefore for any non-zero \( \lambda \in \Lambda_1^+, v \in V \) we have \( a(\lambda^r)(v) = 0 \) for \( r >> 0 \). it follows then from Proposition 0.130 (1) and Proposition 0.125 that \( V(U_\lambda) = V \) for any non-zero \( \lambda \in \Lambda_1^+ \). Therefore \( r_{M_nU_\lambda}(V) = \{0\} \) for any non-zero
6. REPRESENTATION OF $GL(n, F)$.

It follows from Problem 0.123 (1) that $r_{M,U}(V) = \{0\}$ for all proper parabolic subgroups $P$ of $G$.

6.6. Irreducibility implies Admissibility.

**Definition 0.137.** A representation of $G$ is cuspidal if it is both quasi-cuspidal and finitely generated.

Let $G = GL_n(F)$ and $G^0 := \{g \in G|\text{det}(g) \in \mathcal{O}^*\}$. The same arguments as in the proof of Theorem 0.136 prove the following result.

**Theorem 0.138.** [Harish-Chandra] A representation $(\pi, V)$ of $G^0$ is quasi-cuspidal if and only if it is compact.

**Corollary 0.139.** Any irreducible cuspidal representation of $G$ is admissible.

**Proof of the Corollary.** Let $(\rho, W)$ be an irreducible cuspidal representation of $G$. Then $W$ is a finitely generated $G^0$-module. Really since $[G : ZG^0]$ is finite, $W|_{ZG^0}$ is a finitely generated module. On the other hand since $(\rho, W)$ is irreducible it follows from the Schur lemma that $Z$ acts on $W$ by multiplication by scalars. Hence $W$ is finitely generated as a $G^0$-module.

By Harish-Chandra’s theorem, $W|_{G^0}$ is compact. As follows from Proposition 0.56 all finitely generated compact representations are admissible. So $W|_{G^0}$ is admissible. As $G^0$ contains all compact subgroups, the corollary follows.

This corollary is the first step toward our goal of proving

**Theorem 0.140.** Any smooth irreducible representation of $G$ is admissible.

**Proof.** We start with a reminder on the normalized induction and of the Jacquet Functors.

Since we want to apply the results of the previous section to the case when $G$ is a Levi component of some larger group we consider the case when $G$ can be a product of groups of the form $GL(m, F)$.

**Definition 0.141.** Let $P = MU$ be a Parabolic subgroup of $G$.

1. We denote by $r_{M,U} : \mathcal{M}(G) \to \mathcal{M}(M)$ the Jacquet functor which associates with a representation $(\pi, V)$ of $G$ the representation $r_{M,U}(\pi)$ of $M$ on $V/V(U)$ such that

$$r_{M,U}(\pi)(m)(J(v)) := \Delta_p^{-1/2}(m)J(\rho(m)v)$$

where $J : V \to r_{M,U}(V)$ is the natural projection. If $M$ is a standard Levi subgroup of $G$ we often write $r_M$ instead of $r_{M,U}$.

2. We denote by $i_{M,U} : \mathcal{M}(M) \to \mathcal{M}(G)$ the induction functor defined as follows: given a representation $\rho : M \to Aut(V)$ extend it trivially to a representation of $P$ on $V$ and define $i_{M,U}(\rho) := \text{ind}_P^G(\rho)$. If $M$ is a standard Levi subgroup of $G$ we often write $i_M$ instead of $i_{M,U}$. 


Remark 0.142. The Iwasawa decomposition implies that $\text{ind}_P^G = \text{Ind}_P^G$.

Let $P = MU, Q = NV$ be parabolic subgroups of $G$ such that $Q \subset P$ and $N \subset M$ and $Q_M := Q \cap M$. As we know $Q_M$ is a parabolic subgroup of $M$ and $Q_M = N \ltimes V_M, V_M := V \cap M$.

Proposition 0.143. 
1. $r_{M,U}$ is left adjoint to $i_{M,U}$.
2. If $N$ is a Levi subgroup of $M$, then $r_{N,VM} \circ r_{M,U} = r_{N,V}$ and $i_{M,U} \circ i_{N,VM} = i_{N,V}$.
3. $i_{M,U}$ maps admissible representations of $M$ to admissible representations of $G$.
4. $i_{M,U}$ and $r_{M,U}$ are exact.
5. $r_{M,U}$ maps finitely generated representations of $G$ to finitely generated representations of $M$.

Proof. The proof of (1) is contained in Problem 0.70 which is a modification of Lemma 0.67. The part (2) is a simple verification and parts (3) and (4) follow from the equality $\text{ind}_P^G = \text{Ind}_P^G$ since as we already know the functor $\text{ind}_P^G$ is exact and takes admissible representations to admissible ones.

To prove (5) consider a smooth finitely generated $G$-module $(\pi, V)$. As follows from the Iwasawa decomposition $V$ is finitely generated as a $P$-module. Since the action of $P = MU$ on $V$ descends to an action of $P/U = M$ on $V/V(U)$ we see $V/V(U)$ is finitely generated as a $M$-module.

Corollary 0.144. 
1. Let $(\pi, W)$ be an irreducible representation of $G$ and $M < G$ be a standard Levi subgroup, minimal subject to the condition $r_M(\pi) \neq 0$. Then the representation $r_M(\pi)$ of $M$ is cuspidal.
2. For any irreducible representation $(\pi, W)$ of $G$ there exists a parabolic $P = MU$ and an irreducible cuspidal representation $(\tau, R)$ of $M$, such that $W$ is a submodule of $i_M(R)$.

Proof. Let $\rho := M(\pi)$.

1. By part (2) of the Proposition and the choice of $M$ we have $r_{N,M}(\rho) = r_{N,M} \circ r_{M,G}(\pi) = r_{N,G}(\pi) = 0$ for any proper parabolic $Q = NV$ of $M$. So $\rho$ is quasi-cuspidal.

Since $W$ is irreducible it is finitely generated. Thus, by part (4) of the Proposition, $\rho$ is finitely generated. So it is cuspidal.

2. Let $\rho$ be as in (1) and $\tau$ be an irreducible quotient of $\rho$. Then $(\tau, R)$ is an irreducible cuspidal representation of $M$ such that $\text{Hom}_M(r_{N,M}(W), R) \neq \{0\}$.

So [by the Frobenious reciprocity] we get a non-zero map $W \rightarrow i_{M,U}(R)$. As $W$ is irreducible it is an embedding. \qed
Remark 0.145. This use of the adjunction property is typical. It helps to show that something is non-zero, but does not give more detailed information.

Now we can prove Theorem 0.140.

Let \((\pi, V)\) be an irreducible representation of \(G\). By Corollary we can find a parabolic \(P = MU\) and an irreducible cuspidal representation \((\tau, R)\) of \(M\), such that there exists an embedding \(V \hookrightarrow i_{M,U}(L)\). By part \((c)\) of Proposition and Corollary 0.136 the representation \(i_{M,U}(L)\) is admissible. Since \(V \hookrightarrow i_{M,U}(L)\) the representation \(V\) is also admissible.

Corollary 0.146. For any irreducible representation \((\rho, V)\) of \(G\), a congruence subgroup \(K\) and a parabolic subgroup \(P = MU\) of \(G\), we have \(J(V^K) = (r_M(V))^{K \cap M}\).

Definition 0.147. To simplify notations we write \(i_M\) instead of \(i_{M,U}\) and \(r_M\) instead of \(r_{M,U}\).

(1) We denote by \(\kappa_M : \text{Id}_{M(G)} \to i_M \circ r_M\) and \(\tau_M : r_M \circ i_M \to \text{Id}_{M(M)}\) the morphisms of functors coming from the adjunction in Proposition 0.231.

Problem 0.148. The morphism \(\kappa_M(i_M(\pi) : i_M(\pi) \to i_M \circ r_M \circ i_M(\pi)\) is a monomorphism for all \(\pi \in \text{Ob}(M(M))\).

6.7. Uniform Admissibility. As follows from Theorem 0.140 and Lemma 0.19 for any open compact subgroup \(K\) of \(G\), all irreducible representations of the algebra \(H_K\) are finite dimensional. However we did not yet show the existence of a bound \(c(K)\) on dimensions of irreducible representations of \(H_K\).

Theorem 0.149. [Uniform Admissibility] For any open compact subgroup \(K \subset G\) there exists an effectively computable constant, \(c = c(G,K)\), such that all irreducible representations of the algebra \(H_K(G)\) have dimension bounded by \(c(K)\).

Reformulation. For any irreducible representations \(V\) of \(G\) we have \(\dim V^K \leq c(K)\).

The proof is based on the following result from Linear Algebra.

Proposition 0.150. Let \(V\) be a complex vector space of dimension \(m < \infty\) and \(C \subset \text{End}(V)\) a commutative subalgebra generated by \(l\) elements. Then

\[
\dim C \leq m^{2^{-\epsilon_l}} \epsilon_l := \frac{1}{2^{l-1}}
\]

Proof. Since \(C\) is commutative we can decompose \(V\) into a direct sum of \(C\)-invariant subspaces \(V_i\) such that for any \(c \in C\) the restriction \(c_i\) of \(c\) to \(V_i\) has a form \(\lambda_i(c)\text{Id}_{V_i} + a\) where \(a \in \text{End}(V_i)\) is nilpotent.
**Problem 0.151.** Show that it is sufficient to prove the Proposition in the case when all \( c \in C \) are nilpotent.

From now on we assume that all \( c \in C \) are nilpotent. For any \( r > 1 \) we denote by \( d_l(r), l \in \mathbb{Z}_+, r > 1 \) the maximal dimension of commutative subalgebras of \( M_r(\mathbb{C}) \) generated by \( l \) nilpotent elements \( c_1, \ldots, c_l \) where \( [r] \) is the integral part of \( r \).

**Problem 0.152.** Show that the Proposition is implied by the following inductive result.

**Claim 0.153.** \( d_l(r) \leq d_l(r - \frac{d_l(r)}{r}) + d_{l-1}(r) \).

**Proof of the Claim.** We may assume that \( C \subset \text{End} V, V = \mathbb{C}^n \) where \( n := [r] \) and that \( \text{dim}(C) = d_l(r) \). Let \( I \subset \text{End}(V) \) be the ideal generated by \( c_i, 1 \leq i \leq l \) and \( V^k := I^k(\mathbb{C}^n) \). Then

\[
V = V^0 \supset V^1 \supset \ldots \supset V^n = \{0\}
\]

Choose a subspace \( L \subset V^0 \) complementary to \( V^1 \) and define \( m := \text{dim}(L) \).

**Problem 0.154.** Use the equality \( V^k = I^k(L) + V^{k+1} \) to show that \( CL = V \).

Since \( CL = V \) any \( c \in C \) is determined by its restriction to \( L \). So \( \text{dim}(C) = d_l(r) \leq n m \) and therefore \( m \geq d_l(r)/n \).

Let \( C' \subset C \) be the subalgebra generated by \( c_i, 1 \leq i \leq l \) and \( R := c_1 C \). Then \( C = C' + R \). Since \( c_1 \) maps \( V \) to \( V^1 \) the dimension of \( R \) is not greater than the dimension of the restriction of \( C \) on \( V^1 \). So

\[
\text{dim}(R) \leq d_l(n - m) \leq d_l(n - d_l(r)/n)
\]

On the other hand \( \text{dim}(C') \leq d_{l-1}(r) \).

We now prove the theorem.

**Proof.** We know already that any irreducible representations of \( \mathcal{H}_K(G) \) is finite-dimensional. To prove the Theorem it is sufficient to show that

\[
\text{dim}(V) \leq |K_0/K|^{2n}
\]

for any irreducible representations \((\rho, V)\) of \( \mathcal{H}_K(G) \). In other words we have to show that any \( N \)-dimensional representations \((\rho, V)\) of \( \mathcal{H}_K(G) \) where \( N > |K_0/K|^{2n} \) is reducible.

Since \( \mathcal{H}_K = \mathcal{H}_0\mathcal{C}_0, \text{dim}_\mathbb{C}(\mathcal{H}_0) = |K_0/K| \) follows from Lemma 0.125 that

\[
\text{dim}(\rho(\mathcal{H}_K)) \leq |K_0/K|^2\text{dim}_\mathbb{C}(\tilde{C})
\]

where \( \tilde{C} := \rho(C) \subset \text{End}(V) \). On the other hand since \( \tilde{C} \) is a commutative algebra with \( n \) generators it follows from the Proposition that \( \text{dim}(\tilde{C}) \leq N^{2-1/2^{n-1}} \). Since \( N^{1/2^{n-1}} > |K_0/K|^2 \) we see that

\[
\text{dim}_\mathbb{C}(\rho(\mathcal{H}_K)) < \text{dim}(\text{End}(V))
\]
Now we appeal to the Burnside’s Theorem which says that \( \rho : \mathcal{H}_K(G) \to \text{End}V \) is surjective for any finite-dimensional irreducible representations \((\rho, V)\) of \( \mathcal{H}_K(G) \) to see that the representation \( \rho \) is reducible.

Consider the subgroup \( G^0 \subset G \) as before, and fix a congruence subgroup \( K \subset G^0 \). We know that given any irreducible cuspidal representation \((\rho, V)\) of \( G^0 \), and \( v \in V^K \), the function \( g \to \rho(e_K)\rho(g)\rho(e_K)v, g \in G^0 \) has compact support. We will now show how the uniform admissibility theorem can strengthen this result.

**Proposition 0.155.** Given \( K \subset G^0 \subset G \) as above, there exists an open compact subset \( \Omega \subset G^0 \) such that

\[
\text{supp}\ \rho(e_K)\rho(g)\rho(e_K)v \subset \Omega
\]

for all irreducible cuspidal representations \((\rho, V)\) of \( G \) and all \( v \in V \).

**Proof.** It follows from the proof of Harish-Chandra’s theorem that compact representations of \( G^0 \) are exactly those for which \( \lambda \mapsto \rho(a(\lambda))v \) has finite support in \( \Lambda_1^+ \). This is in turn equivalent to the statement that for any non-zero \( \nu \in \Lambda_1^+, v \in V^K \) and any irreducible cuspidal representation \((\rho, V)\) of \( G^0 \) the operator \( \rho(a(\nu)) \) on \( V^K \) is nilpotent. Since \( \dim(V^K) \leq c(K) \) we see that \( \rho(a(\nu^{c(K)})) = 0 \) for all non-zero \( \nu \in \Lambda_1^+ \). But there exists a finite subset \( S \) of \( \Lambda_1^+ \) such that any \( \lambda \in \Lambda_1^+ - S \) has a form \( \lambda = \nu^{c(K)}\mu, \nu \in \Lambda_1^+ - \{0\}, \mu \in \Lambda_1^+ \). So \( \text{supp}\ \rho(e_K)\rho(g)\rho(e_K)v \subset K_0SK_0 \) for any irreducible cuspidal representation \((\rho, V)\) of \( G \) and any \( v \in V \).

**Corollary 0.156.** For any congruence subgroup \( K \) of \( G \) there are only finitely many equivalence classes of irreducible cuspidal representations of \( \mathcal{H}_K(G^0) \).

**Proof.** Since the support of the matrix coefficients of the irreducible cuspidal representations must lie in \( \Omega(G, K) \), the corollary follows from the following general result.

**Problem 0.157.** The matrix coefficients of any set of pairwise non-isomorphic irreducible representations are linearly independent functions.

### 6.8. Decomposing the Categories.

**Definition 0.158.**

1. We denote by \( \text{Irr}_c(G) \subset \text{Irr}(G) \) be the subset of cuspidal irreducible representations.
2. We denote by \( \text{Irr}_i(G) \subset \text{Irr}(G) \) be the set equivalence classes of irreducible representations of \( G \) which a subquotients of representations induced from proper parabolic subgroups of \( G \).
3. \( \mathcal{M}(G)_c := \{ V \in \mathcal{M}(G)|JH(V) \subset \text{Irr}_c(G)\} \).
4. \( \mathcal{M}(G)_i = \{ V \in \mathcal{M}(G)|JH(V) \subset \text{Irr}_i(G)\} \).
Lemma 0.159. \( \text{Irr}_c(G) \cap \text{Irr}_i(G) = \emptyset. \)

Proof. It is sufficient to show that \([D] \notin JH(i_M(\rho))\) for any irreducible cuspidal representation \(D\) of \(G\), any proper standard parabolic \(M < G\) and any \(\rho \in \mathcal{M}(M)\). As follows from the Harish-Chandra theorem the restriction \(D_0\) of \(D\) on \(G^0\) is compact. So it is sufficient to show that \([D_0] \notin JH_G^0(i_M(\rho))\) for any irreducible compact representation \(D_0\) of \(G^0\), any proper standard parabolic \(M < G\) and any \(\rho \in \mathcal{M}(M)\). Since every irreducible compact representation of \(G^0\) splits the category \(\mathcal{M}(G^0)\) we have a decomposition

\[ i_M(\rho) = i_M(\rho)D_0^0 \oplus i_M(\rho)D_0 \]

where \(i_M(\rho)D_0^0\) is a multiple of \(D_0\) and \([D_0] \notin JH_G^0(i_M(\rho))\). So it is sufficient to see that \(\text{Hom}_{G^0}(D_0, i_M(\rho)) = \{0\}\). Since \(MG^0 = G\) we have

\[ \text{Hom}_{G^0}(D_0, i_M(\rho)) = \text{Hom}_{G^0 \cap M}(\rho M(D_0), \rho) = 0 \]

since \(D\) is cuspidal. \(\square\)

Corollary 0.160. \( \text{Irr}(G) \) is a disjoint union of \( \text{Irr}_c(G) \) and \( \text{Irr}_i(G) \).

Theorem 0.161.

\[ \mathcal{M}(G) = \mathcal{M}(G)_c \oplus \mathcal{M}(G)_i \]

Proof. We start with the following result. Let \(V\) be a representation of \(G\) and \(V = V_c \oplus V_i\) a direct sum decomposition into \(G^0\)-invariant subspaces the set \(JH_G^0(V_c)\) consists only of compact representations and the \(G^0\)-representation \(V_i\) has no compact subquotients.

