We say that a subspace $W$ is also graded.

Definition 0.1.  
\begin{enumerate}
  \item A grading on a vector space $V$ is a choice of subspaces $V^n \subset V, 0 \leq n < \infty$ such that $V = \bigoplus_{n=0}^{\infty} V^n$. In this case we say that elements of $V^n$ are homogeneous elements of degree $n$. If $v \in V$ is homogeneous denote by $|v| \in \mathbb{N}$ the number such that $v \in V^{|v|}$. We say that a subspace $W \subset V$ is graded if $W = \bigoplus_{n=0}^{\infty} W^n$ where $W^n := W \cap V^n$. In this case the quotient space $V/W = \bigoplus_{n=0}^{\infty} V^n/W^n$ is also graded.
  
  \item For a graded vector space $V = \bigoplus_{n=0}^{\infty} V^n$ we denote by $\hat{V}$ the completion $\hat{V} = \prod_{n=0}^{\infty} V^n$. In other words elements of $\hat{V}$ are sequences $\hat{v} = (v_0, v_1, ..., v_n, ...), v_n \in V^n$.
  
  \item We consider the topology on $\hat{V}$ such for any $\hat{v} \in \hat{V}$ the sets $U_r(\hat{v}) := \hat{v} + \prod_{n=r}^{\infty} V^n$ constitute the fundamental set of open neighborhoods of $\hat{v}$. It is easy to see that $V$ is dense in $\hat{V}$ and that the operations of the addition and the scalar multiplication on $V$ extend to a continuous operations on $\hat{V}$. [Please check]
  
  \item We say that a Lie algebra $\mathfrak{g}$ is graded if $\mathfrak{g}$ is graded as a vector space [ so $\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{g}^n$, $\mathfrak{g}_0 = \{0\}$ and $[\mathfrak{g}^n, \mathfrak{g}^m] \subset \mathfrak{g}^{n+m}, \forall m, n > 0$. It is easy to see that for any graded Lie algebra $\mathfrak{g}$ the operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ extends to a continuous operation $[\cdot, \cdot] : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ which defines a Lie algebra structure on $\hat{\mathfrak{g}}$.
  
  \item We say that a algebra $A$ is graded if $A$ is graded as a vector space in such a way that $A^n A^n \subset A^{m+n}, \forall m, n > 0$. We say that an ideal $I \subset A$ if $I$ is graded subspace of $A$. In this case the quotient algebra $A/I$ is also graded. [Please check]
  
  \item If $V = \bigoplus_{n=0}^{\infty} V^n$ is a graded vector space we define a grading on $T(V)$ in such a way for any homogeneous elements $v_1, ..., v_r \in V$ the tensor product $v_1 \otimes ... \otimes v_r \in T(V)$ is homogeneous and $|v_1 \otimes ... \otimes v_r| = |v_1| + ... + |v_r|$. [Please check that $T(V)$ has a structure of a graded algebra].
\end{enumerate}

Problem 0.2.  
\begin{enumerate}
  \item Let $V = \bigoplus_{n=0}^{\infty} V^n$ be a graded vector space, $W = \bigoplus_{n=0}^{\infty} W^n$ a graded subspace. Show that $V/W = \bigoplus_{n=0}^{\infty} V^n/W^n$ [ so $V/W$ is a graded vector space] and that for any homogeneous $v \in V$ we have $|\bar{v}| = |v|$ where $\bar{v}$ is the image of $v$ in $V/W$. 
\end{enumerate}
b) Let $A$ be a graded algebra. Show that the multiplication $m : A \times A \to A, (a, b) \to ab$ extends to a continuous operation $m : \hat{A} \times \hat{A} \to \hat{A}$ which defines an algebra structure on $\hat{A}$.

c) Let $g = \bigoplus_{n=1}^{\infty} g^n$ be a graded Lie algebra. Show that the kernel $I$ of the natural homomorphism $T(g) \to U(g)$ is a graded ideal of $T(g)$ and that the imbedding $i : g \hookrightarrow U(g)$ preserves the grading. Moreover the imbedding $i : \hat{g} \hookrightarrow \hat{U}(g)$ extends to a continuous imbedding $\hat{i} : \hat{g} \hookrightarrow \hat{U}(g)$.

