We assume that $\text{char}(k) = 0$.

**Theorem 0.1.** (Cartan’s criterion of solvability). A Lie algebra $\mathfrak{g}$ is solvable iff $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) \equiv 0$ where $K$ is the Killing form.

**Proof.** I give a proof only for the case when $k = \mathbb{C}$. The general case is a part of the homework.

a) Suppose that $\mathfrak{g}$ is solvable. By Lie’s theorem there is a basis in $\mathfrak{g}$ such that all matrices $\text{ad}_x$ are upper-triangular. In this basis matrices $\text{ad}_y$ and $\text{ad}_x \text{ad}_y$, $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$ are strictly upper-triangular. So $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) \equiv 0$.

To prove the implication $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) \equiv 0 \Rightarrow \mathfrak{g}$ is solvable we prove the following result.

**Lemma 0.2.** Let $V$ be a $\mathbb{C}$-vector space, $\mathfrak{g} \subset \text{End}(V)$ a Lie algebra such that $\text{Tr}_V(xy) = 0$, $\forall x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$

Then $\mathfrak{g}$ is solvable.

**Proof.** If $s \in \text{End}(V)$ is a semisimple element there exists a basis $v_i, 1 \leq i \leq d$ of $V$ such that $sv_i = \lambda_i v_i, \lambda_i \in \mathbb{C}, 1 \leq i \leq d$. We define $\bar{s} \in \text{End}(V)$ as a linear operator such that $\bar{s} v_i = \lambda_i v_i, \lambda_i \in \mathbb{C}, 1 \leq i \leq d$ where $\bar{\lambda}$ is the complex conjugate of $\lambda$.

Let $x \in [\mathfrak{g}, \mathfrak{g}]$. Using the Jordan decomposition we can write $x = s + n$ where $s$ is semisimple, $n$ nilpotent and $[s, n] = 0$. When can choose a basis such that $s$ diagonal and $n$ is strictly upper-triangular. So

$$\text{Tr}_V(x \bar{s}) = \sum_{i=1}^{d} \lambda_i \bar{\lambda}_i$$

On the hand $x = \sum_k [y_k, z_k], y_k, z_k \in \mathfrak{g}$. So $\text{Tr}_V(x \bar{s}) = \sum_k \text{Tr}_V([y_k, z_k] \bar{s}) = \sum_k \text{Tr}_V(y_k [z_k, \bar{s}])$. As follows from Theorem 5.59 there exists a polynomial $Q[t] \in \mathbb{C}[t]$ such that $\bar{s} = Q(x)$. So $\text{Tr}_V(x \bar{s}) = - \sum_k \text{Tr}_V y_k Q(\text{ad}_x) z_k$. Since for any $r > 0$ and any $z \in \mathfrak{g}$ we have $\text{ad}_z^r(z) \in [\mathfrak{g}, \mathfrak{g}]$ we see that $\text{Tr}_V(x \bar{s}) = 0$. Since on the other hand $\text{Tr}_V(x s) = \sum_{i=1}^{d} \lambda_i \bar{\lambda}_i$ we see that $\lambda_i = 0, \forall i, 1 \leq i \leq d$. Therefore every $x \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent and it follows from the Engel’s theorem that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. So $\mathfrak{g}$ is solvable.

Now is is easy to prove Theorem 1. If $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) \equiv 0$ then it follows from Lemma 2 that $\text{ad}(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$ is solvable. But $\text{ad}(\mathfrak{g}) = \mathfrak{g}/Z(\mathfrak{g})$. So $\mathfrak{g}$ is solvable. □
Problem 0.3. Show that for any Lie algebra $\mathfrak{g}$ and an ideal $J \subset \mathfrak{g}$ we have
\[ \text{Tr}_{\mathfrak{g}} \text{ad}(x) \text{ad}(y) = \text{Tr}_J \text{ad}(x) \text{ad}(y), \forall x, y \in J \]

Lemma 0.4. Let $\mathfrak{g}$ be a semisimple Lie algebra, $I \subset \mathfrak{g}$ an ideal and $I^\vee := \{ x \in \mathfrak{g} \mid K(x, y) = 0, \forall y \in I \}$. Then $I^\vee$ is an ideal and $\mathfrak{g} = I \oplus I^\vee$.

