In this lecture we describe finite-dimensional representations of the Lie algebra $sl_2(k)$ in the case when $\text{char}(k) = 0$ and $k$ is algebraically closed.

The algebra $sl_2(k)$ has a basis $e, f, h$ such that

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f$$

For any $n$ we denote by $V_n$ the $n + 1$-dimensional space with a basis $v_{n-2k}, 0 \leq k \leq n$ and denote by $\rho_n : sl_2(k) \rightarrow \text{End}(V_n)$ the linear map such that

$$\rho_n(h)v_{n-2k} = (n - 2k)v_{n-2k}, \rho_n(f)v_{n-2k} = v_{n-2k-2},$$

$$\rho_n(e)v_{n-2k} = k(n - k + 1)v_{n-2k+2}$$

**Theorem 0.1.** $\rho_n$ is an irreducible representation of the Lie algebra $sl_2(k)$. Moreover any irreducible representation of $sl_2(k)$ is equivalent to $\rho_n$ for some $n \geq 0$.

**Remark 0.2.** $V_0 = k$ and $\rho_0 \equiv 0$

**Proof.** I leave for you to check that $\rho_n$ is a representation of Lie algebra $sl_2(k)$. To prove the irreducibility of $\rho_n$ we have to show that any invariant non-zero subspace $W \subset V$ is equal to $V$. Choose a non-zero $\rho_n(h)$-eigenvector $w \in W$. Since all eigenvalues of the operator $\rho_n(h)$ are distinct we have $w = cv_{\lambda - 2k}$ for some $k, 0 \leq k \leq n$ and some non-zero $c \in k$. But then vectors $\rho_n(f)^aw, \rho_n(e)^bw \in W$ span $V$.

Let $\rho : sl_2(k) \rightarrow \text{End}(V)$ be an irreducible finite-dimensional representation of $sl_2(k)$. We write $E = \rho(e), F = \rho(f), H = \rho(h)$. For any $\lambda \in k$ we define

$$V[\lambda] := \{v \in V | Hv = \lambda\}$$

**Lemma 0.3.** $EV[\lambda] \subset V[\lambda + 2], FV[\lambda] \subset V[\lambda - 2]$

**Proof.** If $v \in V[\lambda]$ we have $HEv = [H, E]v + EHV = 2Esv + E\lambda v = (\lambda + 2)Esv$. Analogous arguments show that $HFv = (\lambda - 2)Fv$. 

Since $V$ is finite-dimensional and $\text{char}(k) = 0$ there exists $\lambda \in k$ such that $V[\lambda] \neq \{0\}$ and $V[\lambda + 2] = \{0\}$. Choose a non-zero vector $v_{\lambda} \in V[\lambda]$. Since $V[\lambda + 2] = \{0\}$ we have $Ev_{\lambda} = \{0\}$. For any $k \geq 0$ define $v_{\lambda-2k} := F^{k}(v_{\lambda})$.

**Lemma 0.4.**

$$Hv_{\lambda-2k} = (\lambda - 2)k, Fv_{\lambda-2k} = v_{\lambda-2k-2}, Ev_{\lambda-2k} = k(\lambda - k + 1)v_{\lambda-2k+2}$$
Proof. The first equality follows from Lemma 2, the second from the definition of vectors $v_{\lambda-2k}$. We prove the equality 

\[(\star_k) \ E v_{\lambda-2k} = k(\lambda - k + 1)v_{\lambda-2k+2}\]

by induction in $k$. By the definition of $v_{\lambda}$ the equality $(\star_0)$ is true. Assume that $E v_{\lambda-2k+2} = (\lambda-k)(k-1)v_{\lambda-2k+4}$ the equality $(\star_{k-1})$ is true. Then we have

\[
E v_{\lambda-2k} = EF v_{\lambda-2k+2} = [E, F] v_{\lambda-2k+2} + FE v_{\lambda-2k+2} = \\
H v_{\lambda-2k+2} + F(\lambda - k)(k - 1)v_{\lambda-2k+4} = \\
(\lambda - 2k + 2)v_{\lambda-2k+2} + (\lambda - k)(k - 1)F v_{\lambda-2k+4} = \\
k[(\lambda - 2k + 2) + (\lambda - k)(k - 1)]v_{\lambda-2k+2} = k(\lambda - k + 1)v_{\lambda-2k+2}
\]

\[\square\]

Since $V$ is finite-dimensional and [by Lemma 2] $F^k v_{\lambda} \in V[\lambda - 2k]$ there exists $n \geq 0$ such that $F^n v_{\lambda} \neq \{0\}$ but $F^{n+1} v_{\lambda} = \{0\}$.

I claim that $\lambda \in \mathbb{Z}_{\geq 0}$. Really if $\lambda \notin \mathbb{Z}_{\geq 0}$ then $\lambda - k + 1)k \neq 0$ for all $k > 0$. From this we deduce by induction in $k$ that $v_{n-2k} \neq 0, \forall k \geq 0$. But is impossible since $\dim V < \infty$.

