Problem 0.1. *Let A, B be associative unital k-algebras. Consider a 4-linear form $\mu : A \times B \times A \times B \to A \otimes B$ gien by*

$$\mu(a', b', a'', b'') := a'a'' \otimes b'b''$$

a) Explain why the 4-linear form μ defines a bilinear map

$$m : (A \otimes B) \times (A \otimes B) \to A \otimes B$$

(A hint). As we know the 4-linear form μ defines a linear map $\tilde{f} : A \otimes B \otimes A \otimes B \to A \otimes B$ which we can consider as a linear map $\tilde{f} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$. Now we can define $m : (A \otimes B) \times (A \otimes B) \to A \otimes B$ by $m(x, y) := \tilde{f}(x \otimes y)$.

b) Show that m defines a structure of an associative unital k-algebra on the vector space $A \otimes B$ and the maps

$$i_A : A \to A \otimes B, i_B : B \to A \otimes B, i_A(a) := a \otimes 1_B, i_B(b) := 1_A \otimes b$$

are algebra homomorphisms.

c) Show that for any associative k-algebra C and algebra homomorphisms $f_A : A \to C, f_B : B \to C$ such that

$$f_A(a)f_B(b) = f_B(b)f_A(a), \forall a \in A, b \in B$$

there exists unique algebra homomorphism $f : A \otimes B \to C$ such that $f \circ i_A \equiv f_A, f \circ i_B \equiv f_B$

Problem 0.2. *a) Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \to gl(V)$ be a representation. We denote by $\rho^\vee : \mathfrak{g} \to gl(V^\vee)$ the linear map*

$$\rho^\vee(x) := -(\rho(x))^\vee x \in \mathfrak{g}$$

Show that $\rho^\vee : \mathfrak{g} \to gl(V^\vee)$ is a representation of $\mathfrak{g}$.

b) Let $\rho' : \mathfrak{g} \to gl(V'), \rho'' : \mathfrak{g} \to gl(V'')$ be representations of $\mathfrak{g}$. We denote by $\rho' \otimes \rho'' : \mathfrak{g} \to gl(V' \otimes V'')$ the linear map

$$\rho' \otimes \rho''(x) := \rho'(x) \otimes Id_{V''} + Id_{V'} \otimes \rho''(x)$$

Show that $\rho' \otimes \rho'' : \mathfrak{g} \to gl(V' \otimes V'')$ is a representation of $\mathfrak{g}$.

Let $Hom(V', V'')$ be the space of linear maps from V' to V''. We denote by $\underline{Hom}(\rho', \rho'') : \mathfrak{g} \to gl(Hom_k(V', V''))$ the linear map

$$\underline{Hom}(\rho', \rho'')(x)(T) := -T\rho'(x) + \rho''(x)T, T \in Hom(V', V'')$$

Show that $\underline{Hom}(\rho', \rho'') : \mathfrak{g} \to gl(Hom(V', V''))$ is a representation of $\mathfrak{g}$.

d) Construct an isomorphism of Lie-algebra representations

$$\underline{Hom}(\rho', \rho'') \leftrightarrows \rho'^\vee \otimes \rho''$$

Definition 0.3. *a) Let* $\mathfrak{g}$ *be a Lie algebra,* $\rho : \mathfrak{g} \to gl(V)$ *be a representation. The representation* $\rho^\vee$ *is called the representation dual to* ρ.

b) Let $\mathfrak{g}$ *be a Lie algebra,* $\rho : \mathfrak{g} \to gl(V)$ *be a representation. We denote by* $V^\mathfrak{g} \subset V$ *the subspace of* $v \in V$ *such that* $\rho(x)v = 0$ *for all* $x \in \mathfrak{g}$.

c) Let $\rho' : \mathfrak{g} \to gl(V'), \rho'' : \mathfrak{g} \to gl(V'')$ *be representations of* $\mathfrak{g}$. *The representation* $\rho' \otimes \rho''(x)$ *is called the tensor product of* ρ' *and* ρ''.