Problem 0.162. (1) The subspaces \(V_c, V_i \subset V\) are \(G\)-invariant.

(2) \(V_i \in \mathcal{M}(G)_i\).

It follows now from the Harish-Chandra theorem that it is sufficient to prove that the subset \(\text{Irr}_c G^0 \subset \text{Irr} G^0\) of irreducible compact representations, splits \(\mathcal{M}(G^0)\). Let \(V\) be a representation of \(G^0\). As we have seen for any congruence subgroup \(K\) there are only a finite number of irreducible cuspidal representations \(D_1, ..., D_r\) of \(G^0\) with \(K\)-invariant vectors. Since representations \(D_1, ..., D_r\) are compact we have

\[ V = V_{K,c} \oplus V_{c,K} \]

where \(V_{K,c}\) is a direct sum of compact irreducible representations of \(G\) which have a non-zero \(K\)-invariant vector and \(V_{c,K}\) does not have irreducible subquotients of this form.

Consider a decreasing sequence of congruence subgroups, \(K_1 \supset K_2 \supset \ldots\) such that \(\cap_i K_i = \{e\}\) and define

\[ V_c := \cup_i V_{c,K_i}, V_i := \cap_i V_{c,K_i}^\perp. \]

Obviously, \(JH(V_c) \subset \text{Irr}_c G\) and \(JH(V_i) \cap \text{Irr}_c G = \emptyset.\)
7. Examples of cuspidal representations.

It only remains to show that $V = V_e \oplus V_i$. Take $v \in V$ and choose a congruence subgroup $K$ which stabilizes $v$. The decomposition

$$V = V_{e,K} \oplus V_{e,K}^\perp$$

shows that we have a decomposition $v = v_{e,K} + v'$ where $v_{e,K} \in V_{e,K}$, $v' \in V_{e,K}^\perp$. To finish the proof of the Theorem it is sufficient to show that $v' \in V_i$.

Since $v' \in V_i$ it is sufficient to show that the $G$-submodule $V' \subset V$ generated by $v'$ does not contain any compact irreducible representation $D$ without a non-zero $K$-invariant vector. Since $D$ splits the category we have $V' = V'_D \oplus V'_D^\perp$ where $V'_D$ is a multiple of $D$ and $D \notin JH(V'_D^\perp)$. Since $v'$ generates $V'$ the projection of $v'$ to $V'_D$ generates $V_D$. On the other hand $v$ and therefore $v'$ are $K$-invariant while $D$ does not have non-zero $K$-invariant vectors. So $V'_D = \{0\}$. \hfill $\Box$

Definition 0.163. Let $M < G$ be a standard levi subgroup.

1. For any $V \in M(M)$ we denote by $V_c$ the projection of $V$ on the subcategory of quasi-cuspidal representations.
2. We denote by $\kappa_{M,c}(V) : V \to i_M \circ r_M(V_c)$ the composition of $\kappa_M(V_c)$ [see 0.147] with the projection $V \to V_c$.

Lemma 0.164. For any $V \in M(G)$ the map

$$\oplus_{M < G} \kappa_{M,c}(V) : V \to \oplus_{M < G} i_M \circ r_M(V)$$

is an imbedding.

Proof. Let $V_0 := \cap_{M < G} \ker \kappa_{M,c}(V)$. It follows from Problem 0.148 that $r_M(V_0) = \{0\}$ for any standard proper parabolic subgroup $M$ of $G$. So the representation of $G$ on $V_0$ is quasi-cuspidal. Since $(V_0)_c = \{0\}$ we see that $V_0 = \{0\}$. \hfill $\Box$

7. Examples of cuspidal representations.

Let $G = GL(n, F), Z = F^*$ be the center of $G$ and $\psi : k \to C^*$ be a non-trivial additive character.

Definition 0.165. (1) To any matrix $A \in \mathfrak{g} = M_n(k)$ we can associate a function

$$\nu_A : K_1 \to C^*, \nu(k) := \psi(Tr(A(k - 1)))$$

It is clear that $\nu_A$ is a character of $K_1$.

(2) A matrix $A \in \mathfrak{g} = M_n(k)$ is anisotropic if the characteristic polynomial $p_A(x) := det(xI_\mathfrak{g} - A) \in k[x]$ is irreducible.

(3) A character $\nu_A, A \in M_n(k)$ is anisotropic if $A$ is anisotropic.

(4) We denote by by $Z_G(\nu_A) \subset ZK_0$ the stabilizer of $\tilde{\nu}_A$ in $Z(G)K_0$ [which make sense since $K_1$ is a normal subgroup of $Z(G)K_0$. Since $\nu_A$ is a character of $K_1$ we have $K_1 \subset Z_G(\nu_A)$.

Problem 0.166. Show that
(1) A matrix $A \in M_n(k)$ is anisotropic iff it does not stabilize any non-trivial flag $F$ in $k^n$.

(2) A matrix $A$ is anisotropic iff for any proper parabolic subgroup $P = MU$ of $G$ the restriction of $\nu_A$ on $K_1 \cap U$ is not equal to the function 1.

(3) For any $g \in G - Z(G)K_0$ there exists a standard proper parabolic $P = MU$ of $G$ and a subgroup $H$ of $K_2$ such that $gHg^{-1} \subset K_1$ and $K_1 \cap U \subset gHg^{-1}K_2$.

(4) For any congruence subgroup $K_n$ there exists a compact $C_n \subset G$ such that for any $g \in G - Z(G)C_n$ there exists a proper standard parabolic $P = MU$ of $G$ and a subgroup $H$ of $K_n$ such that $gHg^{-1} \subset K_1$ and $K_1 \cap U \subset gHg^{-1}K_2$.

(5) For any $A \in M_n(k)$ there exists a character $\chi : Z_G(\nu_A) \to \mathbb{C}^*$ with the restriction on $K_1$ equal to $\nu_A$.

(6) For any $A \in M_n(k)$ and any unitary character $\chi : Z_G(\nu_A)K_1 \to \mathbb{C}^*$ the induced representation $\text{ind}^G_{Z_G(\nu_A)} \chi$ has a natural $G$-invariant unitary structure.

**Proposition 0.167.** Let $A \in M_n(k)$ be an anisotropic matrix $\nu = \nu_A$ and $\chi : Z_G(\nu_A) \to \mathbb{C}^*$ be a character with the restriction on $K_1$ equal to $\nu_A$. Then the induced representation $(\pi_{\nu, \chi}, V) := \text{Ind}^G_{Z_G(\nu_A)} \chi$ is cuspidal and irreducible.

**Proof.** We first prove that the representation $(\pi_{\nu, \chi}, V)$ is admissible and equal to $\text{ind}^G_{Z_G(\nu_A)} \chi$. It is sufficient to prove that for any $n > 0$ there exists a compact $C_n$ such that $\text{supp}(f) \in Z(G)C_n$ for all $f \in V^K_n$. We take $C_n$ to be the compact set as in Problem 0.166. We have to show that for any $g \in G - Z(G)C_n$ we have $f(g) = 0$ if $f \in V^K_n$. Choose subgroups $P = MU$ and $H$ as in Problem 0.166. Then for any $h \in H$ we have

$$f(g) = f(gh) = f((ghg^{-1})g) = \nu(ghg^{-1})f(g).$$

Since $\nu$ is anisotropic and $K_1 \cap U \subset gHg^{-1}K_2$ there exists $h \in H$ such that $\nu(ghg^{-1}) \neq 1$. Therefore $f(g) = 0$.

Since the representation $(\pi_{\nu, \chi}, V)$ is unitary and admissible it is sufficient to show that $\text{Hom}_G(V, V) = \mathbb{C}$. [See Corollary 0.251]. As follows from the Frobenious reciprocity we have

$$\text{Hom}_G(V, V) = \text{Hom}_{Z_G(\nu_A)}(C_X, V).$$

For a proof of the inequality $\text{dim}(\text{Hom}_Z(C_X, V)) \leq 1$ it is sufficient to show that $\phi(1)(g) = 0$ for any $\phi \in \text{Hom}_{Z_G(\nu_A)}(C_X, V)$ and $g \in G - \tilde{Z}$.

Consider first the case when $g \in G - Z(G)K_0$. As follows from the part (3) of the previous problem there exists a proper parabolic subgroup $P = MU$ and a subgroup $H$ of $K_2$ such that $gHg^{-1} \subset K_1$ and $gHg^{-1}K_2 \supset U \cap K_1$. 


Since \( \phi \in \text{Hom}_{\mathbb{Z}}(\mathbb{C}_\chi, V) \) we see that

\[
\phi(1)(g) = \frac{1}{|K_1 \cap u/K_2 \cap U|} \sum_{u \in u(k)} \psi(\nu(u))\phi(1)(g)
\]

Since the functional \( \nu \) is anisotropic the restriction of \( \psi \circ \nu \) on \( u \) is a non-trivial character of the group \( u \) and we see that

\[
\sum_{u \in u} \psi(\nu(u)) = 0
\]

So \( \phi(1)(g) = 0 \).

Consider now the case when \( g \in Z(G)K_0 - \tilde{Z} \). Since \( g \) does not belong to the stabilizer of \( \tilde{\nu} \) there exists \( k \in K_1 \) such that \( \tilde{\nu}(gkg^{-1} \neq \tilde{\nu}(k) \). Now the same arguments show that \( \phi(1)(g) = 0 \).

Now we prove the cuspidality of the representation \( \pi := \pi_{\nu, \tilde{\chi}} \). Let \( P = MU \) be a proper parabolic subgroup of \( G \) and \( q : V \to r_M(V) \) the canonical projection. Consider \( v \in V, v : G \to \mathbb{C} \) given by the function supported on \( Z(G)K_0 \) and such that \( v(e) = 1 \). Since the representation \( \pi \) is irreducible it is sufficient to show that \( q(\pi(g)v) = 0 \) for all \( g \in G \). Since \( G = PK_0 \) and the projection \( q \) commutes with the action of \( P \) it is sufficient to show that \( q(\pi(k)v) = 0 \) for all \( k \in K_0 \). Since \( q(\pi(u)w) = q(w) \) for all \( w \in V, u \in U \) it is sufficient to show that

\[
\int_{u \in U \cap K_1} \pi(u)\pi(k)v du = 0.
\]

In other words it is sufficient to check that

\[
\int_{u \in k^{-1}U_k \cap K_1} \pi(u)v du = 0.
\]

Since \( \text{supp} \pi(u)v \subset Z(G)K_0 \) for any \( u \in k^{-1}U_k \cap K_1 \) it is sufficient to show that

\[
\int_{u \in k^{-1}U_k \cap K_1} \pi(u)v(e) = 0.
\]

We have

\[
\int_{u \in k^{-1}U_k \cap K_1} \pi(u)v(e) = \sum_{u \in u(k)} \psi(\nu^g(u))
\]

Since the functional \( \nu \) is anisotropic the restriction of the character \( \nu^g(u) \) on \( U \cap K_1 \) is a non-trivial character of the group \( U \cap K_1 \). So \( \sum_{u \in u(k)} \psi(\nu^g(u)) = 0 \).

8. Cuspidal components

8.1. Relations between representations of a group and of its subgroups of finite index.
Claim 0.168. Let $G$ be a group, $H \subset G$ a subgroup of finite index and $\rho : G \to \text{Aut}(V)$ an irreducible representation. Then the restriction $\rho_H$ is semisimple and of finite length. [That is $\rho_H$ is a finite direct sum of irreducible representations].

Proof. Let $L := \cap_{g \in G/H} gHg^{-1}$. It is clear that $L \subset G$ is a normal subgroup of finite index. We show that the restriction $\rho_L$ is semisimple and of finite length and will leave for the reader to prove that the restriction $\rho_H$ is also semisimple.

Since $V$ is irreducible, $V$ is a finitely generated $\mathbb{C}[G]$-module and, since $L$ is of finite index, $V$ is a finitely generated $\mathbb{C}[L]$-module. Hence the Zorn’s Lemma implies the existence of an $L$-irreducible quotient $q : V \to W$.

Let $K \subset V$ be the kernel of $q$. Since $L \subset G$ is a normal subgroup we see that for every $g \in G$ the subspace $\rho(g)(K) \subset V$ is $L$-invariant and the quotient $V/\rho(g)(K)$ is an irreducible representation of $L$. The kernel of the natural map $V \to \oplus_{g \in G/L} V/\rho(g)(K)$ is $G$ invariant, and hence [since $V$ is irreducible] is equal to $\{0\}$. So we see that $(\rho_L, V)$ is a subrepresentation of a finite direct sum of irreducible representations of $L$. Therefore $(\rho_L, V)$ is a finite direct sum of irreducible representations of $L$.

Let $G$ be a group, $H \subset G$ a normal subgroup of finite index, $\rho : G \to \text{Aut}(V)$ an irreducible representation of $G$ and $(\pi, W)$ an irreducible representation of $H$. Since the the restriction $\rho_H$ is completely reducible we can write $V$ as a direct sum $V = V_W \oplus V^W_W$ where $V_W$ is a multiple of $W$ and $W$ does not appear as a subquotient of $V^W_W$. We define $\Phi := \text{Hom}_H(W, V) = \text{Hom}_H(W, V_W)$. It is clear that the map $\phi \otimes w \to \phi(w)$ defines an isomorphism $\Phi \otimes W \to V_W$.

Definition 0.169. (1) For any $g \in G$ we denote by $\pi^g : H \to \text{Aut}(V)$ the representation $\pi^g(h) := \pi(ghg^{-1})$.

(2) We denote by $G_\pi \subset G$ the subgroup of elements $g \in G$ such that the representation $\pi^g$ of $H$ is equivalent to $\pi$.

By the definition for any $g \in G_\pi$ there exists an automorphism $A(g) \in \text{Aut}(W)$ such that $\pi(g^{-1}hg) = A(g)^{-1}\pi(h)A(g), \forall h \in H$. Since $W$ is irreducible an automorphism $A(g)$ is defined uniquely up to a multiplication by a scalar $c \in \mathbb{C}^*$.

Problem 0.170. (1) There exist $c_{g', g''} \in \mathbb{C}^*, g', g'' \in G$ such that $A(g'g'') = c_{g', g''}A(g')A(g'')$. In other words the map $g \to A(g)$ is a projective representation of the group $G_\pi$ on $W$.

(2) The subspace $V_W$ of $V$ is $G_\pi$-invariant.

(3) There exist a projective representation $\nu$ of the group $G_\pi/H$ on $\Phi$ such that $A(g) = \nu(g)u \otimes \pi(g)$ for all $g \in G_\pi$.

(4) Let $a : \text{ind}_{G_\pi}^G(V_W) \to V$ be the morphism corresponding to the imbedding $V_W \hookrightarrow V$ under the Frobenious reciprocity $\text{Hom}_G(\text{ind}_{G_\pi}^G(V_W), V) = \text{Hom}_{G_\pi}(V_W, V)$. 
Then \( a \) is an imbedding.

(5) The representation \( \tau \) of \( G_\pi \) on \( V_W \) is irreducible and \( \rho \) is equivalent to \( \text{Ind}_{G_\pi}^G \tau \).

(6) The projective representation \( \nu \) of \( G_\pi/H \) on \( \Phi \) is irreducible.

(7) If \( G/H \) is cyclic then \( \dim(\Psi) = 1 \) and we can assume that \( \tau \equiv 1 \).

**Definition 0.171.** Let \( H \) be a normal subgroup of \( G \) such that the quotient group \( \bar{G} := G/H \) is commutative.

(1) We denote by \( \bar{G} \) the group \( \text{Hom}(\bar{G}, \mathbb{C}^*) \) of characters of \( \bar{G} \).

(2) For any irreducible representation \( \rho \) of \( G \) we define \( \Psi_\rho \subset \Psi \) as the subgroup of characters \( \psi \) such that the representation \( \psi \otimes \rho \) is equivalent to \( \rho \).

(3) For any \( \psi \in \Psi_\rho \) we choose an intertwining operator \( I_\psi \in \text{Aut}_H(V) \) which defines an equivalence between \( \rho \) and \( \psi \otimes \rho \).

**Lemma 0.172.** Let \( H \) be a normal subgroup of \( G \) such that the quotient group \( G/H \) is cyclic.

(1) The restriction \( \rho_H \) is a direct sum of distinct irreducible representations.

(2) The representation \( \rho_H \) of \( H \) is irreducible iff \( \Psi_\rho = \{1\} \).

(3) There exist constants \( c_{\psi', \psi''} \in \mathbb{C}^*, \psi', \psi'' \in \Psi_\rho \) such that

\[
I_{\psi' \psi''} = c_{\psi', \psi''} I_\psi I_{\psi''}.
\]

(4) The the set \( \{I_\psi \in \text{End}_H(V)\}, \psi \in \Psi_\rho \) is a basis of \( \text{End}_H(V) \).

**Proof.** We use notation of Problem 0.170.

(1) follows from the equality \( \dim(\Psi) = 1 \).

(2) Follows from the Frobenious reciprocity.

(3) Follows from the Schur lemma and

(4) is true since characters of any finite commutative group is a basis in the space of functions on this group.

**Problem 0.173.** Generalize the statements of the Lemma to the case when \( G = \prod_i G_i, H = \prod_i H_i, i \in I \) where \( G_i \) is a finite family of groups and \( H_i \subset G_i \) a family of normal groups such that the quotients \( G_i/H_i \) is cyclic.

**8.2. Relations between representations of a reductive group \( G \) and the subgroup \( G^0 \subset G \).** Let now \( G \) be a connected reductive group. As before we denote by \( G^0 \subset G \) the subgroup generated by all compact subgroups of \( G \). We assume that \( G = \prod_i G_i \) \( \forall i \in I \) where all the quotients \( G_i/G^0_i \) are cyclic. Then the quotient \( \Lambda(G) := G/G^0 \) is isomorphic to \( \mathbb{Z}^l, l \geq 0 \) the natural map \( Z(G)/Z(G) \cap G^0 \rightarrow \Lambda \) is an isomorphism on a subgroup \( \Lambda_Z(G) \subset \Lambda(G) \) of finite index and the subgroup \( Z(G)G^0 \subset G \) is of finite index.