So we can assume that $\lambda = n \in \mathbb{Z}_{\geq 0}$. I claim that the set

$$\{v_{n-2k}, 0 \leq k \leq n, v_{n-2k} := F^k(v_{\lambda})\}$$

is a basis of $V$. Really it follows from Lemma 3 that vectors $v_{n-2k}, 0 \leq k \leq n$ are linearly independent and that they generate an $sl_2(k)$-invariant subspace of $V$. Since $V$ is irreducible vectors $v_{n-2k}, 0 \leq k \leq n$ span $V$. It is clear now that the representation $\rho$ is equivalent to the representation $\rho_n$.

\[\square\]

Definition 0.5. We define the Casimir element $\Delta \in U(sl_2(k))$ by $\Delta := ef + fe + h^2/2 \in U(sl_2(k))$.

Problem 0.6. Show that

a) The representations $\rho_n$ are equivalent to ones constructed in the end of the lecture 1.

b) $\Delta$ belongs to the center of the algebra $U(sl_2(k))$.

c) $\rho_n(\Delta) = \frac{n(n+2)}{2} Id_{V_n}$

Lemma 0.7. Every exact sequence

$$\{0\} \rightarrow V' \rightarrow V \rightarrow k \rightarrow \{0\}$$

of finite-dimensional representations of $sl_2(k)$ splits [that is $V$ is equivalent to the direct sum $V' \oplus k$].
Proof. a) Consider first the case when the representation $V'$ is irreducible. As follows from Theorem 1 it is sufficient to show that any exact sequence of representations of $sl_2(k)$ of the form

$$\{0\} \to V_n \to V \to k \to \{0\}$$

splits. We consider separately the cases $n = 0$ and $n \neq 0$.

a') Assume that $n = 0$. So we have an exact sequence

$$\{0\} \to V_0 \to V \to k \to \{0\}$$

[I write $V_0$ instead of $k$ to distinguish between the subspace $(V_0)$ and the quotient space $(k)$ of $V$.] Choose a basis $v_1, v_2$ of $V$ such that $v_1 \in V_0$. Then we can write elements of $\text{End}(V)$ as $2 \times 2$ matrices. Let $\rho$ be the representation of $sl_2(k)$ on $V$. Since $\rho_0 \equiv 0$ we obtain

$$\rho(x) = \begin{pmatrix} 0 & a(x) \\ 0 & 0 \end{pmatrix}, x \in sl_2(k)$$

So $\rho([x, y]) = [\rho(x), \rho(y)] = 0, \forall x, y \in sl_2(k)$ and therefore $\rho(e) = \rho(f) = \rho(h) = 0$. So $r \equiv 0$ and any linear section $s : k \to V$ defines a splitting $V = V_0 \oplus k$ representations of $sl_2(k)$.

a’”) If $n \neq 0$ consider the operator $A := \rho(\Delta) \in \text{End}(V)$. It is clear that the subspace $V_n \subset V$ is $A$-invariant and it follows from Problem 5 c) that the restriction of $A$ on $V_n$ is equal to $\frac{n(n+2)}{2}Id_{V_n}$. On the other hand $A$ acts on the quotient space $V/V_n = k$ as $\rho_0(\Delta) = 0$. Since $\frac{n(n+2)}{2} \neq 0$ the space $W := \{v \in V | Av = 0\} \subset V$ is a non-zero and $V = V_n \oplus W$. As follows from Problem 5 b) the subspace $W$ is $sl_2(k)$-invariant. So $V = V_n \oplus k$.

Now we prove Lemma 7 by induction in $\text{dim}(V)$. Consider an exact sequence $\{0\} \to V' \to V \to k \to \{0\}$ of representations of $sl_2(k)$. We know that it splits if the representation of $sl_2(k)$ on $V'$ is irreducible. If $V'$ is reducible choose an irreducible subrepresentation $W \subset V'$ and consider the exact sequence $\{0\} \to V'/W \to V/W \to k \to \{0\}$. By the inductive assumptions it splits and there exist an $sl_2(k)$-equivariant section $s : k \leftarrow V/W$. Let $\tilde{V} \subset V$ be the preimage of $\text{Im}(s)$. We have an exact sequence $\{0\} \to V' \to \tilde{V} \to k \to \{0\}$. Using once more the inductive assumption we see that there exists an $sl_2(k)$-equivariant section $\tilde{s} : k \leftarrow V$.

\[\square\]

**Theorem 0.8.** Any finite-dimensional representation of the Lie algebra $sl_2(k)$ is completely reducible.
Proof. We have to show that for any exact sequence
\[ \{0\} \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow \{0\} \]
of representations of $sl_2(k)$ there exists an $sl_2(k)$-equivariant section
\( \tilde{s} : V'' \rightarrow V \) of the projection $p : V \rightarrow V''$.

