Problem 0.1. *Let $\mathfrak{g} = sl_n, n > 1$. Show that*

a) The set $\mathfrak{h}$ of diagonal matrices

$$\begin{matrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & & \\ 0 & & 0\dots & \lambda_n \end{matrix}$$

with trace zero is a Cartan subalgebra of $\mathfrak{g}$. We will write this matrix as $\{\lambda_i\}$.

b) The set $\alpha_{ab} \in \mathfrak{h}^\vee, \alpha_{ab}(\{\lambda_i\}) := \lambda_a - \lambda_b, 1 \leq a \neq b \leq n$ is the set of roots of $\mathfrak{g}$.

c) An element $t = \{\lambda_i\} \in \mathfrak{h}$ is regular iff $\lambda_a \neq \lambda_b$ for $1 \leq a \neq b \leq n$.

d) For any regular $t = \{\lambda_i\} \in \mathfrak{h}$ there exists unique permutation $\sigma_t \in S_n$ such that $\sigma_t(a) < \sigma_t(b) \leftrightarrow \lambda_a < \lambda_b$.

e) Two regular elements $t, t' \in \mathfrak{h}$ belong to the same Weyl chamber iff $\sigma_t = \sigma_{t'}$.

Let $t_0 = (\frac{-n}{2}, \frac{-n+2}{2}, \dots, \frac{n}{2}) \in \mathfrak{h}$. We denote by C_+ the Weyl chamber corresponding to t_0 and for any $\sigma_t \in S_n$ denote by C_σ the Weyl chamber corresponding to $\sigma(t_0)$.

f) The set $\alpha_{ab}, 1 \leq a < b \leq n$ is the set R^+ of positive roots and $\alpha_a := \alpha_{a,a+1}, 1 \leq a < n$ is the set Σ of simple roots.

g) The Weyl group W is equal to the symmetric group S_n and simple reflections s_{α_a} corresponds to simple transpositions $(a, a+1) \in S_n$.

h)

$$l(\sigma) = |\{(a,b)|1 \leq a < b \leq n, \sigma(a) > \sigma(b)\}|\forall \sigma \in S_n$$

Definition 0.2. *Let G, H be groups and G acts on G,*

$$h \to h^g, h \in H, g \in G$$

by automorphisms [that is we have a morphism $G \to Aut(H)$]. We denote by $G \ltimes H$ the group which as a set is equal to the set $G \times H$ of pairs $(g,h), h \in H, g \in G$ and $(g,h)(g',h') := (gg', h^{g'}h')$

Problem 0.3. *a) Show that the Weyl group of C_n and B_n is isomorphic to the semidirect product $S_n \ltimes \mathbb{Z}_2^n$ and the Weyl group of D_n is isomorphic to the semidirect product $S_n \ltimes \tilde{\mathbb{Z}}_2^n$ where $\tilde{\mathbb{Z}}_2^n \subset \mathbb{Z}_2^n$ is the kernel of the summation morphism $\mathbb{Z}_2^n \to \mathbb{Z}_2$.*

b) Carry out the analysis as in the Problem 1 for classical Lie algebras.

The rest of the homework is from the Kirillov's book

Homeworks : 7.7-7.11