**Problem 0.174.** If \( G = GL(n, F) \) then \( G^0 = \{g \in GL(n, F)|\text{det}(g) \in \mathcal{O}^*, \Lambda(G) = \mathbb{Z} \text{ and } \Lambda_Z(G) = n\mathbb{Z}\}. \)
Definition 0.175. An unramified character of $G$ is a character $\psi : G \to \mathbb{C}^*$ which is trivial on $G^0$. The set of unramified characters is denoted $\Psi(G)$.

Remark 0.176. Since $\Lambda(G) := G/G^0$ is isomorphic to $\mathbb{Z}^l, l \geq 0$ we see that $\Psi(G) = \text{Hom}(\Lambda(G), \mathbb{C}^*) = (\mathbb{C}^*)^l$. In this way, we introduce (complex) algebraic geometry into the study of $G$.

Lemma 0.177. For any irreducible representations $(\rho', V)$ and $(\rho'', V')$ of $G$ the space $W := \text{Hom}_{G^0}(\rho', \rho'')$ is finite-dimensional.

Proof. Let $\chi', \chi'' : Z(G) \to \mathbb{C}^*$ be the central characters of representations $(\rho', V')$ and $(\rho'', V'')$. If the restrictions $\chi'_{Z(G)\cap G^0}, \chi''_{Z(G)\cap G^0} : Z(G \cap G^0) \to \mathbb{C}^*$ differ then $W = \{0\}$. So we can assume that $\chi'_{Z(G)\cap G^0} = \chi''_{Z(G)\cap G^0}$. In this case the ratio $\chi'/\chi''$ is a character of the group $Z(G)/Z(G) \cap G^0$. As well known there exists a character $\chi : G/G^0 = \Lambda \to \mathbb{C}^*$ such that the restriction of $\chi$ on $Z(G)/Z(G) \cap G^0$ is equal to $\chi'/\chi''$. Consider now the representation $\tilde{\rho}'' := \rho'' \otimes \chi$. Since the restriction of $\chi$ on $G^0$ is equal to 1 the restriction of $\tilde{\rho}''$ on $G^0$ is equal to the restriction of $\rho''$ on $G^0$ and $\text{Hom}_{G^0}(\rho', \rho'') = \text{Hom}_{G^0}(\rho', \tilde{\rho}'')$. On the other hand since the central characters of representations $(\rho', V')$ and $(\tilde{\rho}'', V'')$ coincide we have

$$W = \text{Hom}_{G^0}(\rho', \rho'') = \text{Hom}_{Z(G)G^0}(\rho', \tilde{\rho}'')$$

and it follows from Lemma 0.168 that $\text{dim}(W) < \infty$. \qed

Proposition 0.178. Let $(\rho, V)$ and $(\rho', V')$ be irreducible representations of $G$. The following conditions are equivalent

1. The representations $\rho|_{G^0}$ and $\rho'|_{G^0}$ of the group $G^0$ are equivalent.
2. $\text{JH}(\rho|_{G^0}) \cap \text{JH}(\rho'|_{G^0}) \neq \emptyset$.
3. $\rho' = \psi \rho$ for some unramified character $\psi \in \Psi$.

Proof. (1) The implications $\Rightarrow (2)$ and (3) $\Rightarrow (1)$ are obvious. Thus, it is enough to show that (2) $\Rightarrow$ (3).

Let $W := \text{Hom}_{G^0}(V, V')$. As follows from the previous Lemma the space $W$ is finite-dimensional and the condition (2) implies that $W \neq \{0\}$. We define a representation $\tau$ of $G$ on $W$ by

$$\tau(g) f = \rho'(g)f \rho(g)^{-1} g \in G, f \in W.$$ 

By the definition of $W$, the restriction $\tau|_{G^0}$ is the identity. Thus, we may think of $\tau$ as a representation of the group $\Lambda(G)$ on $W$. Since the group $\Lambda$ is commutative and the space $W$ is finite-dimensional there exists an eigenvector $h \in W, h \neq 0$. In other words $\tau(g)h = \psi(g)h$ for all $g \in G$ for some character $\psi$ of $G$.

Consider $h$ as a linear map from $V$ to $V'$. Then $h$ intertwines the $(\psi \rho, V)$ with $(\rho', V')$. As both representations are irreducible, we see that $\rho' = \psi \rho$. \qed
8. CUSPIDAL COMPONENTS

Definition 0.179. (1) We define an action of the complex algebraic group \( \Psi(G) \) on the set \( \text{Irr } G \) by
\[
(\psi, \rho) \mapsto \psi \otimes \rho
\]

(2) For any \( \rho \in \text{Irr } G \) we define
\[
\Psi_\rho := \{ \psi \in |\Psi(G)| | \rho \otimes \psi \text{ is equivalent to } \rho \}.
\]

Lemma 0.180. The subgroup \( \Psi_\rho \) is finite for all \( \rho \in \text{Irr } G \).

Proof. It is clear that for any \( \psi \in \Psi_\rho \) the restriction of \( \psi \) on the center \( Z \) is trivial. So \( \Psi_\rho \) is a subgroup of characters of the finite quotient group \( \Lambda/\Lambda_Z \).

Lemma 0.181. Let \( (\rho, V) \) be a cuspidal representation of \( G \) and \( (\pi, W) \in JH(\rho) \). Then there exists a \( G \)-equivariant surjection \( V \rightarrow W \) and a \( G \)-equivariant injection \( W \hookrightarrow V \).

Proof. We will only prove the existence of a \( G \)-equivariant injection \( W \hookrightarrow V \). The proof of the existence of a surjection is completely analogous.

Let \( \Phi := \text{Hom}_{G^0}(W, V) \). As we know \( \Phi \) is a finite-dimensional \( \mathbb{C} \)-vector space and the commutative group \( \Lambda = G/G^0 \) acts on \( \Phi \) by
\[
g(\phi)(w) := \rho^{-1}(g)\phi(\pi(g)(w))
\]
Since \( \pi \in JH(\rho) \) there exists a \( \Lambda \)-invariant subspaces \( \Phi' \subseteq \Phi'' \) of \( \Phi \) such that \( \Lambda \) acts trivially on \( \Phi''/\Phi' \). Since the space \( \Phi \) is a finite-dimensional there exists a non-zero \( \Lambda \)-invariant vector \( \phi \in \Phi = \text{Hom}_{G^0}(W, V) \). Since the map \( \phi \) is \( \Lambda \)-invariant we see that \( \phi \in \text{Hom}_G(W, V) \).

Definition 0.182. (1) A cuspidal component of \( \mathcal{M}(G) \) is an orbit of \( \Psi(G) \) in the set \( \text{Irr}_c G \) of cuspidal irreducible representations of \( G \).

(2) We denote by \( X_c(G) \) the set of cuspidal components of \( \mathcal{M}(G) \).

Remark 0.183. It is easy to see that \( \psi \rho \) is cuspidal whenever \( \rho \) is.

Each cuspidal component \( D \) has the form where by Lemma 0.180 the subgroup \( \Psi_\rho \subset \Psi(G) \) is finite. Therefore, \( D \) has the structure of a connected complex algebraic variety and the action of \( \Psi(G) \) on \( D \) is algebraic.

Theorem 0.184. Let \( D \subset \text{Irr } G \) be a cuspidal component. Then \( D \) splits the category \( \mathcal{M}(G) \).

Proof. We have to show that every \( V \in \mathcal{M}(G) \) can be written \( V = V_D \oplus V_D^\perp \) where \( \text{JH}(V_D) \subset D \) and \( \text{JH}(V_D^\perp) \cap D = \emptyset \). By Proposition 0.178 the restrictions of irreducible objects of \( \mathcal{M}(D) \) on \( G^0 \) all coincide and are finite direct sums of irreducible representations \( \rho_i, 1 \leq i \leq r \) of \( G^0 \). These irreducible representations are cuspidal and therefore compact (Harish-Chandra). Since compact representations split the category \( \mathcal{M}(G) \) there exists a decomposition \( V = V_D \oplus V_D^\perp \) of \( G^0 \)-modules, where \( \text{JH}(V_D|_{G^0}) \subset \rho_1, \ldots, \rho_r \) and \( \text{JH}(V_D^\perp|_{G^0}) \cap \{ \rho_1, \ldots, \rho_r \} = \emptyset \).
It only remains to observe that this decomposition is preserved by the action of $G$. But this follows from the fact that $G$ permutes the $\rho_i$. \hfill $\square$

Using arguments analogous to ones used in the proof of Theorem 0.161 we obtain a proof of the following result.

**Theorem 0.185.** The subset of irreducible cuspidal representations, $\text{Irr}_c G \subset \text{Irr} G$, splits $\mathcal{M}(G)$. In other words, any $V \in \mathcal{M}(G)$, can be uniquely decomposed in the direct sum $V = V_c \oplus V_i$ where $\text{JH}(V_i)$ consists only of cuspidal representations and $V_i$ has no cuspidal subquotients.

**Definition 0.186.** For any $\rho \in \text{Ob}(\mathcal{M}(G))$ we denote by $\rho_{\text{cusp}}$ the projection to the cuspidal summand of $\mathcal{M}(G)$.

### 8.3. A result from the category theory.

**Definition 0.187.** Let $\mathcal{M}$ be an abelian category with arbitrary direct sums.

1. A functor $F$ from $\mathcal{M}$ to the category $\text{Ab}$ of abelian groups is **faithful** if $F(f) \neq 0$ for any non-zero morphism $f \in \text{Hom}_\mathcal{M}(X, Y)$.

2. An object $X$ of $\mathcal{M}$ is **compact** if the functor $\text{Hom}(X, \ast)$ from $\mathcal{M}$ to $\text{Sets}$ commutes with direct limits.

3. A projective object $\Pi$ in $\mathcal{M}$ is a **generator** if the functor $F\Pi : X \to \text{Hom}(\Pi, X)$ from $\mathcal{M}$ to the category $\text{Ab}$ is faithful.

**Problem 0.188.**

1. Let $\Pi$ be a projective object in $\mathcal{M}$ such that $F\Pi(X) \neq 0$ for all non-zero objects $X$ of $\mathcal{M}$. Then $\Pi$ is a generator.

2. If $P$ is a generator of $\mathcal{M}$ then any $X \in \text{Ob}(\mathcal{M})$ can be presented as a cokernel of a morphism $f : P^S \to P^T$ where $S, T$ are sets where $P^S := \oplus_{s \in S} P$.

**Lemma 0.189.** Let $\mathcal{M}$ be an abelian category, $\Pi \in \mathcal{M}$ a compact, projective generator and $A := \text{End}_\mathcal{M} \Pi$. Then the functor

$$\alpha : \mathcal{M} \to \mathbb{C}(A), X \to \text{Hom}_\mathcal{M}(P, X)$$

from $\mathcal{M}$ to the category $\mathcal{M}(A)$ of right $A$-modules is an equivalence of categories.

**Proof.** It is sufficient for a proof of the Lemma to construct a functor $\beta : \mathcal{M}(A) \to \mathcal{M}$ and functorial morphisms $a : \beta \circ \alpha \to \text{Id}_\mathcal{M}$ and $b : \alpha \circ \beta \to \text{Id}_{\mathcal{M}(A)}$ such that the morphisms $a(M) : \beta \circ \alpha(M) \to M$ and $b(X) : X \to \alpha \circ \beta(X)$ are isomorphisms for any $A$-module $M$ and any $X \in \text{Ob}(\mathcal{M})$.

We start with a construction of a functor $\beta : \mathcal{M}(A) \to \mathcal{M}$. For any $A$-module $M$ we define $A^M := \oplus_{m \in M} MA$ and denote by $\pi_M : A^M \to M$ the $A$-morphism given by $\pi_M(a_m) := \sum_{m \in M} a_m m \in M$. Let $N := \text{Ker}(\pi_M) i : N \hookrightarrow A^M$ be the imbedding and $\eta_M := i \circ \pi_N : A^N \to A^M$. Such a map is
given by a matrix \( q_M = (q_{m,n}) \), \( m \in M, n \in N \), \( q_{m,n} \in A \) and for any \( n \in N \) we have \( q_{m,n} = 0 \) for almost all \( m \in M \).

We define \( \Pi^M \in Ob(\mathcal{M}) \) as the direct sum \( \bigoplus_{m \in M} \Pi \) which exists since \( \mathcal{M} \) is an abelian category with arbitrary direct sums. The matrix \( q_M \) defines a morphism \( \tilde{q}_M : \Pi^N \to \Pi^M \) and we define \( \beta(M) \) as the cokernel of \( \tilde{q}_M \).

Problem 0.190. (1) The functor \( \beta \) is right exact.
(2) The functor \( \beta \) commutes with direct sums.
(3) \( \beta(A) = P \).

Now we define a functorial morphism \( a : \beta \circ \alpha \to Id_\mathcal{M} \).

For any \( X \in Ob(\mathcal{M}) \) we have \( \alpha(X) = \text{Hom}_\mathcal{M}(\Pi, X) \) and therefore \( \beta \circ \alpha(X) \) is the quotient of \( \bigoplus_{\phi \in \text{Hom}_\mathcal{M}(\Pi, X)} \Pi \) by the image of \( \tilde{q}_{\text{Hom}_\mathcal{M}(\Pi, X)} \). We define \( \tilde{a} : \bigoplus_{\phi \in \text{Hom}_\mathcal{M}(\Pi, X)} \Pi \to X \) by

\[
\tilde{a}(\sum_{\phi \in \text{Hom}_\mathcal{M}(\Pi, X)} p_\phi) := \sum_{\phi \in \text{Hom}_\mathcal{M}(\Pi, X)} \phi(p_\phi) \in X
\]

Problem 0.191. \( \tilde{a}(X) \circ \tilde{q}_{\text{Hom}_\mathcal{M}(\Pi, X)} = 0 \) for all \( X \in Ob(\mathcal{M}) \).

We see that \( \tilde{a}(X) \) defines a morphism \( a(X) : \beta \circ \alpha(X) \to X \).

Let’s now prove that the morphism \( a(X) : \beta \circ \alpha \to X \) is an isomorphism. Consider first the case \( X = P \). We have \( \alpha(P) = A, \beta \circ \alpha(P) = P \) and I’ll leave for you to check that the morphism \( a(P) : P \to P \) is the identity.

Consider now the case \( X = P^S \) for some set \( S \). Since \( P \) is compact we have \( \alpha(X) = \bigoplus_{s \in S} A \) and, since \( \beta \) commutes with direct sums, we have \( \beta \circ \alpha(X) = X \) and I’ll leave for you to check that the morphism \( a(P) : P \to P \) is the identity.

Now let \( X \) be an arbitrary object of \( \mathcal{M} \). Then we can present \( X \) as a cokernel of a morphism \( f : P^S \to P^T \). Since the functor \( \alpha \) is exact and the functor \( \beta \) is right exact the composition \( \beta \circ \alpha \) is also right exact. Now the five homomorphisms lemma implies that \( a(X) : \beta \circ \alpha \to X \) is an isomorphism.

Problem 0.192. Show that
(1) Construct a functorial morphisms \( b : Id_{\mathcal{M}(A)} \to \alpha \circ \beta \).
(2) Show that the morphism \( b(M) \) are isomorphisms for any \( A \)-module \( M \).

8.4. A description of cuspidal components. In this section we investigate categories \( \mathcal{M}(D) \) of representations corresponding to a cuspidal component

\[
D = \{ \psi \rho, \psi \in \Psi(G) \}
\]

where \( \rho \) is a cuspidal irreducible representation.

Let \( R \) be the algebra of regular functions on the algebraic variety \( \Psi(G) \). \( R \) is naturally a \( G/G^0 \)-module and therefore is also a \( G \)-module.

Example 0.193. If \( G = GL(n, F) \) then \( A = \mathbb{Z} \) and \( R = \mathbb{C}[t, t^{-1}] \).
Problem 0.194. (1) Construct a $G$-equivariant isomorphism 
\[ R = \mathbb{C}[A(G)] = \text{ind}^G_{G_0}(\mathbb{C}). \]

(2) Construct a $G$-equivariant isomorphisms $R \otimes \rho = \text{ind}^G_{G_0}(\mathbb{C}) \otimes \rho = \text{ind}^G_{G_0}(\rho|_{G_0}).$

**Proposition 0.195.** Let $\Pi(D) := R \otimes \rho$. Then

1. $\Pi(D) \in \mathcal{M}(D)$.
2. $\Pi(D)$ is a projective object in $\mathcal{M}(D)$.
3. $\Pi(D)$ is a compact object.
4. $\Pi(D)$ is a generator of the category $\mathcal{M}(D)$.

**Proof.** For (1), just observe that $\text{JH}(\Pi(D)|_{G_0}) \subset \text{JH}(\rho|_{G_0}).$

To prove (2), we must show that the functor $X \mapsto \text{Hom}_G(\Pi(D), X)$ from $\mathcal{M}(D)$ to the category of sets is exact. Since $\Pi(D) = \text{ind}^G_{G_0}(\rho|_{G_0})$ it follows from the Frobenious adjunction [see Problem 0.67 ] that

\[ \text{Hom}_G(\Pi(D), X) = \text{Hom}_{G_0}(\rho|_{G_0}, X|_{G_0}). \]

Since $\rho|_{G_0}$ is a direct sum of compact representation (see Proposition 0.168 and Theorem 0.138) it follows from Problem 0.21 that the functor $X \mapsto \text{Hom}_G(\Pi(D), X)$ is exact.

(3) Follows from the Frobenious adjunction and the finiteness of the decomposition $\rho|_{G_0}$ to a direct sum of irreducible representations.