Proof. $I^\vee$ is an ideal since $K$ is an invariant form [Lemma 5.45]. Therefore $J : I \cap I^\vee$ is also an ideal and
\[ \text{Tr}_I \text{ad}(x) \text{ad}(y) = \text{Tr}_J \text{ad}(x) \text{ad}(y), \forall x, y \in J \]
Since
\[ K(x, y) = 0, \forall x \in I^\vee, y \in I \]
we see that
\[ \text{Tr}_J \text{ad}(x) \text{ad}(y) = 0, \forall x, y \in J \]
It follows then from the Cartan’s criterion that $J$ is solvable. Since $\mathfrak{g}$ is semisimple we see that $J = \{0\}$. \qed

Problem 0.5. Prove the following

Corollary. a) Any ideal in a semisimple Lie algebra is a semisimple Lie algebra.

b) If $\mathfrak{g}$ is semisimple then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Definition 0.6. Let $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be a representation of a Lie algebra such that the form $B_\rho(x, y) := \text{Tr}_V(\rho(x)\rho(y))$ on $\mathfrak{g}$ is non-degenerate. Choose a basis $e_i \in \mathfrak{g}, 1 \leq i \leq d$, denote by $f_i, 1 \leq i \leq d$ the dual basis in respect to the form $B_\rho$ and define $\Delta_\rho := \sum_{i=1}^{d} e_i f_i \in U(\mathfrak{g})$. We say that $\Delta_\rho$ is the Casimir element corresponding to $B_\rho$.

Problem 0.7. $\Delta_\rho$ is well define [that is $\Delta_\rho$ does not depend on a choice of a basis $e_i \in \mathfrak{g}$] and $\Delta_\rho \in Z(U(\mathfrak{g}))$.

Theorem 0.8. Any finite-dimensional representation of a semi-simple algebra $\mathfrak{g}$ is completely reducible.

Proof. As we have seen in Lecture 4 it is sufficient to show that any exact sequence
\[ \{0\} \rightarrow V \rightarrow V' \rightarrow k \rightarrow \{0\} \]
$\mathfrak{g}$-representation such that $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ is irreducible splits.

a) Consider first the case when the map $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ is an imbedding. In this case the bilinear form $B_\rho(x, y) := \text{Tr}_V(\rho(x)\rho(y))$ on $\mathfrak{g}$ is non-degenerate. Really let $I := \{ x \in \mathfrak{g} \mid B_\rho(x, y) = 0, \forall y \in \mathfrak{g} \}$. $I$ is an ideal since the form $B_\rho$ is invariant. By the Cartan’s criterion, $I$ is solvable. Since $\mathfrak{g}$ is semi-simple we see that $I = \{0\}$. 

Since the representation $\rho: g \to \text{End}(V)$ is irreducible and $\Delta_\rho \in Z(U(g))$ we see that $\rho(\Delta_\rho) = c I_d$. I claim that $c = \dim(g)/\dim(V)$. To prove it is sufficient to show that $\text{Tr}_V(\rho(\Delta_\rho)) = d := \dim(g)$. But

$$\text{Tr}_V(\rho(\Delta_\rho)) = \sum_{i=1}^{d} \text{Tr}_V(e_i f_i) = \sum_{i=1}^{d} B_\rho(e_i, f_i) = \sum_{i=1}^{d} 1 = d$$

Now the we can find the splitting of the exact sequence

$$\{0\} \to V \to V' \to k \to \{0\}$$

as in the proof of the part a’') of Lemma 7 in the Lecture 4.

b) Consider now the general case. Let $I := \text{Ker}\rho$. By corollary to lemma 4 we know that $I$ is semisimple and therefore $[I, I] = I$. So the same arguments as in the case of the part a’) of Lemma 7 in the Lecture 4 show that $I$ acts trivially on $V'$. So we can consider $\rho' : g \to \text{End}(V')$ as a representation of a Lie algebra $g/I$. Since the Lie algebra $g/I$ is semisimple and $\rho: g/I \to \text{End}(V)$ is an imbedding we know from a) that the exact sequence

$$\{0\} \to V \to V' \to k \to \{0\}$$

splits.

□

Definition 0.9. a) Let $\Sigma_n$ the the symmetric group. For any $\sigma \in \Sigma_n$ we define

$$l(\sigma) := |\{(i < j), 1 \leq i < j \leq n|\sigma(i) > \sigma(j)\}|, \epsilon(\sigma) := (-1)^{l(\sigma)}$$

b) If $V$ is a vector space we define the action of $\Sigma_n$ on $V^\otimes n$ by

$$\sigma(v_1 \otimes ... \otimes v_n) = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}, \sigma \in \Sigma_n, v_i \in V, 1 \leq i \leq n$$

c) We define

$$S^n(V) = \{t \in V^\otimes n|\sigma(t) = t, \sigma \in \Sigma_n\}$$

$$\Lambda^n(V) = \{t \in V^\otimes n|\sigma(t) = \epsilon(\sigma)t, \sigma \in \Sigma_n\}$$

Problem 0.10. a) Show that

a) $\epsilon(\sigma \sigma') = \epsilon(\sigma)\epsilon(\sigma'), \forall \sigma, \sigma' \in \Sigma_n$.

b) $S^n(V), \Lambda^n(V)$ are $\text{gl}(V)$-invariant subspaces of $V^\otimes n$.

