From now on we will always assume that k is a field of characteristic zero.

Definition 0.1. *a) A grading on a vector space V is a choice of subspaces $V^n \subset V, 0 \leq n < \infty$ such that $V = \oplus_{n=0}^{\infty} V^n$. In this case we say that elements of V^n are homogeneous elements of degree n. If $v \in V$ is homogeneous denote by $|v| \in \mathbb{N}$ the number such that $v \in V^{|v|}$. We say that a subspace $W \subset V$ is graded if $W = \oplus_{n=0}^{\infty} W^n$ where $W^n := W \cap V^n$. In this case the quotient space $V/W = \oplus_{n=0}^{\infty} V^n/W^n$ is also graded.*

b) For a graded vector space $V = \oplus_{n=0}^{\infty} V^n$ we denote by $\hat{V}$ the completion $\hat{V} = \prod_{n=0}^{\infty} V^n$. In other words elements of $\hat{V}$ are sequences $\hat{v} = (v_0, v_1, ..., v_n, ...), v_n \in V^n$.

c) We consider the topology on $\hat{V}$ such for any $\hat{v} \in \hat{V}$ the sets $U_r(\hat{v}) := \hat{v} + \prod_{n=r}^{\infty} V^n$ constitute the fundamental set of open neighborhoods of $\hat{v}$. It is easy to see that V is dense in $\hat{V}$ and that the operations of the addition and the scalar multiplication on V extend to a continuous operations on $\hat{V}$.[Please check]

d) We say that a Lie algebra $\mathfrak{g}$ is graded if $\mathfrak{g}$ is graded as a vector space [so $\mathfrak{g} = \oplus_{n=1}^{\infty} \mathfrak{g}^n$], $\mathfrak{g}_0 = \{0\}$ and $[\mathfrak{g}^n, \mathfrak{g}^m] \subset \mathfrak{g}^{m+n}, \forall m, n > 0$. It is easy to see that for any graded Lie algebra $\mathfrak{g}$ the operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ extends to a continuous operation $[,] : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ which defines a Lie algebra structure on $\hat{\mathfrak{g}}$.

e) We say that a algebra A is graded if A is graded as a vector space in such a way that $A^m A^n \subset A^{m+n}, \forall m, n > 0$. We say that an ideal $I \subset A$ if I is graded subspace of A. In this case the quotient algebra A/I is also graded. [Please check]

f) If $V = \oplus_{n=0}^{\infty} V^n$ is a graded vector space we define a grading on $T(V)$ in such a way for any homogeneous elements $v_1, ..., v_r \in V$ the tensor product $v_1 \otimes ... \otimes v_r \in T(V)$is homogeneous and $|v_1 \otimes ... \otimes v_r| = |v_1| + ... + |v_r|$. [Please check that $T(V)$ has a structure of a graded algebra].

Problem 0.2. *a) Let $V = \oplus_{n=0}^{\infty} V^n$ be a graded vector space, $W = \oplus_{n=0}^{\infty} W^n$ a graded subspace. Show that $V/W = \oplus_{n=0}^{\infty} V^n/W^n$ [so V/W is a graded vector space] and that for any homogeneous $v \in V - W$ we have $|\bar{v}| = |v|$ where $\bar{v}$ is the image of v in V/W.*

b) Let A be a graded algebra. Show that the multiplication $m : A \times A \to A, (a,b) \to ab$ extends to a continuous operation $m : \hat{A} \times \hat{A} \to \hat{A}$ which defines an algebra structure on $\hat{A}$.

c) Let $\mathfrak{g} = \oplus_{n=1}^{\infty} \mathfrak{g}^n$ be a graded Lie algebra. Show that the kernel I of the natural homomorphism $T(\mathfrak{g}) \to U(\mathfrak{g})$ is a graded ideal of $T(\mathfrak{g})$ and that the imbedding $i : \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ preserves the grading. Moreover the imbedding $i : \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ extends to a continuous imbedding $\hat{i} : \hat{\mathfrak{g}} \hookrightarrow \hat{U}(\mathfrak{g})$.

