We assume that $char(k) = 0$.

Theorem 0.1. *(Cartan's criterion of solvability). A Lie algebra* $\mathfrak{g}$ *is solvable iff* $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) \equiv 0$ *where* K *is the Killing form.*

Proof. I give a proof only for the case when $k = \mathbb{C}$. The general case is a part of the homework.

a) Suppose that $\mathfrak{g}$ is solvable. By Lie's theorem there is a basis in $\mathfrak{g}$ such that all matrices ad_x are upper-triangular. In this basis matrices ad_y and $ad_x ad_y, x \in \mathfrak{g}, y \in [\mathfrak{g},\mathfrak{g}]$ are strictly upper-triangular. So $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) \equiv 0$.

To prove the implication $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) \equiv 0 \Rightarrow \mathfrak{g}$ is solvable we prove the following result.

Lemma 0.2. *Let* V *be a* $\mathbb{C}$*-vector space,* $\mathfrak{g} \subset End(V)$ *a Lie algebra such that*

$$Tr_V(xy) = 0, \forall x \in [\mathfrak{g},\mathfrak{g}], y \in \mathfrak{g}$$

Then $\mathfrak{g}$ *is solvable.*

Proof. If $s \in End(V)$ is a semisimple element there exists a basis $v_i, 1 \leq i \leq d$ of V such that $sv_i = \lambda_i v_i, \lambda_i \in \mathbb{C}, 1 \leq i \leq d$. We define $\bar{s} \in End(V)$ as a linear operator such that $\bar{s}v_i = \lambda_i v_i, \bar{\lambda}_i \in \mathbb{C}, 1 \leq i \leq d$ where $\bar{\lambda}$ is the complex conjugate of λ.

Let $x \in [\mathfrak{g},\mathfrak{g}]$. Using the Jordan decomposition we can write $x = s+n$ where s is semisimple, n nilpotent and $[s,n] = 0$. When can choose a basis such that s diagonal and n is strictly upper-triangular. So

$$Tr_V(x\bar{s}) = \sum_{i=1}^{d} \lambda_i \bar{\lambda}_i$$

On the hand $x = \sum_k [y_k, z_k], y_k, z_k \in \mathfrak{g}$. So $Tr_V(x\bar{s}) = \sum_k Tr_V([y_k, z_k]\bar{s}) = \sum_k Tr_V(y_k[z_k], \bar{s}])$. As follows from Theorem 5.59 there exists a polynomial $Q[t] \in t\mathbb{C}[t]$ such that $\bar{s} = Q(x)$. So $Tr_V(x\bar{s}) = -\sum_k Tr_V y_k Q(ad_x) z_k$. Since for any $r > 0$ and any $z \in \mathfrak{g}$ we have $ad_x^r(z) \in [\mathfrak{g},\mathfrak{g}]$ we see that $Tr_V(x\bar{s}) = 0$. Since on the other hand $Tr_V(x\bar{s}) = \sum_{i=1}^{d} \lambda_i \bar{\lambda}_i$ we see that $\lambda_i = 0, \forall i, 1 \leq i \leq d$. Therefore every $x \in [\mathfrak{g},\mathfrak{g}]$ is nilpotent and it follows from the Engel's theorem that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. So $\mathfrak{g}$ is solvable. □

Now is is easy to prove Theorem 1. If $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) \equiv 0$ then it follows from Lemma 2 that $ad(\mathfrak{g}) \subset End(\mathfrak{g})$ is solvable. But $ad(\mathfrak{g}) = \mathfrak{g}/Z(\mathfrak{g})$. So $\mathfrak{g}$ is solvable. □

Problem 0.3. *Show that for any Lie algebra* $\mathfrak{g}$ *and an ideal* $J \subset \mathfrak{g}$ *we have*

$$Tr_{\mathfrak{g}} ad(x)ad(y) = Tr_J ad(x)ad(y), \forall x, y \in J$$

Lemma 0.4. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra,* $I \subset \mathfrak{g}$ *an ideal and* $I^\vee := \{x \in \mathfrak{g} | K(x, y) = 0, \forall y \in I\}$*. Then* $I^\vee$ *is an ideal and* $\mathfrak{g} = I \oplus I^\vee$*.*