(4) Follows from the Frobenious adjunction. \qed

This proposition is a powerful tool for elucidating the structure of $\mathcal{M}(D)$ when combined with the previous lemma which implies that $\mathcal{M}(D) = \mathbb{C}(\text{End}(\Pi(D))).$

Our next goal is to describe the ring $\text{End}(\Pi(D))$ explicitly.

For any $\psi \in \Psi_\rho$ we fix an intertwining operator $I_\psi \in \text{Aut}(V)$ as in Problem 0.168 and define $\nu_\psi := \text{Id} \otimes I_\psi \in \text{Aut}_C(\Pi(D)).$

**Problem 0.196.** a) $\nu_\psi \in A(D) := \text{End}_G(\Pi(D)).$

b) For any $\psi \in \Psi_\rho$, $f \in R$ we have

\[ f\nu_\psi = \nu_\psi f_\psi \]

where $f_\psi$ is the shift of $f$ by $\psi$.

**Lemma 0.197.** Let $D$ be a cuspidal component. Then

1. $A(D)$ is a free $R$-module with generators $\nu_\psi$, $\psi \in \Psi_\rho$

2. As an algebra $A(D)$ is defined by the following relations

(a) $f\nu_\psi = \nu_\psi f_\psi$, $f \in R$, $\psi \in \Psi_\rho$

where $f_\psi$ is the translation of $f$ by $\psi$.

(b) $\nu_\psi \nu_\phi = c_{\psi\phi} \nu_{\psi\phi}$, $\psi, \phi \in \Psi_\rho$.

3. The algebra $A(D)$ is Noetherian.
9. Basic geometric Lemma

Proof. It is clearly sufficient to prove the part (a).

\[ A(D) = \text{Hom}_G(\Pi(D), \Pi(D)) = \text{Hom}_G(\text{ind}_{G^0}^G(\rho_{G^0}), \Pi(D)) = \text{Hom}_{G^0}(\rho, \rho \otimes R) \]

where the second and the third equalities follow from Problem 0.194. So \[ A(D) = \text{End}_{G^0}(\rho) \otimes R. \]

The result now follows from Lemma 0.170 and Problem 0.173.

Corollary 0.198.

1. \( ba \neq 0 \) for any \( b \in B - \{0\}, a \in A - \{0\}. \)

2. \( A(D) = R \) if \( \Psi_{\rho} = \{ e \} \).

8.4.1. General Remarks. It is important to keep in mind that there may be more than one projective generator so that we get different realizations of the category. As an example, we took \( \Pi = \text{ind}_{G^0}^G(\rho|_{G^0}) \) as our projective generator for \( \mathcal{M}(D) \). We could also have taken \( \Pi' = \text{ind}_{G^0}^G(\tau) \) for some \( \tau \subset \rho|_{G^0} \).

9. Basic geometric Lemma

In this section we will prove a very important result which allows a reduction of number of representation-theoretical problems to the cuspidal case. We start with a reminder of the Mackey theory for finite groups.


Problem 0.199.

1. A locally closed subset (i.e. the intersection of an open and a closed subset) of an l-space is an l-space.

2. If \( K \subset X \) is compact and \( K \subset \cup \alpha U_\alpha \) is an open covering, then there exists disjoint open compact \( V_i \subset X, i = 1 \ldots k \) such that \( V_i \subset U_\alpha \) for some \( \alpha \) and \( \cup V_i \supset K \).

3. Let \( G \) be a countable at infinity l-group acting on an l-space \( X \) with a finite number of orbits. Then \( G \) has an open orbit \( X_0 \subset X \).

Definition 0.200. For any l-space \( X \) we denote by \( S(X) \) the algebra of locally constant, compactly supported, complex-valued functions on \( X \). \( S(X) \) will serve as the “test functions” for our analysis on \( X \). Thus, the set \( S^*(X) \) of linear functionals on \( S(X) \) are called distributions. Note that as \( S(X) \) has no topology, there is obviously no continuity assumed.

Lemma 0.201 (Exact Sequence of an Open Subset). Let \( U \subset X \) be open and \( Z = X \setminus U \). Then

\[ 0 \to S(U) \to S(X) \to S(Z) \to 0 \]

is exact.

Proof. For the injection at \( S(U) \) just extend functions on \( U \) by zero to all of \( X \). For the surjection at \( S(Z) \) we must explain how to extend functions from a closed subset. Since \( f \in S(Z) \) is locally constant and compactly supported, we may assume that \( Z \) is compact and has a covering by a finite number of open sets \( U_\alpha \) with \( f|_{U_\alpha} = c_\alpha \) constant. Let \( V_\alpha \) be as in
0.199 (2). Then we can extend \( f \) by defining \( f(x) = c_a \) if \( x \in V_i \subset U_a \) and zero otherwise.

**Definition 0.202.** Let \( G \) be a countable at infinity \( l \)-group acting on an \( l \)-space \( X \) with a finite set \( I \) of orbits.

1. We say that \( i \leq j, i, j \in I \) if \( \Omega_i \) is in the closure \( \Omega_j \) of the orbit \( \Omega_j \).
2. In this way we define a partial order on \( I \).
3. For any \( i \in I \) we define \( S_i = \{ f \in S(X) | f_{\Omega_i} = 0 \} \). It is clear that \( S_i \supset S_j \) if \( i \leq j, i, j \in I \).

**Problem 0.203.** Define a canonical isomorphism between the space \( S_i \) and the space \( S(\Omega_i) \).

**9.2. The formulation and the proof of the Basic geometric Lemma.**

**Definition 0.204.**

1. For any semidirect product \( P = M \ltimes U \) such that \( M \) and \( U \) are unimodular \( l \)-groups and a smooth representation \( \rho : P \to Aut(R) \) we denote by \( R_U \) the space of \( U \)-coinvariants and by \( q_U : R \to R_U \) the natural surjection. Since \( M \) normalizes \( U \) the restriction of \( \rho \) on \( M \) induced a representation \( (m, \tilde{r}) \to m\tilde{r}, m \in M, \tilde{r} \in R_U \) of \( M \) on \( R_U \).
2. We denote by \( c^U_\rho \) the functor \( M(P) \to M(M), (\rho, R) \to (c^U_\rho(\rho), R_U) \) where we define the action of \( M \) on \( R_U \) by
   \[
   c^U_\rho(\rho)(m)(\tilde{r}) = \text{mod}^{1/2}(m)m\tilde{r}
   \]
   where the function \( \text{mod} \) is defined in 0.36.
3. Let \( G \) be an \( l \)-group, \( P, Q \) closed subgroups of \( G, P = M \ltimes U, Q = N \ltimes V \). As before we say that the semidirect products decompositions of subgroups \( P, Q \) are compatible if
   \[
   P \cap Q = (M \cap Q)(U \cap Q) = (N \cap P)(V \cap P)
   \]
   In this case we define \( L = M \cap N, V' = M \cap V \) and \( U' = N \cap U \).
   Let \( P = M \ltimes U, Q = N \ltimes V \) be a compatible pair of subgroups of \( G \).
4. We define functors \( i_{M,U} : M(M) \to M(G), r_{N,V} : M(G) \to M(N) \) by
   \[
   i_{M,U} = \text{Ind}^G_U \circ \text{Inf}^P_M, r_{N,V} := c^N_Q \circ \text{Res}^Q_G
   \]
   As before we often write \( i_U, r_V \) of \( i_{M,U} \) and \( r_{N,V} \).
5. Since the pair \( P, Q \) is compatible we can also define functors \( i_U : M(L) \to M(N) \) and \( r_V : M(M) \to M(L) \).

Assume now that \( G \) is a reductive \( F \)-group and \( P, Q \) are parabolic subgroups. We consider the action of \( P \times Q \) on \( G \) given by \( (p \times q; g) \to pgq^{-1} \). As we know the set \( I \) of \( P \times Q \)-orbits is finite moreover can be identify with the set \( W^{M,N} \). As follows from 0.202 the set \( W^{M,N} \) is partially ordered and
we have a filtration \( \Psi_w \) on the functor \( \Psi := r_{N,V} \circ i_{M,U} : \mathcal{M}(M) \to \mathcal{M}(N) \). In particular for any \( w \in W^{M,N} \) we can define the functor

\[
\tilde{\Psi}_w = (\sum_{w' < w} \Psi_{w'})/\Psi_w
\]

Since pairs \((w(P), Q)\) are compatible for all \( w \in W^{M,N} \) we can define functors

\[
\tilde{\Psi}_w := i_{N \cap w(M),N \cap w(U)} \circ Ad(w) \circ r_{M \cap w^{-1}(N),M \cap w^{-1}(V)}
\]

**Proposition 0.205.** For any \( w \in W^{M,N} \) the functors \( \Psi_w \) and \( \tilde{\Psi}_w \) are isomorphic.

**Proof.** We start with the simplest case when \( G = SL(2,F), P = Q = B = TU \).

**Lemma 0.206.** Let \( G = SL(2,F), P = Q = B = TU \). Then for any character \( \chi \) of \( T \) we have an exact sequence

\[
\{0\} \to \mathbb{C}_\chi^w \to r_{T,U}(V(\chi)) \to \mathbb{C}_\chi \to \{0\}
\]

**Proof.** Let \( V(\chi) := i_{T,U}(\mathbb{C}_\chi) \). Then

\[
V(\chi) = \{ f : G \to \mathbb{C} | f(tug) = \chi(t)mod_{U'}(t)f(g) \}, \quad t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \quad t \in F^*
\]

and \( G \) acts on \( V(\chi) \) by right shifts. We have \( U \setminus G = F^2 - \{(0,0)\} \) where the map \( G \to F^2 - (0,0) \) is given by

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to (\gamma, \delta)
\]

So we have

\[
V(\chi) = \{ f(x,y), (x,y) \neq (0,0) | f(tx,ty) = |t|^{-1} \chi(t)f(x,y) \}, \quad t \in F^*
\]

there exists \( r > 0 \) such that \( i_\chi(k)(f) = k, k \in K_r \}

where \( G \) acts by \( i_\chi(g)(f)(x,y) = f(\alpha x + \gamma y, \beta x + \delta y) \) for

\[
g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

We write \( W = \{ e, s \}, Y := \{(0,y) \subset F^2 - (0,0)\} \) and define

\[
V(\chi)_s := \{ f \in V(\chi) | f|_Y = 0 \}
\]

By the construction we have an exact sequence

\[
\{0\} \to V(\chi)_s \to V(\chi) \to V(\chi)/V(\chi)_s \to \{0\}
\]

where the subspace \( V(\chi)_s \subset V(\chi) \) is \( B \)-invariant. Since the functor of \( U \)-coinvariants is exact we obtain an exact sequence

\[
\{0\} \to r_{T,U}(V(\chi)_s) \to r_{T,U}(V(\chi) \to r_{T,U}(V(\chi)/V(\chi)_s) \to \{0\}
\]

By the definition we have

\[
\Psi_s(\mathbb{C}_\chi) = r_{T,U}(V(\chi)_s), \Psi_e(\mathbb{C}_\chi) = r_{T,U}(V(\chi)/V(\chi)_s)
\]
So for the proof of the Lemma we have to check that \( r_{T,U}(V(\chi)e) = \mathbb{C}_\chi \) and \( r_{T,U}(V(\chi)_s) = \mathbb{C}_\chi^w \).

The restriction to ~\( Y \) defines an isomorphism of the \( B \)-representation \( \Psi_e(\chi) \) with the one-dimensional space on which \( tu \in Q \) acts by multiplication by \( |t|^{-1}\chi(t) \). So \( \Psi_e(\chi) = \mathbb{C}_\chi \).

Let \( \mathbb{C}(F) \) be the space of functions on \( F \) and consider the operator \( \nu : V(\chi)_s \to \mathbb{C}(F) \) of the restriction \( \nu(f)(y) := f(1,y) \) of \( f \) to the line \( l := \{(1,y), y \in F\} \).

**Claim 0.207.** The operator \( \nu \) defines an isomorphism \( \nu : V(\chi)_s \to \mathcal{S}(F) \).

**Proof.** Let us fix \( f \in V(\chi)_s \). Since the representation \( V(\chi)_s \) is smooth there exists a congruence subgroup \( K_r \subset G \) such that \( i_\chi(k)(f) = k \) for \( k \in K_r \). By the definition \( f_{\mathcal{Y}} = 0 \). Therefore \( f_{K_r} = 0 \). But it is easy to see that the complement \( l - l \cap K_r \mathcal{Y} \) coincides with the set \( \{(1, y), ||y|| \leq ||b||q^r\} \) which is compact. So we see that \( \nu(f) \in \mathcal{S}(F) \). On the other hand for any \( \phi \in \mathcal{S}(F) \) we can consider the function \( f_\phi : F^2 - \{(0,0)\} \to \mathbb{C} \) given by \( f_\phi(x,y) = \chi(x)||x||^{-1}\phi(y/x) \).

By the construction the group \( B \) acts on the space \( V(\chi)_s \) and therefore the isomorphism \( \nu \) defines a representation \( \tau : B \to \text{Aut}(\mathbb{C}(F)) \).

Let us describe \( \tau(\hat{t})(\phi), \phi \in \mathbb{C}(F) \). Since \( \tau(\hat{t})(\phi) := \nu(i_\chi(\hat{t})(f_\phi) \) we have

\[
\tau(\hat{t})(\phi)(y) = i_\chi(\hat{t})(f_\phi)(1,y) = f_\phi(t^{-1},ty) = \chi(t^{-1})||t||\phi(t^2y)
\]

**Problem 0.208.**

\[
\tau(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})(\phi)(y) = \phi(y+u)
\]

As we know that space \( \mathcal{S}(F)_U = \mathbb{C} \) and the map \( q : \mathcal{S}(F) \to \mathcal{S}(F)_U = \mathbb{C} \) is given by \( \phi \to \int_F \phi(y)dy \). So we see that \( \text{dim}(r_{T,U}(V(\chi))) = 1 \). Let us describe the action \( \theta \) of \( T \) on the space \( r_{T,U}(V(\chi)) \).

By the definition we have

\[
\theta(\hat{t})q(\phi) = ||t||q(\tau(\hat{t})(\phi))
\]

for any \( \phi \in \mathcal{S}(F) \). But

\[
q(\tau(\hat{t})(\phi)) = \int_F (\hat{t})(\phi)(y)dy = \int_F \chi(t^{-1})||t||\phi(t^2y) = \chi(t^{-1})||t||^{-1}\int_F (\phi)(y)dy
\]

So \( \theta(\hat{t})q(\phi) = \chi(t^{-1})q(\phi) \) and we see that \( \theta(\hat{t}) = \chi(t^{-1}) \).

Now we consider the case when \( G = \text{GL}(n,F), P = Q = B = TU \). Since the proof is completely parallel to the proof of the previous Lemma I will omit details. In the proof we will use notations and results from Section 5.2. In particular we know that \( B \) is the union of \( B \)-orbits \( B_w \subset B, w \in W \) and \( B_{w'} \) is in the closure \( \bar{B}_w \) if and only if \( w' \preceq w \) in the partial order on \( W \) defined in in section 5.2. Let \( r : U\backslash G \to B\backslash G = B \) be the natural projection
and $X_w := r^{-1}(B_w)$. It is clear that $X_{w'}$ is in the closure $\bar{X}_w$ if and only if $w' \leq w$.

Let $\chi$ be a character of $T, V(\chi) := i_T(\chi)$. Then $V(\chi)$ is the space of functions $f$ on $U \setminus G$ such that $f(tx) = \Delta^{1/2}(t)\chi(t)f(x), x \in U \setminus G, t \in T$ for which there exists $r > 0$ such that $f(xk) \equiv f(x)$ for all $k \in K_r$. The representation of the group $G = GL(n, F)$ on the space $V(\chi)$ comes from the action of $G$ on $U \setminus G$.

For any $w \in W$ we denote by $V_w(\chi) \subset V(\chi)$ of functions $f$ such that $f|_{X_w} \equiv 0$. It is clear that $V_{w_0}(\chi) = \{0\}$ and $V_{w'}(\chi) \subset V_w(\chi)$ if and only if $w' \geq w$. We define

$$\hat{V}_w(\chi) := \cap_{w' < w} V_{w'}(\chi), \hat{\bar{V}}_w(\chi) := \hat{V}_w(\chi)/V_w(\chi)$$

**Problem 0.209.**

1. The restriction to $wU_w \subset X_w$ defines an isomorphism $\nu : \hat{V}_w(\chi) \to S(U_w)$.

Using the isomorphism $\nu$ we define can use the action of $i_T(\chi)$ of $B$ on $\Psi_w(\chi)$ to define a representation $\tau_w : B \to G(\hat{S}(U_w))$ by

$$\tau(b)(\phi) = \nu(i_T(\chi)(b)\nu^{-1}(\phi))$$

2. $\tau_w(u_0)(\phi)(u) = \phi(uu_0), \phi \in S(U_w), u, u_0 \in U_w$.

3. $\tau_w(t)(\phi)(u) = \Delta^{1/2}(t^w)\chi(t^w)(\phi)(t^{-1}t^wu(t^{-1}t)^{-1})$

**Lemma 0.210.** The space $r_T(i_T(\chi))$ has a filtration $\Psi_w(\chi), w \in W$ by $T$-invariant subspaces such that the quotient spaces $\bar{\Psi}_w(\chi) := \Psi_w(\chi)/\sum_{w' < w} \Psi_{w'}(\chi)$ are one-dimensional and $t \in T$ acts on $\bar{\Psi}_w(\chi)$ by the multiplication by $\chi(t^w)$.

**Proof.** As in the proof of the previous Lemma we can define a filtration of the space $r_T(i_T(\chi))$ has a filtration $r_T(i_T(\chi))$ by $T$-invariant subspaces such that the quotient spaces $\bar{\Psi}_w(\chi)$ equal to $r_T(\tau_w)$. Since the group $U_w \subset U$ acts transitively on the space $U_w$ we see that the spaces $\bar{\Psi}_w(\chi)$ as one-dimensional. The lemma follows now from Problem 0.90.