d) For any homogeneous $u', u'' \in U(g)$ the product $u' \otimes u'' \in U(g) \otimes U(g)$ is homogeneous and $|u' \otimes u''| = |u'| + |u''|$. 

e) Let $A$ be a graded algebra, $M \subset \hat{A}$ the set of elements of the form $\bar{u} = (0, u_1, ..., u_n, ..)$, $u_n \in A^n$. Show that the maps

$$
\exp : M \to 1 + M, \exp(x) := \sum_{n=0}^{\infty} x^n/n!
$$

$$
\ln : 1 + M \to M, \ln(1 + x) := \sum_{n=1}^{\infty} (-1)^{n-1} x^n/n
$$

are well defined and $\exp \circ \ln = Id_{1+M}, \ln \circ \exp = Id_M$.

f) Define an algebra homomorphism $U(\hat{g}) \to \hat{U}(g)$ and check whether it is always an isomorphism.

Example Let $V = k[t], V^n = kt^n$. In this case $\hat{V}$ is the ring $k[[t]]$ of Taylor power series.

Let $g = \bigoplus_{n=1}^{\infty} g^n$ be a graded Lie algebra, $U(g)^n \subset U(g)$ the grading as in Problems 2 c). The isomorphism $U(g) \otimes U(g) = U(g \oplus g)$ provides the definition of a grading $U(g \oplus g)^n \subset U(g \oplus g)$.

Lemma 0.3. $\Delta(U(g)^n) \subset U(g \oplus g)^n$

Proof. Since the graded algebra $U(g)$ is generated as a graded algebra by the graded subspace $g \subset U(g)$ it is sufficient to check that $|\Delta(x)| = |x|$ for any homogeneous element $x \in g$. But this follows from the part d) of the previous problem and the equality $\Delta(x) = x \otimes 1 + 1 \otimes x$. \qed

Let $g = \bigoplus_{n=1}^{\infty} g^n$ be a graded Lie algebra. As follows from the Lemma 3 the diagonal map $\Delta : U(g) \to U(g) \otimes U(g)$ extends to a continuous map $\hat{\Delta} : \hat{U}(g) \to \hat{U}(g) \otimes \hat{U}(g)$.

Definition 0.4. We say that an element $\hat{u} \in \hat{U}(g)$ is primitive if $\Delta(\hat{u}) = \hat{u} \otimes 1 + 1 \otimes \hat{u}$ and we say that $\hat{u}$ is of a group if $\Delta(\hat{u}) = \hat{u} \otimes \hat{u}$. 

Lemma 0.5. The set of primitive elements of $\hat{U}(\mathfrak{g})$ coincides with $\hat{\mathfrak{g}} \subset \hat{U}(\mathfrak{g})$

Proof. Let $\hat{u} = (t_0, t_1, ..., t_n, ...) \in U(\mathfrak{g})^n(X)$ be a primitive element. Since

$$\hat{\Delta}(u_0, u_1, ..., u_n, ...) = (\Delta(u_0), \Delta(u_1), ..., \Delta(u_n), ...)$$

where [by Lemma 3] $\Delta(u_n) \in U(\mathfrak{g})^n(X)$ we have $\Delta(u_n) \in U(\mathfrak{g})$ are primitive. It follows from Theorem 5.4 in Serre that $u_n \in \mathfrak{g}^n(X)$. So $\hat{u} \in \hat{\mathfrak{g}}$. □

Lemma 0.6. The map $\exp : \mathcal{M} \to 1 + \mathcal{M}$ defines a bijection between primitive elements in $\mathcal{M}$ and group elements in $1 + \mathcal{M}$

Proof. a) Let $\hat{u} \in \mathcal{M}$ an element such that $\Delta(\hat{u}) = \hat{u} \otimes 1 + 1 \otimes \hat{u}$. We have to show that $\Delta(\exp(\hat{u})) = \exp(\hat{u}) \otimes \exp(\hat{u})$. Since $\Delta$ is an algebra isomorphism we have to show $\Delta(\exp(\hat{u})) = \exp(\Delta(\hat{u})) = \exp(\hat{u} \otimes 1 + 1 \otimes \hat{u})$.