Proof. $I^\vee$ is an ideal since K is an invariant form [Lemma 5.45]. Therefore $J : I \cap I^\vee$ is also an ideal and

$$Tr_{\mathfrak{g}} ad(x)ad(y) = Tr_J ad(x)ad(y), \forall x, y \in J$$

Since

$$K(x, y) = 0, \forall x \in I^\vee, y \in I$$

we see that

$$Tr_J ad(x)ad(y) = 0, \forall x, y \in J$$

It follows then from the Cartan's criterion that J is solvable. Since $\mathfrak{g}$ is semisimple we see that $J = \{0\}$. ☐

Problem 0.5. *Prove the following*

Corollary. *a) Any ideal in a semisimple Lie algebra is a semisimple Lie algebra.*

b) If $\mathfrak{g}$ *is semisimple then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$*.*

Definition 0.6. *Let* $\rho : \mathfrak{g} \to End(V)$ *be a representation of a Lie algebra such that the form* $B_\rho(x, y) := Tr_V(\rho(x)\rho(y))$ *on* $\mathfrak{g}$ *is non-degenerate. Choose a basis* $e_i \in \mathfrak{g}, 1 \leq i \leq d$*, denote by* $f_i, 1 \leq i \leq d$ *the dual basis in respect to the form* B_ρ *and define* $\Delta_\rho := \sum_{i=1}^d e_i f_i \in U(\mathfrak{g})$*. We say that* Δ_ρ *is the Casimir element corresponding to* B_ρ*.*

Problem 0.7. Δ_ρ *is well define [that is* Δ_ρ *does not depend on a choice of a basis* $e_i \in \mathfrak{g}$*] and* $\Delta_\rho \in Z(U(\mathfrak{g}))$*.*

Theorem 0.8. *Any finite-dimensional representation of a semi-simple algebra* $\mathfrak{g}$ *is completely reducible.*

Proof. As we have seen in Lecture 4 it is sufficient to show that any exact sequence

$$\{0\} \to V \to V' \to k \to \{0\}$$

$\mathfrak{g}$-representation such that $\rho : \mathfrak{g} \to End(V)$ is irreducible splits.

a) Consider first the case when the map $\rho : \mathfrak{g} \to End(V)$ is an imbedding. In this case the bilinear form $B_\rho(x, y) := Tr_V(\rho(x)\rho(y))$ on $\mathfrak{g}$ is non-degenerate. Really let $I := \{x \in \mathfrak{g} | B_\rho(x, y) = 0, \forall y \in \mathfrak{g}\}$. I is an ideal since the form B_ρ is invariant. By the Cartan's criterion, I is solvable. Since $\mathfrak{g}$ is semi-simple we see that $I = \{0\}$.

Since the representation $\rho : \mathfrak{g} \to End(V)$ is irreducible and $\Delta_\rho \in Z(U(\mathfrak{g}))$ we see that $\rho(\Delta_\rho) = cId_V$. I claim that $c = dim(\mathfrak{g})/dim(V)$. To prove it is sufficient to show that $Tr_V(\rho(\Delta_\rho)) = d := dim(\mathfrak{g})$. But

$$Tr_V(\rho(\Delta_\rho)) = \sum_{i=1}^{d} Tr_V(e_i f_i) = \sum_{i=1}^{d} B_\rho(e_i, f_i) = \sum_{i=1}^{d} 1 = d$$

Now the we can find the splitting of the exact sequence

$$\{0\} \to V \to V' \to k \to \{0\}$$

as in the proof of the part a") of Lemma 7 in the Lecture 4.

b) Consider now the general case. Let $I := Ker\rho$. By corollary to lemma 4 we know that I is semisimple and therefore $[I, I] = I$. So the same arguments as in the case of the part a') of Lemma 7 in the Lecture 4 show that I acts trivially on V'. So we can consider $\rho' : \mathfrak{g} \to End(V')$ as a representation of a Lie algebra $\mathfrak{g}/I$. Since the Lie algebra $\mathfrak{g}/I$ is semisimple and $\rho : \mathfrak{g}/I \to End(V)$ is an imbedding we know from a) that the exact sequence

$$\{0\} \to V \to V' \to k \to \{0\}$$

splits. □

Definition 0.9. *a) Let* Σ_n *the the symmetric group. For any* $\sigma \in \Sigma_n$ *we define*

$$l(\sigma) := |\{(i < j), 1 \leq i < j \leq n | \sigma(i) > \sigma(j)\}|, \epsilon(\sigma) := (-1)^{l(\sigma)}$$

b) If V *is a vector space we define the action of* Σ_n *on* $V^{\otimes n}$ *by*

$$\sigma(v_1 \otimes ... \otimes v_n) = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}, \sigma \in \Sigma_n, v_i \in V, 1 \leq i \leq n$$

c) We define

$$S^n(V) = \{t \in V^{\otimes n} | \sigma(t) = t, \sigma \in \Sigma_n\}$$

$$\Lambda^n(V) = \{t \in V^{\otimes n} | \sigma(t) = \epsilon(\sigma)t, \sigma \in \Sigma_n\}$$

Problem 0.10. *a) Show that*

a) $\epsilon(\sigma\sigma') = \epsilon(\sigma)\epsilon(\sigma'), \forall \sigma, \sigma' \in \Sigma_n$.

b) $S^n(V), \Lambda^n(V)$ *are* $gl(V)$*-invariant subspaces of* $V^{\otimes n}$.