d) For any homogeneous $u', u'' \in U(\mathfrak{g})$ the product $u' \otimes u'' \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is homogeneous and $|u' \otimes u''| = |u'| + |u''|$.

e) Let A be a graded algebra, $\mathcal{M} \subset \hat{A}$ the set of elements of the form $\bar{u} = (0, u_1, ..., u_n, ..), u_n \in A^n$. Show that the maps

$$\exp : \mathcal{M} \to 1 + \mathcal{M}, \exp(x) := \sum_{n=0}^{\infty} x^n/n!$$

$$\ln : 1 + \mathcal{M} \to \mathcal{M}, \ln(1+x) := \sum_{n=1}^{\infty} (-1)^{n-1} x^n/n$$

are well defined and $\exp \circ \ln = Id_{1+\mathcal{M}}, \ln \circ \exp = Id_{\mathcal{M}}$.

f) Define an algebra homomorphism $U(\hat{\mathfrak{g}}) \to \hat{U}(\mathfrak{g})$ and check whether it is always an isomorphism.

Example Let $V = k[t], V^n = kt^n$. In this case $\hat{V}$ is the ring $k[[t]]$ of Taylor power series.

Let $\mathfrak{g} = \oplus_{n=1}^{\infty} \mathfrak{g}^n$ be a graded Lie algebra, $U(\mathfrak{g})^n \subset U(\mathfrak{g})$ the grading as in Problems 2 *c*). The isomorphism $U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \oplus \mathfrak{g})$ provides the definition of a grading $U(\mathfrak{g} \oplus \mathfrak{g})^n \subset U(\mathfrak{g} \oplus \mathfrak{g})$.

Lemma 0.3. $\Delta(U(\mathfrak{g})^n) \subset U(\mathfrak{g} \oplus \mathfrak{g})^n$

Proof. Since the graded algebra $U(\mathfrak{g})$ is generated as a graded algebra by the graded subspace $\mathfrak{g} \subset U(\mathfrak{g})$ it is sufficient to check that $|\Delta(x)| = |x|$ for any homogeneous element $x \in \mathfrak{g}$. But this follows from the part *d*) of the previous problem and the equality $\Delta(x) = x \otimes 1 + 1 \otimes x$. □

Let $\mathfrak{g} = \oplus_{n=1}^{\infty} \mathfrak{g}^n$ be a graded Lie algebra. As follows from the Lemma 3 the diagonal map $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ extends to a continuous map $\hat{\Delta} : \hat{U}(\mathfrak{g}) \to \hat{U}(\mathfrak{g}) \otimes \hat{U}(\mathfrak{g})$.

Definition 0.4. *We say that an element $\hat{u} \in \hat{U}(\mathfrak{g})$ is primitive if $\Delta(\hat{u}) = \hat{u} \otimes 1 + 1 \otimes \hat{u}$ and we say that $\hat{u}$ is of a group if $\Delta(\hat{u}) = \hat{u} \otimes \hat{u}$.*

Lemma 0.5. *The set of primitive elements of $\hat{U}(\mathfrak{g})$ coincides with $\hat{\mathfrak{g}} \subset \hat{U}(\mathfrak{g})$*

Proof. Let $\hat{u} = (t_0, t_1, ..., t_n, ...), t_n \in U(\mathfrak{g})^n(X)$ be a primitive element. Since

$$\hat{\Delta}(u_0, u_1, ..., u_n, ...) = (\Delta(u_0), \Delta(u_1), ..., \Delta(u_n), ...)$$

where [by Lemma 3] $\Delta(u_n) \in U(\mathfrak{g})^n(X)$ we have $\Delta(u_n) \in= u_n \otimes 1 + 1 \otimes u_n$. So $\hat{u}$ is primitive iff all elements $u_n \in U(\mathfrak{g})$ are primitive. It follows from Theorem 5.4 in Serre that $u_n \in \mathfrak{g}^n(X)$. So $\hat{u} \in \hat{\mathfrak{g}}$. □

Lemma 0.6. *The map* $\exp : \mathcal{M} \to 1 + \mathcal{M}$ *defines a bijection between primitive elements in* $\mathcal{M}$ *and group elements in* $1 + \mathcal{M}$