Since the elements $\hat{u} \otimes 1, 1 \otimes \hat{u} \in \hat{U}(\mathfrak{g})$ commute we have $\exp(\hat{u} \otimes 1 + 1 \otimes \hat{u}) = \exp(\hat{u} \otimes 1) \exp(1 \otimes \hat{u}) = \exp(\hat{u}) \otimes \exp(\hat{u})$

b) Let $\hat{u} \in 1 + \mathcal{M}$ a group element. We have to show that $\ln(\hat{u})$ is primitive. I leave for you to finish the proof. □

Corollary 0.7. (Campbell-Hausdorff) For any $x, y \in \hat{\mathfrak{g}}$ there exists $z \in \hat{\mathfrak{g}}$ such $\exp(x) \exp(y) = \exp(z)$

Proof. By Lemma 6 we $\exp(x), \exp(y)$ are group elements. Since $\Delta$ is an algebra isomorphism we see that $\exp(x) \exp(y)$ is also a group element. So by Lemma 5 there exists $z \in \hat{\mathfrak{g}}$ such $\exp(x) \exp(y) = \exp(z)$. □

To show that there exists a universal formula for $z(x, y)$ we introduce the notion of a free Lie algebra.

Definition 0.8. a) Let $X$ be a set.

a) A free Lie algebra on $X$ is a pair $(i, L(X))$ where $L(X)$ is a Lie algebra and $i : X \to L(X)$ is a map such that for any Lie algebra $\mathfrak{g}$ and any map $j : X \to \mathfrak{g}$ there exists unique Lie algebra homomorphism $f : L(X) \to \mathfrak{g}$ such that $j = f \circ i$.

b) Let $V_X$ be the space with a basis $e_x, x \in X, V_X \hookrightarrow \text{Ass}_X := T(V_X)$ be the tensor algebra and $i : X \to \text{Ass}_X$ be the imbedding $i(x) := e_x, x \in X$. We denote by $\tilde{L}(X) \subset \text{Ass}_X$ the Lie subalgebra generated by $e_x, x \in X$. (That is $\tilde{L}(X) \subset \text{Ass}_X$ is the subspace spanned by the commutators of $e_x, x \in X$.)
Lemma 0.9. \((i, \tilde{L}(X))\) is a free Lie algebra on \(X\).

Proof. We have to show that for any Lie algebra \(\mathfrak{g}\) and a map \(j : X \to \mathfrak{g}\) there exists unique Lie algebra homomorphism \(f : \tilde{L}(V) \to \mathfrak{g}\) such that \(f(e_x) = j(x), \forall x \in X\). Since the Lie algebra \(\tilde{L}(V)\) is generated by \(e_x, x \in X\) the uniqueness of \(f\) is obvious. To prove the existence of \(f\) consider the algebra homomorphism \(\tilde{f} : \text{Ass}_X \to U(\mathfrak{g})\) such that \(\tilde{f}(e_x) = j(x) \in \mathfrak{g} \subset U(\mathfrak{g})\) [see Lemma 4 in the Lecture 2]. It is clear that the restriction \(f\) of \(\tilde{f}\) on \(\tilde{L}(X) \subset \text{Ass}_X\) is a Lie algebra homomorphism and \(f(e_x) = j(x), \forall x \in X\). □

By the definition of \(U(L(X))\) the imbedding \(L(X) \hookrightarrow \text{Ass}_X\) extends uniquely to an algebra homomorphism \(\phi : U(L(X)) \to \text{Ass}_X\). On the other hand the linear map \(V_X \to L(X) \subset U(L(X))\) extends uniquely to an algebra homomorphism \(\psi : \text{Ass}_X \to U(L(X))\).