We start the proof of the general case with the following observation.

**Claim 0.211.** Let $G, P, Q, w$ be as in the Proposition. We denote by $w_G, w_M, w_N$ the longest elements of $W_G, W_M, W_N$.

1. There exists a character $\kappa(M, N, w) : M \cap w^{-1}(N) \to \mathbb{R}_+$ such that the functor $\Psi_w$ is isomorphic to the functor

$$i_{N \cap w(M), N \cap w(U)} \circ Ad(w) \circ \otimes \kappa(M, N, w) \circ r_{M \cap w^{-1}(N), M \cap w^{-1}(V)}$$

where we consider $\otimes \kappa(M, N, w)$ as an automorphism of the category $\mathcal{M}(M \cap w^{-1}(N))$

2. There exists an algebraic homomorphism

$$\theta(M, N, w) : M \cap w^{-1}(N) \to \mathbb{G}_m$$

such that $\kappa(M, N, w) = |\theta(M, N, w)|^{1/2}$.
(3) For any \( L < M, K < N, w \in W^{M,N} \) we have
\[
\theta(M, N, w) = \theta(L, K, w'w'') \quad \text{with} \quad w' := w_Gw_M, w'' := w_Nw_G
\]

I will not give a proof but only observe that (1) is completely analogous to the problem ??, (2) follows from (1) since the only reasons for the appearance of \( \kappa \) are either our twisting by \( \text{mod}^{1/2} \) or the characters which come as determinants of changes of variables. The part (3) is a tautology.

The Proposition is equivalent to the equality \( \theta(M, N, w) = 1 \). Since the claim is purely algebraic we can assume that \( G \) is split. Using the part (3) of the Claim we reduce the statement to the case \( P = Q = B \).

It is easy to check that that for any \( w', w'' \in W \) such that \( l(w'w'') = l(w') + l(w'') \) we have \( \theta(B, B, w) = \theta(B, B, w')w'(\theta(B, B, w'')) \). So the proof reduced to the case \( G = SL(2) \) which was analyzed in the previous example.

\[\Box\]

**Corollary 0.212.** [Basic geometric Lemma] For any standard parabolic subgroups \( P = M \ltimes U, Q = N \ltimes V \) of a reductive \( F \)-groups \( G \) the functor \( r_N \circ i_M : \mathcal{M}(M) \to \mathcal{M}(N) \) has a filtration by subfunctors \( \Psi_w, w \in W^{M,N} \) such that for any \( w \in W^{M,N} \) the quotient \( \Psi_w \) is isomorphic to the functor \( i_{N\cap w(M),N\cap w(U)} \circ Ad(w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)} \).

We say that the functor \( r_N \circ i_M : \mathcal{M}(M) \to \mathcal{M}(N) \) is glued from functors \( i_{N\cap w(M),N\cap w(U)} \circ Ad(w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)} \).

**Corollary 0.213.** For any \( M, N < G \) and a quasicuspidal representation \( \rho \) of \( M \) we have

(1) If \( N \) does not have standard subgroups associated with \( M \) then \( r_N \circ i_M(\rho) = \{0\} \).

(2) If \( N \) is not associated with \( M \) then \( r_N \circ i_M(\rho) \) does not have any non-zero quasicuspidal subquotients.

(3) If \( N \sim M \) then the representation \( r_N \circ i_M(\rho) \) is glued from representations \( w(\rho), w \in W(M, N)/W_M \).

(4) There exists a proper Zariski closed subset \( X \) of \( \Psi_M \) such that \( \text{End}_G(i_M(\rho \otimes \psi)) = C \) for \( \psi \in \Psi_M - X \).

**Proof.** The parts (1), (2) and (3) are immediate consequences of the Basic geometric Lemma.

Let \( D = \{ \rho \otimes \psi \}, \psi \in \Psi_M \) be the cuspidal component containing \( \rho \) and \( W(M, D) = \{ w \in W(M, M)/W_M | D^w = D \} \). For any \( w \in W(M, M)/W_M \) we fix \( \kappa_w \in \Psi_M \) such that \( \rho^w = \rho \otimes \kappa_w \) and define \( Y_w := \{ \psi \in \Psi_M | \psi^w \otimes \psi^{-1} \in \kappa_w \Psi_M \} \).

Since the subgroup \( \Psi_\rho \) is finite [see Lemma 0.180] \( Y_w \) is a proper Zariski closed subset of \( \Psi_M \) for all \( w \neq \{ e \} \). Consider \( X := \cup_{w \in W(M, N)/W_M} Y_w \subset \Psi_M \). As follows from (3) the representation \( r_M \circ i_M(\rho \otimes \psi) \) is glued from representations \( w(\rho \otimes \psi), w \in W(M, N)/W_M \). By the construction the representation \( w(\rho \otimes \psi) \) is not equivalent to the representation \( \rho \otimes \psi \) for all
As follows from Lemma 0.231 there exists a standard Levi subgroup \( M \). For any associated pair \( w \in W(M, N)/W_M - \{ e \} \). Therefore

\[
\dim(\text{Hom}_M(r_M \circ i_M(\rho \otimes \psi), \rho \otimes \psi)) \leq \dim(\text{Hom}_M(\rho \otimes \psi, \rho \otimes \psi)) = 1.
\]

But then it follows from the Frobenious reciprocity that

\[
\dim(\text{End}_G(i_M(\rho \otimes \psi))) \leq 1.
\]

\[\square\]

9.3. Some applications.

**Definition 0.214.** Let \( M, M' < G \) be standard Levi subgroups.

1. \( W(M, M') := \{ w \in W | w(M) = M' \}, \) where \( w(M) := wMw^{-1} \). We write \( M' \sim M \) if \( W(M, M') \neq \emptyset \).

2. Given representations \( \rho \in \mathcal{M}(M), \rho' \in \mathcal{M}(M') \). We define

\[
W(\rho, \rho') := \{ w \in W | w(\rho) = \rho' \}
\]

where \( w(M) \) is a standard Levi subgroup of \( M' \) and \( w(\rho) \) is defined by \( w(\rho)(m') := \rho(w^{-1}m'w), m' \in M' \). We write \( \rho' \sim \rho \) if \( W(\rho, \rho') \neq \emptyset \).

3. \( W(M, \star) := \bigcup_{M' \sim M} W(M, M') \).

4. \( l(M) \) is the cardinality of the set \( W(M, \star)/W_M \).

5. We denote by \( l' = l'_M \) the function on \( \mathcal{M}(G) \) given by

\[
l'(\tau) = \sum_{L \sim M} l(r_L(\tau))
\]

where \( l(\rho) \) is the length of the representation \( \rho \).

**Problem 0.215.**
1. \( l(M) = 2 \) iff \( M \) is a maximal Levi subgroup of \( G \).

2. For any associated pair \( M, M' \) of standard Levi subgroups and \( w \in W(M, M') \) there exists chains \( M = M_0, M_1, ..., M_r = M', L_1, L_2, ..., L_r \) of standard Levi subgroups of \( G \) and a decomposition \( w = w_1...w_r \) such that
   (a) \( M_{i-1}, M_i \) are maximal Levi subgroups of \( L_i, 1 \leq i \leq r \).
   (b) \( w_i \in W_{L_i} \).
   (c) \( w_i(M_{i-1}) = M_i, 1 \leq i \leq r \).

**Claim 0.216.** Let \( M \) be a standard Levi subgroup of \( G, \rho \in \text{Irr}_c(M), \pi = i_M(\rho) \) and \( \pi_0 \) an irreducible subquotient of \( \pi \). Then \( l'(\pi_0) > 0 \).

**Proof.** As follows from Lemma 0.231 there exists a standard Levi subgroup \( M' < G \) such that \( r_M(\pi_0) \) is a non-zero quasi-cuspidal representation of \( M' \). As follows from Corollary 0.213 we have \( M' \sim M \).

**Lemma 0.217.** Let \( M, M' \) be standard Levi subgroups of \( G, \rho \in \text{Irr}_c(M), \rho' \in \text{Irr}_c(M') \) and \( \pi = i_M(\rho), \pi' = i_M(\rho') \). Then

1. \( \dim(\text{Hom}_G(\pi, \pi')) \leq |W(\rho, \rho')/W_M| \)

2. The length \( l(\pi) \) of the representation \( \pi \) is finite and moreover \( l(\pi) \leq l(M) \).
Proof. (1) By the Frobenius reciprocity we have
\[ \text{Hom}_G(\pi, \pi') = \text{Hom}_{M'}(r_{M'} \circ i_M(\rho), \rho') \]
As follows from 0.213 the representation \( r_{M'} \circ i_M(\rho) \) has a filtration parameterized by \( \bar{w} \in W(M, M')/W_M \) with subquotients equal to \( \bar{w}(\rho) \). Therefore
\[ \dim(\text{Hom}_G(\pi, \pi')) \leq \sum_{\bar{w} \in W(M, M')/W_M} \dim(\text{Hom}_{M'}(\bar{w}(\rho), \rho')) \]
Since \( \dim(\text{Hom}_{M'}(\bar{w}(\rho), \rho')) = 0 \) for \( \bar{w} \notin W(\rho, \rho') \) and \( \dim(\text{Hom}_{M'}(\bar{w}(\rho), \rho')) = 1 \) for \( \bar{w} \in W(\rho, \rho') \) we see that \( \dim(\text{Hom}_G(\pi, \pi')) \leq |W(\rho, \rho')/W_M| \).
(2) As follows from the exactness of the functor \( r_M \) we have
\[ l'(\tau) = l'(\tau') + l'(\tau/	au') \]
for any subrepresentation \( \tau' \) of \( \tau \). By the previous Claim \( l'(\pi_0) > 0 \) for any non-zero subquotient \( \pi_0 \) of \( \pi \). So \( l(\pi) \leq l'(\pi) \). But it follows from 0.213 that \( l'(\pi) = |W(M, \ast)/W_M| = l(M) \). \( \square \)

Lemma 0.218. Let \( M < G \) be a standard Levi subgroup such that \( l(M) = 2 \) and \( \rho \in \text{Irr}_c(M) \) a representation such that \( \dim(\text{Hom}_G(i_M(\rho), i_M(\rho))) > 1 \).
Then
\begin{enumerate}
\item \( W(M, M) \neq W_M \).
\item \( w(M) = M \) and \( w(\rho) = \rho \) for any \( w \in W(M, M) - W_M \).
\end{enumerate}
Proof. (1) By the Frobenius reciprocity we have
\[ \text{Hom}_G(i_M(\rho), i_M(\rho))) = \text{Hom}_M(r_M \circ i_M(\rho), \rho) \]
If \( W(M, M) = W_M \) then (see 0.213) \( r_M \circ i_M(\rho) = \rho \) and
\[ \text{Hom}_M(r_M \circ i_M(\rho), \rho) = \mathbb{C} \]
So \( W(M, M) - W_M \neq \emptyset \).
Since \( l(M) = 2 \) we have \( W(M, M) - W_M = wW_M \) for any \( w \in W(M, M) - W_M \). As follows from 0.213 we have an exact sequence
\[ \{0\} \to w(\rho) \to r_M \circ i_M(\rho) \to \rho \to \{0\} \]
So
\[ \dim(\text{Hom}_G(i_M(\rho), i_M(\rho))) \leq \dim(\text{Hom}_M(\rho, \rho) + \dim(\text{Hom}_M(w(\rho), \rho)) \]
By the assumption \( \dim(\text{Hom}_G(i_M(\rho), i_M(\rho))) > 1 \). So \( w(\rho) = \rho \). \( \square \)

Theorem 0.219. Let \( M, M' \) be standard Levi subgroups of \( G \), \( \rho \in \text{Irr}_c(M), \rho' \in \text{Irr}_c(M') \), \( \pi = i_M(\rho), \pi' = i_M(\rho') \). The following conditions are equivalent.
\begin{enumerate}
\item \( M \sim M' \) and \( \rho \sim \rho' \).
\item \( \text{Hom}_G(\pi, \pi') \neq \{0\} \).
\item \( JH(\pi) = JH(\pi') \).
\item \( JH(\pi) \cap JH(\pi') \neq \emptyset \).
\end{enumerate}
Proof. It is clear that (2) \(\Rightarrow\) (4) and (3) \(\Rightarrow\) (4). We start with a proof of the implication (4) \(\Rightarrow\) (1). Since \(JH(\pi) \cap JH(\pi') \neq \emptyset\) we can find \(\tau \in JH(\pi) \cap JH(\pi')\). Let \(N < G\) be a standard Levi subgroup standard Levi subgroup such that \(r_N(\tau)\) is a non-zero quasi-cuspidal representation of \(N\). As follows immediately from Corollary 0.213 we have
\[
\{w(\rho)\}, w \in W(M, N)
\]
and also
\[
\{w'(\rho')\}, w' \in W(M', N)
\]
But this implies that \(\rho \sim \rho'\).

We now show that (1) \(\Rightarrow\) (2). Since \(\rho \sim \rho'\) it follows from Corollary 0.213 that \(\rho' \in JH(r_{M'} \circ i_M(\rho))\). Now Lemma 0.181 implies that \(r_{M'} \circ i_M(\rho)\) has a quotient isomorphic to \(\rho'\). In other words
\[
\text{Hom}_{M'}(r_{M'} \circ i_M(\rho), \rho') \neq \{0\}
\]
and therefore it follows from the Frobenious reciprocity that
\[
\text{Hom}_G(\pi, \pi') = \text{Hom}_{M'}(r_{M'} \circ i_M(\rho), \rho') \neq \{0\}.
\]

It is clear that for a proof of Theorem it is sufficient now to prove the implication (1) \(\Rightarrow\) (3). We start a proof with the case when \(l(M) = 2\).

Lemma 0.220. If \(l(M) = 2\) then (1) \(\Rightarrow\) (3).

Proof. If \(M = M', \rho = \rho'\) then there is nothing to prove. So we assume that \(\rho \neq \rho'\) and therefore \(l'(\pi) = l'(\pi') = 2\). Since we know already that (1) \(\Rightarrow\) (2) we can choose non-zero \(G\)-equivariant morphisms \(a : \pi \rightarrow \pi'\) and \(a' : \pi' \rightarrow \pi\). Since \(l(\pi) \leq l(M) = 2\) we see that either \(\pi\) is irreducible or \(l(\pi) = 2\). If the morphism \(a\) is an isomorphism then there is nothing to prove. So we assume that \(a\) is not an isomorphism.

If \(\pi\) is irreducible then \(a\) is an imbedding and
\[
l'(\pi'/a(\pi)) = l'(\pi') - l'(\pi) = 0
\]
and it follows from Claim 0.216 that \(a\) is onto. So \(a\) is an isomorphism and \(JH(\pi) = JH(\pi')\). The same arguments work if \(\pi'\) is irreducible. So

From now on we assume that both \(\pi\) and \(\pi'\) are reducible. Then \(l(\pi) = l'(\pi') = 2\) and we have exact sequences
\[
\{0\} \rightarrow \pi_1 \rightarrow \pi \rightarrow \pi_2 \rightarrow \{0\}
\]
\[
\{0\} \rightarrow \pi'_1 \rightarrow \pi' \rightarrow \pi'_2 \rightarrow \{0\}
\]
where \(\pi_1, \pi'_1, \pi_2, \pi'_2\) are irreducible representations of \(G\) such that \(l'(\pi_1) = l'(\pi'_1) = l'(\pi_2) = l'(\pi'_2) = 1\).

Claim 0.221. (1) If \(\dim \text{Hom}_G(\pi, \pi) > 1\) then \(M = M'\) and \(\rho = \rho'\).

(2) If \(\dim \text{Hom}_G(\pi, \pi) = 1\) then \(\text{Im}(a')\) is the only irreducible submodule of \(\pi\).
Proof. (1) If $\dim \text{Hom}_G(\pi, \pi) > 1$ then it follows from Corollary 0.218 that there exists $w \in W(M, M) - W_M$ such that $w(\rho) = \rho$. On the other hand since $l(M) = 2$ and $W(M, M) - W_M \neq \emptyset$ we see that $M' = M$ and $\rho' = \rho$.  

(2) Since $l'(\text{Ker}(a')) > 0$ and $l(\pi') = 2$ we have $l'(\text{Im}(a')) = 1$ and therefore $l(\text{Im}(a')) = 1$. So $\text{Im}(a')$ is an irreducible submodule of $\pi$. Suppose that there exists another irreducible submodule $\tau$ of $\pi$. Since $l(\pi) = 2$ we have $\pi = \text{Im}(a') \oplus \tau$. But then $\dim \text{Hom}_G(\pi, \pi) > 1$. \hfill $\Box$

From now on we assume that $\dim \text{Hom}_G(\pi, \pi) = 1$.