Definition 0.11. If $g$ is a Lie algebra, $\rho : g \to \text{gl}(V)$ a representation we denote by $\rho^\otimes n$ the representation of $g$ on $V^\otimes n$, by $S^n(\rho)$ the restriction of $\rho^\otimes n$ to the subspace $S^n(V) \subset V^\otimes n$ of symmetric tensors and by $\Lambda^n(\rho)$ the restriction of $\rho^\otimes n$ to the subspace $\Lambda^n(V) \subset V^\otimes n$ [of anti-symmetric tensors].
Problem 0.12.  
a) Show that

\[ \rho_n : \mathfrak{sl}_2(k) \to \mathfrak{gl}(V_n), n \geq 0 \text{ be the irreducible } n+1\text{-dimensional representation. Then } V_n \otimes V_1 \sim V_{n+1} \oplus V_{n-1}. \text{ [That is the tensor product } V_n \otimes V_1 \text{ is equivalent to direct sum } V_{n+1} \oplus V_{n-1}.] \]

b) For any \( n \geq 0 \) there exists unique [up to a multiplication by a non-zero scalar] \( \mathfrak{sl}_2(k) \)-invariant non-zero bilinear form \( Q_n \) on \( V_n \). It is non-degenerate and symmetric for even \( n \) and anti-symmetric for odd \( n \).

c) The representation \( S^n(\rho_1) \) of \( \mathfrak{sl}_2(k) \) is equivalent to the representation \( \rho_n \).

d) For any nilpotent matrix \( X \in \mathfrak{gl}_n(k) \) there exist matrices \( H, Y \in \mathfrak{gl}_n(k) \) such that

\[ [H, X] = 2X, [H, Y] = -2Y, [X, Y] = H \]

(in other words, there exists a representation \( \rho : \mathfrak{sl}_2(k) \to \mathfrak{gl}_n(k) \) such that \( \rho(e) = X \).

A hint. Bring \( X \) to the Jordan normal form.

e) Let \( H', Y' \in \mathfrak{gl}_n(k) \) be another pair of elements such that

\[ [H', X] = 2X, [H', Y'] = -2Y', [X, Y'] = H' \]

Then there exists an invertible \( n \times n \) matrix \( A \) such that

\[ A^{-1}XA = X, A^{-1}YA = Y', A^{-1}HA = H' \]

f) Let \( b \subset \mathfrak{sl}_2(k) \) be the subalgebra of upper-triangular matrices, \( \rho : \mathfrak{sl}_2(k) \to \mathfrak{gl}(V) \) a finite-dimensional representation and \( v \in V \) is such that \( \rho(x)v = 0 \) for all \( x \in b \). Show that \( \rho(x)v = 0 \) for all \( x \in \mathfrak{g} \).

\( g^* \) find \( c_i, i \geq 0 \) such that \( V_n \otimes V_m \sim \oplus c_i \rho_i \).

A hint. Use the classification of representations of \( \mathfrak{sl}_2(k) \)

Problem 0.13. Let \( \mathfrak{sl}_r(k) \subset \mathfrak{gl}(k^r), r > 1 \) be the Lie algebra of \( r \times r \)-matrices \( A \) such that \( \text{tr}(A) = 0 \). We denote by \( \rho_{st} : \mathfrak{sl}_r(k) \to \mathfrak{gl}(k^r) \) the representation \( \rho_{st}(A) := A, A \in \mathfrak{sl}_r(k) \). Show that

a) \( \mathfrak{sl}_r(k) \) is a simple Lie algebra.

b) the representations \( S^n(\rho_{st}) \) of the Lie algebra \( \mathfrak{sl}_r(k) \) are irreducible for all \( n \geq 0 \).

c) the representations \( \Lambda^i(\rho_{st}), 1 \leq i \leq r \) of the Lie algebra \( \mathfrak{sl}_r(k) \) are irreducible and that the representation \( \Lambda^r(\rho_{st}) \) is equal to the trivial representation \( \rho_{tr} \).
d) the representation $\Lambda^{-i}(\rho_{st})$ is equivalent to the representation $\Lambda^i(\rho_{st})$ dual to $\Lambda^i(\rho_{st})$. 