Lemma 0.10. Homomorphisms \(\phi\) and \(\psi\) provide an isomorphism between associative algebras \(U(L(X))\) and \(\text{Ass}_X\).

Proof. It is sufficient to show that \(\phi \circ \psi = \text{Id}_{\text{Ass}_X}, \psi \circ \phi = \text{Id}_{U(L(X))}\). By the construction \(\phi \circ \psi : \text{Ass}_X \to \text{Ass}_X\) is an algebra homomorphism such that \(\phi \circ \psi(e_x) = e_x, \forall x \in X\). Since the set \(e_x, x \in X\) generates the algebra \(\text{Ass}_X\) we see that \(\phi \circ \psi = \text{Id}_{\text{Ass}_X}\). The analogous arguments show that \(\psi \circ \phi = \text{Id}_{U(L(X))}\) □

We will identify the algebras \(U(L(X))\) and \(\text{Ass}_X\). In particular we consider the diagonal map \(\Delta : U(L(X)) \to U(L(X)) \otimes U(L(X))\) as an algebra homomorphism \(\Delta : \text{Ass}_X \to \text{Ass}_X \otimes \text{Ass}_X\).

Definition 0.11. a) A Lie polynomial in two variables is an element \(P \in L(X), X = (x, y)\). For any Lie algebra, a Lie polynomial \(P\) in two variables \(a, b \in \mathfrak{g}\) we define the evaluation \(P(a, b) \in \mathfrak{g}\) as follows. By the definition of a free Lie algebra \(L(X)\) there exists unique homomorphism \(f_{ab} : L(X) \to \mathfrak{g}\) of graded Lie algebras such that \(f_{ab}(x) = a, f_{ab}(y) = b\). For any Lie polynomial \(P\) in two variables we define \(P(a, b) := f_{ab}(P)\).

b) A formal Lie polynomial in two variables is a element \(P \in \tilde{L}(X), X = (x, y)\). For any graded Lie algebra \(\mathfrak{g}\), a formal Lie polynomial \(P(x, y)\) in two variables \(a, b \in \mathfrak{g}, |a| = |b| = 1\) we can define the evaluation \(P(a, b) \in \mathfrak{g}\) [please give a definition].

Theorem 0.12. There exists a a formal Lie polynomial \(Q(x, y)\) such that for any graded Lie algebra \(\mathfrak{g}\) and any homogeneous elements \(a, b \in \mathfrak{g}\) we have \(\exp(a) \exp(b) = \exp(z), z = Q(a, b)\).
Proof. Follows from Corollary 7

On can write explicitly a formula for $Q(x,y)$. An algorithm for the computation of $Q(x,y)$ is in the end of portion of the Serre’s book which I posted.

Problem 0.13. Show that

a) $Q(x,y) = x + y + 1/2[x,y] + 1/12[x,[x,y]] + 1/12[y,[y,x]] + ...$ where we omit terms of degree bigger then three. [ You can prove this equality without looking in the Serre’s book].

b) $L^1(X)$ is the span of $e_x, x \in X$ and $L^n(X) = [L^1(X), L^{n-1}(X)]$.

c) The center of $L(X)$ is equal to $\{0\}$ if $\text{Card}(X) > 1$.

d) Let $g$ be a graded Lie algebra, $u', u'' \in \hat{U}(g)$ group elements. Then $u'u'' \in \hat{U}(g)$ is also a group element.

e) Let $g$ be the graded Lie algebra with a basis $x, y, z$ such that $[x,y] = z, [x,z] = [z,y] = 0, |x| = |y| = 1, |z| = 2$.

Using the part a) describe the subgroup $G \subset \hat{U}(g)$ of group elements.

f)**. Let $X$ be a finite set, $d = |X|$. Show that $\text{dim}L(X)^n = 1/n \sum_{m|n} \mu(m)d^{n/m}$ where the function $\mu$ is defined as follows

$(n) = 1$ if $n$ is a square-free positive integer with an even number of distinct prime factors.

$(n) = -1$ if $n$ is a square-free positive integer with an odd number of distinct prime factors.

$(n) = 0$ if $n$ is not square-free.