By the Frobenius reciprocity we have $\text{Hom}(r_M(\pi_1), \rho) = Hom(\pi_1, \pi) \neq \{0\}$. As follows from the definition of the function $l'$ the equality $l'(\pi_1) = 1$ implies that $r_M(\pi_1) = \rho$ and $r_M'(\pi_1) = \{0\}$. Therefore [by the Frobenius reciprocity] we have

$$\text{Hom}(\pi_1, \pi') = \text{Hom}(r_M'(\pi_1), \rho') = \{0\}.$$ 

So $a(\pi_1) = \{0\}$ and [by the Claim] $a$ defines an isomorphism $\bar{a} : \pi/\pi_1 \to \pi'$. Analogous arguments show the existence of an isomorphism $\bar{a}' : \pi'/\pi'_1 \to \pi_1$. So $JH(\pi) = JH(\pi')$. \hfill $\Box$

Now we prove the implication (1) $\Rightarrow$ (3) in the general case. Choose $w \in W(M, M')$ such that $w(\rho) = \rho'$. As follows from Problem 0.215 there exists chains $M = M_0, M_1, ..., M_r = M', L_1, L_2, ..., L_r$ of standard Levi subgroups of $G$ and a decomposition $w = w_r...w_1$ such that

(1) $M_{i-1}, M_i$ are maximal Levi subgroups of $L_i, 1 \leq i \leq r.$

(2) $w_i \in W_{L_i}.$

(3) $w_i(M_{i-1}) = M_i, 1 \leq i \leq r.$

Let $\rho_k := w_k...w_1(\rho) \in \mathcal{M}(M_k), 1 \leq k \leq r.$ It is sufficient to show that

$$JH(i_{M_k}(\rho)) = JH(i_{M_{k-1}}(\rho_{k-1}))$$

for all $k, 1 \leq k \leq r.$ Since

$$i_M^G = i_L^G \circ i_M^{L_k} i_{M_{k-1}}^G = i_L^G \circ i_{M_{k-1}}^{L_k}$$

the equality

$$JH(i_{M_k}(\rho)) = JH(i_{M_{k-1}}(\rho_{k-1}))$$

would follow from the equality

$$JH(i_{M_k}(\rho)) = JH(i_{M_{k-1}}(\rho_{k-1})), 1 \leq k \leq r.$$ 

Since $M_{k-1}, M_k, 1 \leq k \leq r$ are standard maximal Levi subgroups of $L_k$ the equality

$$JH(i_{M_k}(\rho)) = JH(i_{M_{k-1}}(\rho_{k-1}))$$

follows from 0.220. \hfill $\Box$
9.4. Cuspidal datas.

Definition 0.222. (1) A cuspidal data of \( G \) is a pair \((M, \rho)\) where \( M < G \) is a standard Levi subgroup and \( \rho \in \text{Irr}_c(M) \).

(2) Two cuspidal datas \((M, \rho)\) and \((M', \rho')\) are associated if there exists \( w \in W \) such that \( w(M) = M' \) and \([w(\rho)] = [\rho']\). In this case we write \((M, \rho) \sim (M', \rho')\).

(3) We denote by \( X(G) \) the set of cuspidal datas up to associate.

(4) A component of \( \mathcal{M}(G) \) is an equivalence class of pairs \((M, D)\) where \( M < G \) is a standard Levi subgroup and \( D \in X_r(M) \) is cuspidal component of \( \mathcal{M}(M) \) where two pairs \((M, D), (M', D')\) are equivalent if there exists \( w \in W \) such that \( w(M) = M' \) and \( w(D) = D' \).

(5) We denote by \( \bar{X}(G) \) the set of components of \( \mathcal{M}(G) \).

(6) For any component \( \Omega \in \bar{X}(G) \) we denote by \( X_\Omega \subset X(G) \) the set of cuspidal datas in \( \Omega \).

Remark 0.223. By the definition for any \( \Omega \in \bar{X}(G) \) the set \( X_\Omega \) is equal to the quotient \( D/W_{\mathcal{M}(M, D)} \) of some cuspidal component \( D \in X_r(M), M < G \). Therefore the structure of an \( \mathbb{C} \)-algebraic variety on \( D \) induces a structure of an \( \mathbb{C} \)-algebraic variety on \( X_\Omega \).

Lemma 0.224. Let \( \pi \) be an irreducible representation of \( G \).

(1) There exists a cuspidal data \((M, \rho)\) of \( G \) such that \( \rho \in JH(r_M(\pi)) \).

(2) Let \((M, \rho)\) be a cuspidal data of \( G \). Then \( \rho \in JH(r_M(\pi)) \) iff \( \pi \in JH(i_{M}(\rho')) \) for any cuspidal data \((M', \rho')\) associated with \((M, \rho)\).

(3) Let \((M, \rho), (M', \rho')\) be cuspidal datas of \( G \) such that \( \rho \in JH(r_M(\pi)) \) and \( \rho' \in JH(r_{M'}(\pi)) \) for some \( \pi \in \text{Irr}(G) \). Then \((M, \rho)\) and \((M', \rho')\) are associated.

Proof. (1) Choose \( M \) to be a minimal standard Levi subgroup such that \( r_M(\pi) \neq \{0\} \). Then \( r_M(\pi) \) is a non-zero cuspidal representation of \( M \) and any \( \rho \in JH(r_M(\pi)) \) satisfies the condition of (1).

(2) Assume that \( \rho \in JH(r_M(\pi)) \). As follows from Lemma 0.181 we have \( \text{Hom}_{M}(r_M(\pi), \rho) \neq \{0\} \). Then by the Frobenius duality \( \text{Hom}_{M}(\pi, i_{M}(\rho)) \neq \{0\} \) and therefore \( \pi \in JH(i_{M}(\rho)) \). If \((M', \rho')\) is cuspidal data associated with \((M, \rho)\) then by Theorem 0.219 we have \( JH(i_{M}(\rho)) = JH(i_{M'}(\rho')) \).

Conversely assume that \( \pi \in JH(i_{M}(\rho)) \). Then as, follows from Corollary 0.213, all the irreducible subquotients of \( r_M(\pi) \) are of the form \( w(\rho), w \in W(M, M) \).

(3) Assume now that \((M, \rho), (M', \rho')\) are cuspidal datas such that \( \rho \in JH(r_M(\pi)) \) and \( \rho' \in JH(r_{M'}(\pi)) \). As follows from (2) we have \( \rho' \in JH(r_{M'} \circ i_{M}(\rho)) \). Now the result follows from Corollary 0.213 c).

Definition 0.225. (1) For any \( \pi \in \text{Irr}(G) \) we denote by \( pr(\pi) \in X(G) \) the associated class of cuspidal datas \((M, \rho)\) such that \( \rho \in JH(r_M(\pi)) \). As follows from the last Lemma the map \( pr : \text{Irr}(G) \rightarrow X(G) \) is well defined.
(2) For any component \( \Omega \in \tilde{X}(G) \) of \( M(G) \) we denote by \( \text{Irr}_\Omega \) the preimage \( pr^{-1}(X_\Omega) \subset \text{Irr}(G) \) and denote by \( M(\Omega) \) the full subcategory of \( M(G) \) of objects \( V \) such that \( JH(V) \subset \text{Irr}_\Omega \).

(3) For and cuspidal date \( (M, \rho) \) we define
\[
X(\rho) := \{ \psi \in \Psi_M | \text{the representation } \pi_\psi \text{ is not irreducible} \}.
\]

**Corollary 0.226.** The map \( pr : \text{Irr}(G) \to X(G) \) is finite-to-one and surjective.

**Proof.** The surjectivity follows immediately from the definition of the map \( pr \) and the finiteness of fibers \( pr^{-1}(\pi) \) follows from Lemma 0.217 (2).

**Theorem 0.227.** [Decomposition Theorem]. For any \( \Omega \in \tilde{X}(G) \) the set \( \text{Irr}_\Omega \) splits the category \( M(G) \).

**Proof.** We start with the following definition.

**Definition 0.228.**
(1) Let \( V \) be a \( G \)-module. For any \( \Omega \in \tilde{X}(G) \) we define \( V(\Omega) \) as the maximal submodule of \( V \) such that \( JH(V(\Omega)) \subset \text{Irr}_\Omega \).

(2) We say that \( V \) is split if \( V = \bigoplus_{\Omega \in \tilde{X}(G)} V(\Omega) \).

**Lemma 0.229.** Any submodule of a split module is split.

**Proof.** Let \( V' \subset V \) be a submodule of a split module. We want to show that \( V' = \bigoplus_{\Omega \in \tilde{X}(G)} V'(\Omega) \). As follows from Lemma 0.25 it is sufficient to show that \( JH(W) = \emptyset \) for
\[
W = V' / \bigoplus_{\Omega \in \tilde{X}(G)} V'(\Omega).
\]
Fix \( \Omega_0 \in \tilde{X}(G) \) and consider
\[
p_{\Omega_0} : V \to \bigoplus_{\Omega \neq \Omega_0} V(\Omega)
\]
Since \( V \) is split, \( \text{Ker} p_{\Omega_0} = V(\Omega_0) \). Consequently, \( \text{Ker}(p_{\Omega_0}) = V'(\Omega_0) \).

Since \( V'(\Omega) = V(\Omega) \cap V' \) for all \( \Omega \in \tilde{X}(G) \) we have
\[
JH(W) \subset \bigcup_{\Omega \in \tilde{X}(G)} \text{Irr}_\Omega.
\]
So
\[
JH(W) \subset \bigcap_{\Omega \in \tilde{X}(G)} \text{Irr}_\Omega = \emptyset
\]
Since \( \text{Irr}(G) \) is a disjoint union of \( \text{Irr}_\Omega, \Omega \in \tilde{X}(G) \) we see that \( JH(W) = \emptyset \).

Now we remind some results on the the decomposition of the cuspidal part.

**Definition 0.230.**
(1) We define \( M(\text{cusp}) := \prod_{M \subset G} M(M)_{\text{cusp}} \).

(2) We define the functor \( I : M(\text{cusp}) \to M(G) \) by \( I((M, \rho_M)) := \bigoplus_{M \subset G} I(M)(\rho_M) \).
We define the functor $R : \mathcal{M}(G) \to \mathcal{M}(\text{cusp})$ by $R(\pi) := \{r_M(\pi)_c\}, M < G$ where $V_c$ is the cuspidal part of $V$ as in Definition 0.163.

**Lemma 0.231.**

1. Let $(\pi, W)$ be an irreducible representation of $G$ and
2. For any smooth irreducible representation $(\pi, W)$ of $G$ there exists a standard parabolic $P = MU$ and an irreducible cuspidal representation $(\tau, W')$ of $M$, such that there exists an embedding $W \hookrightarrow i_M(W)$.
3. $R$ is the left adjoint to $I$.
4. Functors $R$ and $I$ are exact.
5. Functors $R$ and $I$ are faithful.
6. The adjoint map $\kappa(\pi) : \pi \mapsto IR(\pi)$ is a monomorphism.

**Proof.** The parts (1) and (2) are familiar. It is also clear that the functor $I$ is faithful. Let us first of all show that functor $R(\theta) \neq \{0\}$ for any non-zero maps non-zero $\theta \in \mathcal{M}(G)$. Choose $\pi \in JH(\theta)$. Since the functor $R$ is exact it is sufficient to show that $R(\pi) \neq \{0\}$. Let $M$ be a Levi subgroup, minimal subject to the condition $r_{M,U}(\pi) \neq 0$. Then $r_{M,U}(\pi)$ is cuspidal. So $R(\pi) \neq \{0\}$.

Let us now show that $\kappa(\pi) \neq 0$ for any non-zero $\pi \in \mathcal{M}(G)$. By the adjunction we have a bijection

$$a : \text{Hom}_{\mathcal{M}(G)}(\pi, IR(\pi)) \to \text{Hom}_{\mathcal{M}(\text{cusp})}(R(\pi), R(\pi))$$

such that $a(\kappa) = Id_{R(\pi)}$. Since $R(\pi) \neq \{0\}$ we see that $\kappa(\pi) \neq 0$.

Let $\tau := \text{Ker} \kappa$. We want to show that $\tau = \{0\}$. Assume that $\tau \neq \{0\}$. Since the functors $R$ and $I$ are exact the map $RI(\kappa) : RI(\tau) \to RI(\pi)$ is a monomorphism and therefore the composition $\tau \to IR(\tau) \to IR(\pi)$ does not vanish. But it is zero by the definition of $\tau$. So $\tau = \{0\}$.

Since the adjoint map $\kappa(\pi) : \pi \mapsto IR(\pi)$ is a monomorphism we see that $R$ is faithful. $IR(\pi) = \{0\}$. Since the functors $R$ and $I$ are faithful we see that $\tau = \{0\}$. We are now ready to prove the theorem. As we have shown it is sufficient to prove that any $M \in \mathcal{M}(G)$ is a submodule of a split module. But as follows from Lemma 0.231 any smooth $G$-module $V$ is a submodule of $IR(V) = \bigoplus_{M < G} i_M(\tau_M)$ where $\tau_M$ is a cuspidal representation of $M$. Thus, to prove the theorem, it is enough to prove that the $G$-modules $i_M(\tau_M)$ are split for cuspidal $M$-modules $\tau_M$. Since $\tau_M$ is cuspidal, it follows from Theorem 0.149 that we may write $\tau_M = \bigoplus_D \tau(D)$ where the $D$ run through the cuspidal components of $M$. This reduces our problem further; we must prove that $i_M(\tau_M)$ splits when $\tau_M \in \mathcal{M}_D(M)$ for some cuspidal component $D$. But it follows from Corollary 0.213 that $JH(i_M(\tau_M)) \subset X_\Omega$ where $X_\Omega$ is the component of $\mathcal{M}(G)$ corresponding to the pair $(M, D)$. So $i_M(\tau_M) = i_M(\tau_M)(\Omega)$.
Lemma 0.232. Let $(M, (\rho, V))$ be a cuspidal data, $K \subset G$ be a congruence subgroup such that $V^{K \cap M} \neq \{0\}$ and $\Omega \in \bar{X}(G)$ be the corresponding component. Then $L^K \neq \{0\}$ for any irreducible $(\pi, L) \in \operatorname{Irr}_\Omega$.

Proof. As follows from Corollary 0.213 $r_M(L) \neq \{0\}$ the representation $r_M(\pi)$ of $M$ has a subquotient isomorphic $\rho \otimes \psi$ for some character $\psi : M/M_0 \to \mathbb{C}^*$. Therefore $(r_M(L))^{K \cap M} \neq \{0\}$. But then it follows from Lemma 0.125 that $L^K \neq \{0\}$. \hfill \Box

Corollary 0.233. Let $K \subset G$ be a congruence subgroup.

1. The set

$$\operatorname{Irr}_K(G) := \{ (\pi, V) \in \operatorname{Irr}(G) | V^K \neq \{0\} \}$$

splits the category $\mathcal{M}(G)$.

2. If a representation $(\rho, V)$ of $G$ is generated by the subspace $V^K$ then the same is true for any subquotient of $V$.

Proof. (1) As follows from the decomposition theorem it is sufficient to show that the set $\operatorname{Irr}_K(G) \cap \operatorname{Irr}_\Omega$ splits the category $\mathcal{M}(\Omega)$. But by the Lemma 0.232 either $\operatorname{Irr}_K(G) \cap \operatorname{Irr}_\Omega = \operatorname{Irr}_\Omega$ or $\operatorname{Irr}_K(G) \cap \operatorname{Irr}_\Omega = \emptyset$.

2. Let $V = V_1 \oplus V_2$ be the decomposition such that $JH(V_1) \subset \operatorname{Irr}_K(G)$ and $JH(V_1) \cap \operatorname{Irr}_K(G) = \emptyset$. Since $V$ is generated by the subspace $V^K$ we see that $V_2$ is generated by the subspace $V_2^K$. By the definition of $V_2$ we have $V_2^K = \{0\}$. So $V = V_1$ and therefore $JH(L) \subset \operatorname{Irr}_K(G)$ for every irreducible subquotient $L$ of $V$.

Assume now that the claim (2) is false and $V' \subset V'' \subset V$ be submodules such that $(V''/V')^K = \{0\}$. Since $JH(V''/V')$ subset $JH(V) \subset \operatorname{Irr}_K(G)$ and $(V''/V')^K = \{0\}$ we see that $V''/V' = \{0\}$. \hfill \Box

Remark 0.234. (1) The Corollary is true for all open subgroups $K$ of $G$ such that $K = (K \cap U)(K \cap M)(K \cap U)$ for any $M < G$. For example you can take $K$ to be the Iwahori subgroup.

2. The result is not true when $K = K_0$. Consider the case when $G = SL(2, F)$ and $V$ the space of smooth measures on $F^1 = B \backslash G$. The integration defines an exact sequence

$$\{0\} \to V_0 \to V \to \mathbb{C} \to \{0\}$$

which [see ??] does not admit a splitting. On the other hand $V_0$ does not have $K_0$-invariant vectors.

9.5. Noetherian properties.

Definition 0.235. We say that an object $V$ of an abelian category $\mathcal{C}$ is Noetherian if any increasing chain $V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$ of subobjects of $V$ stabilizes.

Remark 0.236. Let $A$ be a Noetherian ring and $V$ an $A$-module. Then $V$ is Noetherian as an object of the category of $A$-modules iff $V$ is finitely generated.
Lemmas 0.237. Let $V$ be a smooth $G$-module. The following conditions are equivalent

1. $V$ is Noetherian.
2. $V$ is finitely generated.
3. There exists a congruence subgroup $K$ such that $GV^K = V$ and the $H_K$-module $V^K$ is finitely generated.

Proof. It is clear that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Assume that $V$ is finitely generated, $V = G(W), \dim(W) < \infty$. Then there exists a congruence subgroup $K$ such that $W \subset V^K$. Then $\mathcal{H}_K(W) = e_KG(W) = e_K(V) = V^K$.

(3) $\Rightarrow$ (1). Let $V \in \mathcal{M}(G)$ be such that there exists a congruence subgroup $K$ such that $GV^K = V$ and the $H_K$-module $V^K$ is finitely generated.

Claim 0.238. The module $R(V)$ [see Lemma 0.231] is Noetherian.

Proof. By the definitions $R(V) = \prod_{\{M < G, D_M \in \mathcal{X}_e(M)\}} R(D_M)(V)$ where for any $M < G$ and a cuspidal component $D_M \in \mathcal{X}_e(M)$ we denote by $R(D_M)(V)$ the projection of $r_M(V)$ on the factor $\mathcal{M}(D_M)$ of $\mathcal{M}(M)$. As follows from the previous Corollary and the Uniform Admissibility Theorem 0.149 $R(D_M)(V) = \{0\}$ for almost all pairs $(M, D_M)$. So it is sufficient to show that the representations $R(D_M)(V) \in \mathcal{M}(D_M)$ are Noetherian. As follows from Corollary 0.233 the $M$-module $R(D_M)(V)$ is generated by the subspace $RK \cap M(D_M)(V) = J(V^K)$ [see Proposition 0.130] is finite dimensional. So the $M$-module $R(D_M)(V)$ is finitely generated. On the other hand it follows from Lemma 0.197 that the category $\mathcal{M}(D_M)$ is equivalent to the category of right modules over a Noetherian ring $A(D_M)$. So $R(D_M)(V) \in \mathcal{M}(D_M)$ are Noetherian objects.