Proof. a) Let $\hat{u} \in \mathcal{M}$ an element such that $\Delta(\hat{u}) = \hat{u} \otimes 1 + 1 \otimes \hat{u}$. We have to show that $\Delta(\exp(\hat{u})) = \exp(\hat{u}) \otimes \exp(\hat{u})$. Since Δ is an algebra isomorphism we have to show

$$\Delta(\exp(\hat{u})) = \exp(\Delta(\hat{u})) = \exp(\hat{u} \otimes 1 + 1 \otimes \hat{u})$$

Since the elements $\hat{u} \otimes 1, 1 \otimes \hat{u} \in \hat{U}(\mathfrak{g})$ commute we have

$$\exp(\hat{u} \otimes 1 + 1 \otimes \hat{u}) = \exp(\hat{u} \otimes 1) \exp(1 \otimes \hat{u}) = \exp(\hat{u}) \otimes \exp(\hat{u})$$

b) Let $\hat{u} \in 1 + \mathcal{M}$ a group element. We have to show that $\ln(\hat{u})$ is primitive. I leave for you to finish the proof. □

Corollary 0.7. *(Campbell-Hausdorff) For any* $x, y \in \hat{\mathfrak{g}}$ *there exists* $z \in \hat{\mathfrak{g}}$ *such* $\exp(x)\exp(y) = \exp(z)$

Proof. By Lemma 6 we $\exp(x), \exp(y)$ are group elements. Since Δ is an algebra isomorphism we see that $\exp(x)\exp(y)$ is also a group element. So by Lemma 5 there exists $z \in \hat{\mathfrak{g}}$ such $\exp(x)\exp(y) = \exp(z)$. □

To show that there exists a universal formula for $z(x, y)$ we introduce the notion of a *free Lie algebra.*

Definition 0.8. *a) Let* X *be a set.*

a) A free Lie algebra on X *is a pair* $(i, L(X))$ *where* $L(X)$ *is a Lie algebra and* $i : X \to L(X)$ *is a map such that for any Lie algebra* $\mathfrak{g}$ *and any map* $j : X \to \mathfrak{g}$ *there exists unique Lie algebra homomorphism* $f : L(X) \to \mathfrak{g}$ *such that* $j = f \circ i$.

b) Let V_X *be the space with a basis* $e_x, x \in X, V_X \hookrightarrow Ass_X := T(V_X)$ *be the tensor algebra and* $i : X \to Ass_X$ *be the imbedding* $i(x) := e_x, x \in X$. *We denote by* $\tilde{L}(X) \subset Ass_X$ *the Lie subalgebra generated by* $e_x, x \in X$. *(That is* $\tilde{L}(X) \subset Ass_X$ *is the subspace spanned by the commutators of* $e_x, x \in X$.*)*

Lemma 0.9. *$(i, \tilde{L}(X))$ is a free Lie algebra on X.*

Proof. We have to show that for any Lie algebra $\mathfrak{g}$ and a map $j : X \to \mathfrak{g}$ there exists unique Lie algebra homomorphism $f : \tilde{L}(V) \to \mathfrak{g}$ such that $f(e_x) = j(x), \forall x \in X$. Since the Lie algebra $\tilde{L}(V)$ is generated by by $e_x, x \in X$ the uniqueness of f is obvious. To prove the existence of f consider the algebra homomorphism $\tilde{f} : Ass_X \to U(\mathfrak{g})$ such that $\tilde{f}(e_x) = j(x) \in \mathfrak{g} \subset U(\mathfrak{g})$ [see Lemma 4 in the Lecture 2]. It is clear that the restriction f of $\tilde{f}$ on $\tilde{L}(X) \subset Ass_X$ is a Lie algebra homomorphism and $f(e_x) = j(x), \forall x \in X$. □

By the definition of $U(L(X))$ the imbedding $L(X) \hookrightarrow Ass_X$ extends uniquely to an algebra homomorphism $\phi : U(L(X)) \to Ass_X$. On the other hand the linear map $V_X \to L(X) \subset U(L(X))$ extends uniquely to an algebra homomorphism $\psi : Ass_X \to U(L(X))$.