Consider the exact sequence
\[ \{0\} \rightarrow V' \otimes V''^\vee \rightarrow V \otimes V''^\vee \rightarrow V'' \otimes V''^\vee \rightarrow \{0\} \]
Let $kId_{V''} \subset V'' \otimes V''^\vee = \text{End}(V'')$ be the subspace of scalar operators
and \( \{0\} \rightarrow V' \otimes V''^\vee \rightarrow W \rightarrow kId_{V''} \rightarrow \{0\} \) be the exact sequence
where $W := q^{-1}(kId_{V''}) \subset V \otimes V''^\vee$ and $q : V \otimes V''^\vee \rightarrow V'' \otimes V''^\vee$ is
the map induced by the projection $p : V \rightarrow V'' = V/V'$. As follows
from Lemma 7 there exists an $sl_2(k)$-equivariant section $s : kId_{V''} \hookrightarrow V \otimes V''^\vee$ of the projection $q$. Let $s' := s \otimes Id_{V''} : V'' \hookrightarrow V \otimes V''^\vee \otimes V''$
and
\[ \tilde{s} := (Id_V \otimes Tr_{V''}) \circ s' : V'' \rightarrow V \]
where
\[ Tr_{V''} : V''^\vee \otimes V' = \text{End}(V'') \rightarrow k \]
is the trace map. I’ll leave for you to prove that $\tilde{s} : V'' \rightarrow V$ is an
$sl_2(k)$-equivariant section of the projection $p : V \rightarrow V''$. □

Problem 0.9. a) Complete the proof of Theorem 8.

b) Extend the proof of Theorems 1 and 8 to the case when $k$ is an
arbitrary field of characteristic zero.

c) Let $\rho : sl_2(k) \rightarrow \text{End}(V)$ be a finite-dimensional representation.
Show that the operators $\rho(e), \rho(f) \in \text{End}(V)$ are nilpotent and there
exists unique representation $\tilde{\rho} : SK(2, k) \rightarrow \text{Aut}(V)$ such that
\[ \tilde{\rho}( \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} ) = \exp(a \rho(e)) \]
and
\[ \tilde{\rho}( \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} ) = \exp(b \rho(f)) \]

From now on we assume that $k$ is an arbitrary closed field of arbitrary
characteristic.

d) Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ a finite-dimensional
irreducible representation and $A \in \text{End}_{\mathfrak{g}}(V)$ \( \text{ that is } A \) is a linear
operator such that $A \rho(x) = \rho(x)A, \forall x \in \mathfrak{g} \]. Show that $A = cId_V$ for
some $c \in k$.

e) Give an example of a reducible 2-dimensional representation $\rho$ of
a Lie algebra $\mathfrak{g}$ such that $\text{End}_{\mathfrak{g}}(V) = kId_V$. 
f) Let $ZU(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Show that for any irreducible representation $\rho : \mathfrak{g} \to \text{End}(V)$ and any $z \in ZU(\mathfrak{g})$ we have $\rho(u) = c_{\rho}(z)\text{Id}, c_{\rho}(z) \in k$.

g) Assume that $\text{char}(k) = p >$. Show that $e^p, f^p, h^p - h \in Z\text{sl}_2(k)$.

h) Show that any irreducible representations of $\text{sl}_2(k)$ such that $\rho(e^p) = \rho(f^p) = \rho(h^p - h) = 0$ is equivalent to a representation $\rho_n$ from the problem 12 in the first lecture for unique $n < p$.

i) Show that any exact sequences $\{0\} \to V_i \to V \to V_j \to \{0\}$ of representations of $\text{sl}_2(k)$ splits if $i + j \neq p - 2$.

j) Construct an example of a non-split exact sequences $\{0\} \to V_i \to V \to V_j \to \{0\}$ for $i + j = p - 2$.

k) The Lie algebra $\text{sl}_2(k)$ is solvable if $\text{char}(k) = 2$. Where does the proof of theorem of Lie fail?


From now on [at least for a while] we will follow the book of A.Kirillov available on www.math.sunysb.edu/~kirillov/liegroups/. We start from the chapter 5.3 [page 75]. A couple of remarks. When written

a) complex read an algebraically closed field $\bar{k}$ of characteristic zero.

b) real read a field $k$ of characteristic zero such that $\bar{k}$ is the closure of $k$.

c) complexification of a $k$-Lie algebra $\mathfrak{g}$ read the $\bar{k}$-Lie algebra $\mathfrak{g}_k \otimes \bar{k}$.

An explanation [the proof of Proposition 5.31]. One should say

We claim that $W$ is stable under the action of any $h \in \mathfrak{g}'$;

$\lambda(h)v^k = \sum_{l<k} a_{kl}(h)v^l$ and moreover if $\lambda(\text{ad}_i h) = 0, \forall j > 0$ then $hv^k = \lambda(h)v^k, \forall k \geq 0$.

The proof is by induction in $k$. 