To prove that $V$ is Noetherian consider an increasing chain $V_1 \subset V_2 \subset ... \subset V_n \subset ...$ of subobjects of $V$. Since the object $R(V)$ is Noetherian the chain $R(V_1) \subset R(V_2) \subset ... \subset R(V_n) \subset ...$ stabilizes. Since the functor $R$ is faithful this implies the stabilization of the chain $V_1 \subset V_2 \subset ... \subset V_n \subset ...$ also stabilizes.

Corollary 0.239. (1) The functors $r_M$ map Noetherian objects into Noetherian objects.

(2) The functors $i_M$ map Noetherian objects into Noetherian objects.

Proof. (1) Let $V \in \mathcal{M}(G)$ be a Noetherian object. Then by 0.237 $V$ is finitely generated, $V = G(W), \dim(W) < \infty$ and we may assume that $W$ is $K_0$-invariant. Choose a congruence subgroup $K$ such that $W \subset V^K$. Since $K_0W = W$ the decomposition $G = MU_K$ implies that $Mq(W) = r_M(V)$ where $q : V \rightarrow r_M(V)$ is the natural projection. So the module $r_M(V)$ is finitely generated and therefore it is Noetherian.

(2) Let $W$ be a Noetherian object of $\mathcal{M}(M)$. We want to show that any chain increasing $V_1 \subset V_2 \subset ... \subset V_n \subset ...$ of subobjects of $V := i_M(W)$ stabilizes. Since the functor $R$ is faithful it is sufficient to show that the
chain $R(V_1) \subset R(V_2) \subset \ldots \subset R(V_n) \subset \ldots$ stabilizes. In other words we have to show that for any standard subgroup $N$ the chain $r_N(V_1)c \subset r_N(V_2)c \subset \ldots \subset r_N(V_n)c \subset \ldots$ stabilizes. But now the stabilization of these chains follow from (1), the inductive assumptions and the Basic Geometric Lemma 0.212 which says that objects $r_N(V_i)c$ are glued from a finite number of Noetherian objects.

\[ \square \]

10. Irreducibility of induced representations

**Definition 0.240.** Let $M < G$ be a standard Levi subgroup and $(\rho, V) \in \mathcal{M}(M)$.

(1) We write $(\pi, W) := i_M(\rho, V)$ and for any character $\psi \in \Psi_M$ write $(\pi_\psi, W_\psi) := i_M(\rho \otimes \psi, V)$.

(2) We use the Iwasawa decomposition $G = PK_0, P = MU$ to identify the restriction of the representation $\pi_\psi$ on $K_0$ with the representation

$$\text{ind}^{K_0}_{K_0 \cap P}(\text{inf}^{K_0 \cap P}_{K_0 \cap M}(\text{Res}_{K_0 \cap M}(\rho))).$$

(3) Using this isomorphism we identify spaces $W_\psi, \psi \in \Psi_M$ with the fixed space $W$ in such a way that the restriction of $\pi_\psi$ on $K_0$ does not depend on $\psi$.

As we know the representation $i_M(\rho)$ is admissible and therefore the space $W^K$ is finite-dimensional for any congruence subgroup $K$.

**Lemma 0.241.** For any $h \in \mathcal{H}_K$ the operator $\hat{h}(\psi) := \pi_\psi(h)$ is a regular function on $\Psi_M$ with values in $\text{End} W^K$.

**Proof.** Since the support of $h$ is compact there exists a congruence subgroup $K' \subset K$ such that $x^{-1}k'x \in K$ for all $x \in \text{supp}(h)$. Then $\pi_\psi(x)(W^{K'}) \subset W^K$ for all $x \in \text{supp}(h)$. It is sufficient to show that for any $x \in \text{supp}(h)$ the operator $\pi_\psi(x) : W^{K'} \to W^K$ is a regular function on $\Psi_M$ with values in $\text{Hom}(W^{K'}, W^K)$. But for any $f \in W_\psi = \text{ind}^{K_0}_{K_0 \cap P}(V)$ we have $\pi_\psi(x)(f)(k) = \psi(\theta_M(kx))\pi(x)(f)(k)$. \[ \square \]

**Lemma 0.242.** Let $X(\rho, K) := \{ \psi \in \Psi_M | \text{the representation } \pi_\psi : \mathcal{H}_K \to \text{End}(W^K) \text{ is not irreducible}\}$.

Then $X(\rho, K) \subset \Psi_M$ is a Zariski closed subset.

**Proof.** We start with the following general result.

**Problem 0.243.** Let $k$ be an algebraically closed field, $L$ be a finite-dimensional $k$-vector space, $Y$ an algebraic $k$-manifold, $f_i, i \in I$ a family of $L$-valued regular function on $Y = Y(k)$ and $Z \subset Y$ the set of $z \in Y$ such that $\{f_i(z)\}, i \in I$ do not generate $L$. Then there exists a Zariski closed subset $Z \subset Y$ such that $Z = Z(k)$.
The proof of the Lemma is based on the Bernside’s theorem which says that there is no proper subalgebras of the algebra \( \text{End}(W^K) \) which act irreducibly on \( W^K \). Therefore
\[
X(\rho, K) := \{ \psi \in \Psi_M | \text{the span of } \hat{h}(\psi), h \in \mathcal{H}_K \text{ is equal to } \text{End}(W^K) \}.
\]
We now apply Problem to the case when \( k = \mathbb{C}, Y = \Psi_M, L = \text{End}_\mathbb{C}(W^K), I = G, f_g(\psi) := \pi_\psi(e_K g e_K) \).

\[ \square \]

**Corollary 0.244.** For any cuspidal data \( (M, \rho) \) the subset \( X(\rho) \) [see 0.225] is Zariski closed.

**Proof.** Follows from Corollary 0.233. \[ \square \]

The main goal of this section is to prove that \( X(\rho) \neq \Psi_M \). The proof is based on the analysis of the unitary structure.

**10.1. The Unitary Structure.** Let \( (\rho, V) \) be a smooth representation of \( G \).

**Definition 0.245.** For any \( v \in V, \lambda \in \hat{V} \) we define the matrix coefficient \( m_{\lambda, v}(g) \) as a function on \( G \) given by
\[
m_{\lambda, v}(g) = \hat{v}(\pi(g)v), v \in V, \tilde{v} \in \hat{V}.
\]

1. Assume now that \( (\pi, V) \) is an irreducible representation of \( G \) such that the restriction to the center \( Z(G) \) is equal to \( \chi Id_V \) where \( \chi : Z(G) \to \mathbb{C}^* \) is a unitary character. We say that \( V \) is square integrable modulo center if
\[
\int_{G/Z} |m_{\lambda, v}(g)|^2 dg < \infty.
\]

2. A unitary structure on a \( G \)-module \( (\pi, V) \) is a positive definite, \( G \)-invariant Hermitian scalar product \( Q : V \otimes V \to \mathbb{C} \).

3. Let \( P = MU \) be a parabolic subgroup of \( G \) and \( (\rho, W, <, >) \) be a unitary representation of \( M \). We define the unitary structure \( Q \) on \( (\pi, V) = (i_M(\rho), i_M(W)) \) by
\[
Q(f_1, f_2) = \int_{x \in P \backslash G} < f_1(x), f_2(x) >
\]
where the linear functional \( f \) is as in Definition ??.

**Remark 0.246.** We do not assume that \( V \) is complete with respect to this structure.

**Problem 0.247.** Let \( (\pi, V) \) be an irreducible representation of \( G \) which is essentially square integrable modulo center. Then

1. \( (\pi, V) \) admits a unitary structure.
2. \( |m_{\lambda, v}(g)|^2 \in L^1(G/Z(G)) \) for all \( v \in V, \lambda \in \hat{V} \).

The essential uniqueness of an invariant scalar product for irreducible representations follows from the following version of the Schur’s lemma.
Lemma 0.248 (Schur’s Lemma). Let $V$ be an irreducible $G$-module. Then any two $G$-invariant unitary structures on $V$ are proportional.

Proof. Let $V^+$ be anti-linear dual of $V$, that is the space of anti-linear functionals. An invariant scalar product $Q : V \otimes \overline{V} \to \mathbb{C}$ defines a $G$-equivariant semi-linear map $V \to V^+$. As $V$ is smooth, we obtain a semi-linear map $V \to (V^+)_sm$. Since $V$ is admissible we see that $V^+_sm$ is also admissible and irreducible. Therefore any non-zero $G$-equivariant semi-linear map $V \to (V^+)_sm$ is a bijection. It follows now from the Schur’s lemma that any two non-zero $G$-equivariant semi-linear maps $V \to (V^+)_sm$ are proportional. □

10.2. Applications of the unitarity.

Lemma 0.249. Any smooth admissible unitary representation $(\pi, V)$ of $G$ is completely reducible. [That is, $V = \bigoplus V_i$ where the $V_i$ are irreducible unitary subrepresentations.]

Remark 0.250. The assumption of the admissibility is important.

Proof. Suppose $W \subset V$ is a submodule. Then the orthogonal complement, $W^\perp \subset V$ is also a submodule and $W \cap W^\perp = 0$. It remains to check that $W + W^\perp = V$. For this it is enough to check that for any compact open subgroup $K \subset G$ we have $W^K + (W^\perp)^K = V^K$. Since [by the admissibility] the space $V^K$ is finitely-dimensional we have $V^K = W^K \oplus (W^K)^\perp \cap V^K$. So it is sufficient to show that $(W^K)^\perp \cap V^K \subset W^\perp$. Since the group $K$ is compact we have $W = W^K \oplus L$ where $\int_K \pi(k)ldk = 0$ for $l \in L$. Then $<l,v> = 0$ for all $v \in V^K$ and we see that $<w,v> = 0$ for all $v \in W^K \cap V^K$, $w \in W$. □

Corollary 0.251. Let $V$ be an admissible unitary representation $V$ of $G$ such that $\text{End}_G(V) = \mathbb{C}$. Then $V$ is irreducible.

This Corollary provides a method for establishing the irreducibility of some representations. Here is an important example.

Let $\rho$ be an irreducible cuspidal representation of a Levi subgroup $M < G$. We denote by $X(\rho) \subset \Psi_M$ the set characters $\psi$ such that the representation $\pi_\psi := i_M(\rho \otimes \psi)$ of $G$ is reducible.

Theorem 0.252. $X(\rho)$ is a proper Zariski closed subset of $\Psi_M$.

Proof. Since $\rho$ is irreducible there exists a character $\chi : Z(M) \to \mathbb{C}^*$ such that $\rho(z) = \chi(z)I_d$ for $z \in Z(M)$. Since the subgroup $Z(G) \cap M^0 \subset Z(G)$ is compact there exists $\psi \in \Psi_M$ such that $|\chi(z)\psi(z)| = 1$ for all $z \in Z(M)$. Therefore [see Problem 0.84] the representation $\rho \otimes \psi$ admits a unitary structure. By replacing $\rho$ by $\rho \otimes \psi$ we may assume that $\rho$ admits a unitary structure.

As follows from Corollary ?? the subset $X(\rho) \subset \Psi_M$ is Zariski closed. So we only have to show that $X(\rho) \neq \Psi_M$.

As follows from Corollary 0.213 there exists a proper Zariski closed subset $Y$ of $\Psi_M$ such that $\text{End}_G(i_M(\rho \otimes \psi)) = \mathbb{C}$ for $\psi \in \Psi_M - Y$. Since
the subset $\Psi^*_{M} \subset \Psi_{M}$ of unitary characters is Zariski dense in $\Psi_{M}$ the set $\Psi^*_{M} - \Psi^*_{M} \cap Y$ is not empty. But for any $\psi \in \Psi^*_{M} - \Psi^*_{M} \cap Y$ the representation $\pi_{\psi}$ is unitary and $\text{End}_{G}(i_{M}(\rho \otimes \psi)) = \mathbb{C}$. It follows now from Corollary 0.251 that $\pi_{\psi}$ is irreducible. \hfill $\square$

11. The second adjointness.

11.1. The comparison of orbits of $P$ and $\bar{P}$. We start this section with some results on the structure of of $P$ orbits on $G/Q$ for parabolic subgroups $P, Q$ of $G$.

Definition 0.253. For a pair $M, N$ of standard Levi subgroups we define

$$W^{M,N} := \{ w \in W | w(M \cap B) \subset B, w^{-1}(N \cap B) \subset B \}$$

Problem 0.254. Let $P, Q$ be standard parabolic subgroups of $G$.

1. For any $w \in W$ the intersection $W_{M}wW_{N} \cap W^{M,N}$ consists of one element. In other words any double coset $W_{M}wW_{N}$ contains unique representative in $W^{M,N}$ and we can identify the sets $W^{M,N}$ and $W_{M}\sigma W/W_{N}$.

2. For any standard parabolic subgroups $P, Q$ of $G$ the imbedding $W \hookrightarrow G$ induces the bijection $W_{M}\backslash W/W_{N} \rightarrow \bar{P}\backslash G/Q$.

3. So the set $W^{M,N}$ parametrizes the both double cosets $\bar{P}\backslash G/Q$ and $P\backslash G/Q$ and we defined a bijection $\kappa : P\backslash G/Q \rightarrow \bar{P}\backslash G/Q$.

4. $\kappa$ reverses the order on partially ordered sets $P\backslash G/Q$ and $\bar{P}\backslash G/Q$.

5. The map $w \rightarrow w^{-1}w_{G}$ defines bijection $\kappa' : P\backslash G/Q \rightarrow Q\backslash G/\bar{P}$ which reverses the order on partially ordered sets.

6. There exists unique $w_{G} \in W_{G}$ such that $w \leq w_{G}$ for all $w \in W_{G}$ and the map $w \rightarrow w w_{G}$ reverses the partial order on $W_{G}$.

7. Assume that $P = MU$ is a maximal parabolic subgroup of $G$. Then

(a) $W(M, \ast) = W_{M} \cup w_{G}W_{M}$.

(b) If $W(M, M) = W_{M}$ then the parabolic subgroups $P$ and $\bar{P}$ of $G$ are not conjugate and $N := w_{G}(M) < G$ is a standard Levi subgroup of $G$ associated with $M$. Moreover any standard Levi subgroup of $G$ associated with $M$ is either equal to $M$ or is equal to $N$. In particular $l_{G}(M) = 2$.

(c) If $W(M, M) \neq W_{M}$ then the parabolic subgroups $P$ and $\bar{P}$ are conjugate, $|W(M, M)/W_{M}| = 2$ and any standard Levi subgroup of $G$ associated with $M$ is equal to $M$. So in this case we also have $l_{G}(M) = 2$.

Remark 0.255. In the case when $G = GL(n, F)$, $P = Q = B$ any double coset $BwB$ has a form $BwB$ where $w \in W = S_{n}$ where the symmetric group $S_{n} \subset GL(n, F)$ is realized as the subgroup of permutation matrices. In this case standard parabolic subgroups $P$ correspond to partitions $\sigma_{P}$ of $n$ and we denote by $W_{P} \subset W$ the subgroup of permutations preserving this partition.
Problem 0.256. a) Show that one can identify the sets \( P\sigma G/Q \) and \( W_P \sigma W/W_Q \).

b) Prove all the statements of the last Claim in the case \( G = GL(n) \).

c) Show that any parabolic subgroup of \( G \) is conjugate to a standard parabolic by an element of \( K_0 \) and that \( G/P \) is compact.

Let \( P = MU \) be a parabolic subgroup of \( G \) and \( \bar{P} = M\bar{U} \) be the opposite parabolic. For any parabolic \( Q = NV \) of \( G \) the set \( W^{M,N} \) parametrizes the both double cosets \( P\backslash G/Q \) and double cosets \( P\backslash G/Q \). So we defined a bijection \( \bar{\kappa} : P\backslash G/Q \to \bar{P}\backslash G/Q \).

Definition 0.257. (1) For any parabolic subgroups \( P,Q \) of \( G \) we define a partial order on the finite set \( P\backslash G/Q \) in such a way that \( i \leq j \) if \( i \in P\backslash G/Q \) and \( j \in P\backslash G/Q \) if \( \Omega_i \) is in the closure \( \Omega_j \) of \( \Omega_j \) where \( \Omega_i, \Omega_j \subset G \) are \( P \times Q \)-orbits corresponding to \( i, j \in P\backslash G/Q \).

(2) We define a partial order on \( W \) using the bijection \( W \to \mathcal{B}_0\backslash G/P_0 \).

(3) For any standard parabolic subgroup \( P = MU \) we denote by \( \bar{P} = M\bar{U} \) be the opposite parabolic.

Claim 0.258. (1) For any standard parabolic subgroups \( P,Q \) of \( G \) the imbedding \( N_G(M_0) \to G \) induces the bijection \( W_M\backslash W/W_N \to P\backslash G/Q \). So the set \( W^{M,N} \) parametrizes the both double cosets \( P\backslash G/Q \) and double cosets \( P\backslash G/Q \) and we defined a bijection \( \bar{\kappa} : P\backslash G/Q \to \bar{P}\backslash G/Q \).

(2) \( \bar{\kappa} \) reverses the order on partially ordered sets \( P\backslash G/Q \) and \( \bar{P}\backslash G/Q \).

(3) The map \( w \to w^{-1}w_G \) defines bijection \( \bar{\kappa}' : P\backslash G/Q \to Q\bar{\backslash}G/P \) which reverses the order on partially ordered sets.

(4) There exists unique \( w_G \in W_G \) such that \( w \leq w_G \) for all \( w \in W_G \) and the map \( w \to w_G \) reverses the partial order on \( W_G \).

(5) Assume that \( P = MU \) is a maximal parabolic subgroup of \( G \). Then

(a) \( W(M,\ast) = W_M \cup w_G W_M \).