Lemma 0.10. *Homomorphisms ϕ and ψ provide an isomorphism between associative algebras $U(L(X))$ and Ass_X.*

Proof. It is sufficient to show that $\phi \circ \psi = Id_{Ass_X}, \psi \circ \phi = Id_{U(L(X))}$. By the construction $\phi \circ \psi : Ass_X \to Ass_X$ is an algebra homomorphism such that $\phi \circ \psi(e_x) = e_x, \forall x \in X$. Since the set $e_x, x \in X$ generates the algebra Ass_X we see that $\phi \circ \psi = Id_{Ass_X}$. The analogous arguments show that $\psi \circ \phi = Id_{U(L(X))}$ □

We will identify the algebras $U(L(X))$ and Ass_X. In particular we consider the diagonal map $\Delta : U(L(X)) \to U(L(X)) \otimes U(L(X))$ as a algebra homomorphism $\Delta : Ass_X \to Ass_X \otimes Ass_X$.

Definition 0.11. *a) A Lie polynomial in two variables is a element $P \in L(X), X = (x, y)$. For any Lie algebra, a Lie polynomial P in two variables and elements $a, b \in \mathfrak{g}^1$ we define the evaluation $P(a, b) \in \mathfrak{g}$ as follows. By the definition of a free Lie algebra $L(X)$ there exists unique homomorphism $f_{a,b} : L(X) \to \mathfrak{g}$ of graded Lie algebras such that $f_{a,b}(x) = a, f_{a,b}(y) = b$. For any Lie polynomial P in two variables we define $P(a, b) := f_{a,b}(P)$.*

b) A formal Lie polynomial in two variables is a element $P \in \hat{L}(X), X = (x, y)$. For any graded Lie algebra $\mathfrak{g}$, a formal Lie polynomial $P(x, y)$ in two variables and any elements $a, b \in \mathfrak{g}, |a| = |b| = 1$ we can define the evaluation $P(a, b) \in \hat{\mathfrak{g}}$ [please give a definition]

Theorem 0.12. *There exists a a formal Lie polynomial $Q(x, y)$ such that for any graded Lie algebra $\mathfrak{g}$ and any homogeneous elements $a, b \in \mathfrak{g}$ we have $\exp(a) \exp(b) = \exp(z), z = Q(a, b)$*

Proof. Follows from Corollary 7 □

On can write explicitly a formula for $Q(x, y)$.An algorithm for the computation of $Q(x, y)$ is in the end of portion of the Serre's book which I posted.

Problem 0.13. *Show that*

a) $Q(x, y) = x + y + 1/2[x, y] + 1/12[x, [x, y]] + 1/12[y, [y, x]] + ...$ *where we omit terms of degree bigger then three. [You can prove this equality without looking in the Serre's book].*

b) $L^1(X)$ *is the span of* $e_x, x \in X$ *and* $L^n(X) = [L^1(X), L^{n-1}(X)]$.

c) The center of $L(X)$ *is equal to* $\{0\}$ *if* $Card(X) > 1$.

d) Let $\mathfrak{g}$ *be a graded Lie algebra,* $u', u'' \in \hat{U}(\mathfrak{g})$ *group elements. Then* $u'u'' \in \hat{U}(\mathfrak{g})$ *is also a group element.*

e) Let $\mathfrak{g}$ *be the graded Lie algebra with a basis* x, y, z *such that*

$$[x, y] = z, [x, z] = [z, y] = 0, |x| = |y| = 1, |z| = 2$$

Using the part a) describe the subgroup $G \subset \hat{U}(\mathfrak{g})$ *of group elements.*

$f)^{\star\star}$. *Let* X *be a finite set,* $d = |X|$. *Show that* $dimL(X)^n = 1/n \sum_{m|n} \mu(m)d^{n/m}$ *where the function* μ *is defined as follows*

(n) = 1 if n is a square-free positive integer with an even number of distinct prime factors.
(n) = -1 if n is a square-free positive integer with an odd number of distinct prime factors.
(n) = 0 if n is not square-free.