Definition 0.11. *If* $\mathfrak{g}$ *is a Lie algebra,* $\rho : \mathfrak{g} \to gl(V)$ *a representation we denote by* $\rho^{\otimes n}$ *the representation of* $\mathfrak{g}$ *on* $V^{\otimes n}$*, by* $S^n(\rho)$ *the restriction of* $\rho^{\otimes n}$ *to the subspace* $S^n(V) \subset V^{\otimes n}$ *of symmetric tensors and by* $\Lambda^n(\rho)$ *the restriction of* $\rho^{\otimes n}$ *to the subspace* $\Lambda^n(V) \subset V^{\otimes n}$ *[of anti-symmetric tensors].*

Problem 0.12. *a) Show that*

a) Let $\rho_n : sl_2(k) \to gl(V_n), n \geq 0$ *be the irreducible* $n+1$*-dimensional representation. Then* $V_n \otimes V_1 \sim V_{n+1} \oplus V_{n-1}$*. [That is the tensor product* $V_n \otimes V_1$ *is equivalent to direct sum* $V_{n+1} \oplus V_{n-1}$*.]*

b) For any $n \geq 0$ *there exists unique [up to a multiplication by a non-zero scalar]* $sl_2(k)$*-invariant non-zero bilinear form* Q_n *on* V_n*. It is non-degenerate and symmetric for even* n *and anti-symmetric for odd* n*.*

c) The representation $S^n(\rho_1)$ *of* $sl_2(k)$ *is equivalent to the representation* ρ_n*.*

d) For any nilpotent matrix $X \in gl_n(k)$ *there exist matrices* $H, Y \in gl_n(k)$ *such that*

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$$

(in other words, there exists a representation $\rho : sl_2(k) \to gl_n(k)$ *such that* $\rho(e) = X$*).*

A hint. *Bring* X *to the Jordan normal form.*

e) Let $H', Y' \in gl_n(k)$ *be another pair of elements such that*

$$[H', X] = 2X, [H', Y'] = -2Y', [X, Y'] = H'$$

Then there exists an invertible $n \times n$*-matrix* A *such that*

$$A^{-1}XA = X, A^{-1}YA = Y', A^{-1}HA = H'$$

f) Let $\mathfrak{b} \subset sl_2(k)$ *be the subalgebra of upper-triangular matrices,*$\rho : sl_2(k) \to gl(V)$ *a finite-dimensional representation and* $v \in V$ *is such that* $\rho(x)v = 0$ *for all* $x \in \mathfrak{b}$*. Show that* $\rho(x)v = 0$ *for all* $x \in \mathfrak{g}$*.*

$g^\star$*) find* $c_i, i \geq 0$ *such that* $V_n \otimes V_m \sim \oplus c_i \rho_i$*.*

A hint. Use the classification of representations of $sl_2(k)$

Problem 0.13. *Let* $sl_r(k) \subset gl(k^r), r > 1$ *be the Lie algebra of* $r \times r$*-matrices* A *such that* $tr(A) = 0$*. We denote by* $\rho_{st} : sl_r(k) \to gl(k^r)$ *the representation* $\rho_{st}(A) := A, A \in sl_r(k)$*.Show that*

a) $sl_r(k)$ *is a simple Lie algebra.*

b) the representations $S^n(\rho_{st})$ *of the Lie algebra* $sl_r(k)$ *are irreducible for all* $n \geq 0$*.*

c) the representations $\Lambda^i(\rho_{st}), 1 \leq i \leq r$ *of the Lie algebra* $sl_r(k)$ *are irreducible and that the representation* $\Lambda^r(\rho_{st})$ *is equal to the trivial representation* ρ_{tr}*.*

d) the representation $\Lambda^{r-i}(\rho_{st})$ *is equivalent to the representation* $\Lambda^i(\rho_{st})^\vee$ *dual to* $\Lambda^i(\rho_{st})$.