(b) If \( W(M,M) = W_M \) then the parabolic subgroups \( P \) and \( \bar{P} \) of \( G \) are not conjugate and \( N := w_G(M) \subset G \) is a standard Levi subgroup of \( G \) associated with \( M \). Moreover any standard Levi subgroup of \( G \) associated with \( M \) is either equal to \( M \) or is equal to \( N \). In particular \( l_G(M) = 2 \).

(c) If \( W(M,M) \neq W_M \) then the parabolic subgroups \( P \) and \( \bar{P} \) are conjugate, \( |W(M,M)/W_M| = 2 \) and any standard Levi subgroup of \( G \) associated with \( M \) is equal to \( M \). So in this case we also have \( l_G(M) = 2 \).

Remark 0.259. In the case when \( G = GL(n,F) \), \( P = Q = B \) any double coset \( BgB \) has a form \( BwB \) where \( w \in W = S_n \) where the symmetric group \( S_n \subset GL(n,F) \) is realized as the subgroup of permutation matrices. In this case standard parabolic subgroups \( P \) correspond to partitions \( \sigma_P \) of \( n \) and we denote by \( W_P \subset W \) the subgroup of permutations preserving this partition.
11. THE SECOND ADJOINTNESS.

**Problem 0.260.** a) Show that one can identify the sets \( P\sigma G/Q \) and \( WP\sigma W/W_Q \).

b) Prove all the statements of the last Claim in the case \( G = GL(n) \).

c) Show that any parabolic subgroup of \( G \) is is conjugate to a standard parabolic by an element of \( K_0 \) and that \( G/P \) is compact.

11.2. The construction of second adjointness. Let \( P = MU \) be a parabolic subgroup of \( G \). We know that \( r_{M,U} \) is the left adjoint to \( i_{M,U} \). This implies that for any representation \( \pi \) of \( M \) we have a canonical isomorphism

\[
\text{Hom}_M(r_{M,U} \circ i_{M,U}(\pi), \pi) = \text{Hom}_G(i_{M,U}(\pi), i_{M,U}(\pi)).
\]

In particular we have a canonical morphism \( r_{M,U} \circ i_{M,U} \to \text{Id}_{M(M)} \) of functors. In fact, the existence of such a morphism is implies by the existence of the filtration of the functor \( r_{M,U} \circ i_{M,U} \) described in the Basic Geometric Lemma. Really subquotients in this filtration correspond to orbits of \( P \) acting on \( X = P\backslash G \). There is a distinguished orbit of the action of \( P \) on \( P\backslash G \), namely point \( P \) which is the only closed orbit. Since \( P \) is closed it corresponds to the quotient of \( r_{M,U} \circ i_{M,U} \). It is easy to see that this quotient is equal to \( \text{Id}_{M(M)} \). We see that the adjointness property is related to the existence of the distinguished orbit.

Let \( \overline{P} = M\overline{U} \) be the parabolic opposite to \( P \).

As follows from Problem 0.258 there is unique open orbit of the action of \( \overline{P} \) on \( G/P \) and the functor associated with this orbit is equal to \( \text{Id}_{M(M)} \).

Set, \( \tau_M = r_{M,U} \). We have shown that for any representation \( \tau \in M(M) \) there is a canonical imbedding \( \tau \to \bar{r}_M \circ i_M(\tau) \). Now for any \( \varphi \in \text{Hom}_G(i_{M,U}(\tau), \pi) \). we define a morphism \( \beta(\varphi) \in \text{Hom}_M(\tau, \tau_M(\pi)) \) as the composition

\[
\beta(\varphi) : \tau \to \bar{r}_M \circ i_M(\tau) \xrightarrow{\tau_M(\varphi)} \tau_M(\pi)
\]

In other words, we defined a map

\[
\beta_{G,M}(\tau, \pi) : \text{Hom}_G(i_M(\tau), \pi) \to \text{Hom}_M(\tau, \tau_M(\pi))
\]

We will often write \( \beta(\tau, \pi) \) or simply \( \beta \) instead of \( \beta_{G,M}(\tau, \pi) \).

**Theorem 0.261.** \( \beta \) is an isomorphism. In other words, \( \tau_M \) is the right adjoint functor to \( i_M \).

**Proof.** We assume that the result is known for all proper Levi subgroups of \( G \).

**Lemma 0.262.** (1) Let

\[
\tau'' \to \tau' \to \tau \to \{0\}
\]

be an exact sequence in \( M(M) \) such that the morphisms \( \beta(\tau'', \pi) \) and \( \beta(\tau', \pi) \) are isomorphisms. Then \( \beta(\tau, \pi) \) is also an isomorphism.
(2) Let  
\[ \{0\} \to \pi \to \pi' \to \pi'' \]
be an exact sequence in \( \mathcal{M}(G) \) such that the morphisms \( \beta(\tau, \pi'') \) and 
\( \beta(\tau, \pi') \) are isomorphisms. Then \( \beta(\tau, \pi) \) is also an isomorphism.

**Proof.** The result follows from the exactness of functors \( r_M \) and \( i_M \) and the five-homomorphisms lemma. \( \square \)

**Corollary 0.263.** It is sufficient to check the validity of the Theorem in the case when \( \tau \) is a projective object of \( \mathcal{M}(M) \) and \( \pi \) is an injective object of \( \mathcal{M}(G) \).

**Proposition 0.264.** \( \beta(\tau, \pi) \) is an isomorphism if \( \tau \) is projective and \( \pi \) is equal to \( i_N(\rho) \) where \( N < G \) is a proper Levi subgroup of \( G \) and \( \rho \) is an injective object of \( \mathcal{M}(N) \).

**Proof.** As follows from the Frobenious reciprocity we have equalities 
\[
\text{Hom}_G(i_M(\tau), \pi) = \text{Hom}_G(i_M(\tau), i_N(\rho)) = \text{Hom}_N(r_N(i_M(\tau)), \rho)
\]
By the Basic Geometric Lemma the functor \( \text{Hom}_N(r_N(i_M(\tau))) \) admits a 
increasing filtration parametrized by double cosets \( P\setminus G/Q \) and the subquotients are isomorphic to \( \tilde{\Psi} \) where 
\[
i_{N\cap w(M), N\cap w(U)} \circ \text{Ad}(w) \circ r_{M\cap w^{-1}(N), M\cap w^{-1}(V)}
\]
Since the object \( \rho \) is injective we see that the space \( \text{Hom}_N(r_N(i_M(\tau)), \rho) \) admits an increasing filtration parametrized by double cosets \( P\setminus G/Q \) and 
the subquotients are isomorphic to 
\[
\text{Hom}_N(i_{N\cap w(M), N\cap w(U)} \circ \text{Ad}(w) \circ r_{M\cap w^{-1}(N), M\cap w^{-1}(V)}(\tau), w) \in W^{M,N}
\]
On the other hand the functor \( \tilde{r}_M \circ i_N \) admits a decreasing filtration 
parametrized by double cosets \( Q\setminus G/P \). By Problem 0.258 this set coincides with the set \( \{w^{-1}w_G\}, w \in W^{M,N} \) and we can consider this filtration as an 
increasing filtration parametrized by the set \( P\setminus G/Q \) and the corresponding 
subquotients are isomorphic to \( \tilde{\Psi}' \) where 
\[
\tilde{\Psi}'(\tau) = i_{M\cap w^{-1}(N), M\cap w^{-1}(V)} \circ \text{Ad}(w^{-1}) \circ r_{N\cap w(M), N\cap w(U)}(\tau)) \in W^{M,N}
\]
Since \( \tau \) is a projective object of \( \mathcal{M}(M) \) we see that the space \( \text{Hom}_N(\tau, \tilde{r}_M \circ i_N(\rho)) \) admits an increasing filtration parametrized by double cosets \( P\setminus G/Q \) and 
the subquotients are isomorphic to 
\[
\text{Hom}_M(\tau, i_{M\cap w^{-1}(N), M\cap w^{-1}(V)} \circ \text{Ad}(w^{-1}) \circ r_{N\cap w(M), N\cap w(U)}(\rho)), w) \in W^{M,N}
\]
**Problem 0.265.** (1) The functor \( \beta(\tau, i_{N,V}(\rho)) \) is compatible with 
the increasing filtrations on \( \text{Hom}_G(i_M(\tau), \pi) \) and \( \text{Hom}_M(\tau, r_M(\pi)), w \in W^{M,N} \).
(2) For all \( w \in W^{M,N} \) the induced map from
\[
\text{Hom}_N(i_{N \cap w(M), N \cap w(U)} \circ Ad(w) \circ r_{M \cap w^{-1}(N), M \cap w^{-1}(V)}(\tau, \rho)
\]
to
\[
\text{Hom}_M(\tau, i_{M \cap w^{-1}(N), M \cap w^{-1}(V)} \circ Ad(w^{-1}) \circ r_{N \cap w(M), N \cap w(U)}(\rho))
\]
is given by \( \beta_{N, N \cap w(M)}(Ad(w) \circ r_{M \cap w^{-1}(N), M \cap w^{-1}(V)}(\tau, \rho)) \).

Now Proposition follows from the inductive assumptions.

Corollary 0.266. \( \beta(\tau, \pi) \) is an isomorphism if \( \pi \) is equal to \( i_N(\rho) \), \( N \neq G \) is a proper Levi subgroup of \( G \), \( \rho \in M(M) \).

Proof. The part (1) follows arguments used in the proof of Lemma 0.262.

Since \( \pi \in M^+_c(G) \) the map
\[
\oplus_{\{M < G, M \neq G\} \cap M} i_M \circ r_{M,U}(\pi)
\]
is an imbedding. Applying this argument once more we find an exact sequence
\[
\{0\} \rightarrow \pi \rightarrow \pi' \rightarrow \pi''
\]
such that \( \pi' \) and \( \pi'' \) are direct sums of representations of the form \( i_{N,V}(\rho) \) where \( Q = NV \) is a proper parabolic subgroup of \( G \), \( \rho \in M(M) \).

The Claim follows from Lemma 0.262.

Now we can finish the proof of the Theorem. As follows from the last Corollary and the decomposition theorem saying that \( M(G) = M_c(G) \oplus M^+_c(G) \) it is sufficient to prove the theorem in the case when \( \pi \) is quasi-cuspidal. But it is clear that in this case both sides are equal to \( \{0\} \).

Corollary 0.267. The functor \( i_M : M(G) \rightarrow M(M) \) maps projective objects into projective.

11.3. The Bernstein’s morphism. Let \( P = MU \subset G \) be a standard parabolic, \( \bar{P} = M \bar{U} \) be the opposite parabolic, \( H \subset P \times \bar{P} \) be the preimage of the diagonal under the projection \( P \times \bar{P} \rightarrow P/U \times P/\bar{U} = M \times M \) and \( X_M := H \setminus G \times G \). We use the second adjointness to construct a \( G \times G \)-equivariant morphism \( B_M : S(X_M) \rightarrow S(G) \) introduced by J. Bernstein.

For any \( V \in M(G \times G) \) we write
\[
\bar{r}_M \times r_M(V) := r_{M \times M, U \times U}(V) \in M(M \times M).
\]
Let \( Y := P\bar{P} \subset G \). Then \( Y \) is an open \( \bar{P} \times P \) subset of \( G \).

Problem 0.268. We can identify the \( M \times M \) representation \( \bar{r}_M \times r_M(S(Y)) \) with of the regular representations of \( M \times M \) on \( S(M) \).

We denote by \( j_M \) the composition \( j_M : S(M) \rightarrow \bar{r}_M \times r_M(S(Y)) \rightarrow \bar{r}_M \times r_M(S(G)). \)
**Definition 0.269.** (1) By the construction

\[ S(X_M) = \text{ind}_H^{G \times G} \mathbb{C} = \text{ind}_{P \times \bar{P}}^{G \times G} S(M) \]

where the action of \( P \times \bar{P} \) on \( S(M) \) is the composition of the regular representations of \( M \times M \) on \( S(M) \) and the projection \( P \times \bar{P} \to M \times M \). As follows from Theorem 0.261 we have an isomorphism

\[ \text{Hom}_{G \times G}(S(X_M), S(G)) = \text{Hom}_{M \times M}(S(M), \check{r}_M \times r_M(S(G))). \]

Therefore the canonical imbedding \( j_M \) defines a \( G \times G \)-equivariant morphism \( B_M : S(X_M) \to S(G) \).

(2) Let \( \mathbb{C}(G), \mathbb{C}(X_M) \) be the space of \( \mathbb{C} \)-valued functions on \( G \) and \( X(M) \) which are two-sided invariant under some open compact subgroup of \( G \). Then \( \mathbb{C}(G) = \widehat{S(G)} \), \( \mathbb{C}(X_M) = \widehat{S(X_M)} \) and we denote by \( B_M : \mathbb{C}(G) \to \mathbb{C}(X_M) \) the morphism dual to \( B_M \).

(3) Let \( \mathbb{C}(G), \mathbb{C}(X_M) \) be the space of \( \mathbb{C} \)-valued functions on \( G \) and \( X(M) \) which are two-sided invariant under some open compact subgroup of \( G \). Then \( \mathbb{C}(G) = \widehat{S(G)} \), \( \mathbb{C}(X_M) = \widehat{S(X_M)} \) and we denote by \( B_M : \mathbb{C}(G) \to \mathbb{C}(X_M) \) the morphism dual to \( B_M \).

(4) For any pair \( W_1, W_2 \) of smooth representations of \( M \) denote by \( \mathcal{P}(W_1, W_2) \) the space of \( M \)-invariant \( \mathbb{C} \)-valued bilinear forms on \( W_1 \times W_2 \).

(5) We denote by \( \kappa : \text{Hom}_{G \times G}(V_1 \otimes V_2, \mathbb{C}(X_M)) \to \mathcal{P}(r_M(V_1), r_M(V_2)) \) given by

\[ \kappa(\phi)(J(v_1), \check{J}(v_2)) := \phi(v_1 \otimes v_2)(e) \]

where \( J : V_1 \to r_M(V_1), \check{J} : V_2 \to \check{r}_M(V_2) \) are the canonical projections.

**Problem 0.270.** (1) Show that the bilinear form \( \kappa(\phi)(w_1, w_2), w_1 \in r_M(V_1), w_2 \in \check{r}_M(V_2) \) is well defined. [That is the number \( \phi(v_1 \otimes v_2)(e) \) does not depend on a choice \( v_1 \in V_1, v_2 \in V_2 \) such that \( w_1 = J(v_1), w_2 = \check{J}(v_2) \).]

(2) The map

\[ \text{Hom}_{G \times G}(V_1 \otimes V_2, \mathbb{C}(X_M)) \to \mathcal{P}(r_M(V_1), r_M(V_2)), \phi \to \kappa(\phi) \]

is a bijection.

**Definition 0.271.** Let \( V_1, V_2 \) be smooth representations of \( G \) and \( <, > \) \( V_1 \times V_2 \to \mathbb{C} \) be a \( G \)-invariant pairing. We denote by \( <, >_M : r_M(V_1) \times \check{r}_M(V_2) \to \mathbb{C} \) the \( M \)-invariant bilinear form as in the last problem.

Let \( M < G \) be a standard Levi subgroup, \( (\sigma, V) \) be a smooth representation of \( G \) and \( <, > : V \times \check{V} \to \mathbb{C} \) be the canonical \( G \)-invariant pairing. We denote by \( J : V \to r_M(V), \check{J} : \check{V} \to \check{r}_M(V) \) the canonical projections.
As follows from the Corollary the canonical pairing \( \langle \cdot, \cdot \rangle : V \times \bar{V} \to \mathbb{C} \) induces a pairing \( \langle \cdot, \cdot \rangle_M : r_M(V) \times \bar{r}_M(\bar{V}) \to \mathbb{C} \) and therefore a morphism \( \kappa_{M,V} : \bar{r}_M(\bar{V}) \to r_M(V) \).

**Lemma 0.272.** The \( \kappa_{M,V} \) is an isomorphism.

**Proof.** Consider the morphism \( a : \bar{r}_M(\bar{V}) \to \bar{r}_M(\bar{V}) \) which is the image of the identity \( \text{Id}_{r_M(V)} \) under the composition

\[
\text{Hom}_M(r_M(V), r_M(V)) \to \text{Hom}_G(V, i_M \circ r_M(V)) \to \text{Hom}_G(i_M(\bar{r}_M(V)), V) = \text{Hom}_G(i_M(\bar{r}_M(V)), \bar{V}) \to \text{Hom}_M(\bar{r}_M(V), \bar{r}_M(\bar{V}))
\]

where the first map comes from the Frobenious reciprocity, the second is the duality, the third comes from the inverse of the natural morphism \( i_M(r_M(V)) \to i_M \circ r_M(V) \) and the last map is equal to \( \beta(r_M(V), V) \). It is easy to check that \( a \) is the inverse of \( \kappa_{M,V} \).

**Proposition 0.273.** For any admissible representation \((\sigma, V)\) of \( G \) the \( M \)-invariant pairings \( \langle \cdot, \cdot \rangle_M \) and \( g \langle \cdot, \cdot \rangle_M \) [see Lemma ??] between \( r_M(V) \) and \( \bar{r}_M(\bar{V}) \) coincide.

**Proof.** Let \( b(V) : \bar{r}_M(\bar{V}) \to \bar{r}_M(\bar{V}) \) be the isomorphism coming from the pairing \( \langle \cdot, \cdot \rangle_M \) and \( \bar{\beta} : \text{Hom}_G(i_M(\bar{r}_M(V)), \bar{V}) \to \text{Hom}_M(\bar{r}_M(V), \bar{r}_M(\bar{V})) \) be the map defined as the composition of the isomorphism \( \text{Hom}(i(\tau), \bar{\sigma}) = \text{Hom}(\tau, r(\bar{\sigma})) \) and the morphism \( b(V) \). It is sufficient to show that \( \beta_{G,M} = \beta_{G,M} \). By the definition

\[
\beta_{G,M} = \beta_{G,M}.
\]