d) We denote by $Hom_\mathfrak{g}(\rho', \rho'')$ *the space of linear maps* $T : V' \to V''$ *such that* $T\rho'(x) = \rho''(x)T$. *We call elements of* $Hom_\mathfrak{g}(\rho', \rho'')$ *morphsims of* $\mathfrak{g}$*-representations.*

e) Let V', V'', V *be vector spaces and* $S : V' \to V$, $T : V \to V''$ *be linear maps. We say that the sequence* $V' \to V \to V''$ *is exact if* $Im(S) = Ker(T)$. *If* V', V'', V *are representations of* $\mathfrak{g}$ *and* S, T *are morphisms of* $\mathfrak{g}$*-representations we say that* $V' \to V \to V''$ *is an exact sequence of representations if* $V' \to V \to V''$ *is an exact sequence of vector spaces.*

f) A representation $\rho : \mathfrak{g} \to gl(V)$ *is completely reducible if* V *can be written in the form* $V = \oplus V_i$ *where* V_i *are irreducible subrepresentations of* V.

Problem 0.4. *Show that*

a)

$$Hom(\rho', \rho'') = (\underline{Hom}(\rho', \rho''))^\mathfrak{g}$$

b) Let $\{0\} \to V' \to V \to V''$ *be an exact sequence of representations of* $\mathfrak{g}$*-representations. Then the sequence*

$$\{0\} \to V'^\mathfrak{g} \to V^\mathfrak{g} \to V''^\mathfrak{g}$$

is exact.

c) Give an example of an exact sequence

$$\{0\} \to V' \to V \to V'' \to \{0\}$$

of $\mathfrak{g}$*-representations such that the sequence*

$$\{0\} \to V'^\mathfrak{g} \to V^\mathfrak{g} \to V''^\mathfrak{g} \to \{0\}$$

is not exact.

d) Show that for any exact sequence

$$\{0\} \to V' \to V \to V''$$

of $\mathfrak{g}$*-representations such that* (ρ, V) *is completely reducible then the sequence*

$$\{0\} \to V'^{\mathfrak{g}} \to V^{\mathfrak{g}} \to V''^{\mathfrak{g}} \to \{0\}$$

exact.

Let $\mathcal{H}_n := \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$ *as a vector space and define*

$$[,] : H_n \times H_n \to H_n$$

by

$$[(x'_1, ..., x'_n; y'_1, ..., y'_n, t'), (x''_1, ..., x''_n; y''_1, ..., y''_n, t'') = (0, ..., 0; 0, ..., 0, t)$$

where $t := \sum_{i=1}^{n} x'_i y''_i - x''_i y'_i$

e) Show that $\mathcal{H}_n$ *is a Lie algebra and describe it's center.*

f) describe the Lie algebra $\mathcal{D}(\mathcal{H}_1)$ *[see the problem 4 in the first home-work].*

g^**) describe the Lie algebra* $\mathcal{D}(\mathcal{H}_n)$.

h) Let $\mathfrak{g}$ *be Lie algebra,* $\mathfrak{a} \lhd \mathfrak{g}$ *an ideal. Is it always possible to find a subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}$ *as a vector space? Let* $\mathfrak{h}, \mathfrak{a}$ *be Lie algebras and* $\phi : \mathfrak{h} \to \mathcal{D}(\mathfrak{a})$ *a Lie algebra homomorphism. Let* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}$ *as a vector space and* $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is given by*

$$[(h', a'), (h'', a'')] = ([h', h''] + \phi(a')(h'') - \phi(a'')(h'), [a', a''])$$

i) Show that $(\mathfrak{g}, [,])$ *is a Lie algebra.*

A hint for a solution of g*). Let $Z = \{0\} \times \{0\} \times \mathbb{C}$ be the center of the Lie algebra $\mathcal{D}(\mathcal{H}_n)$. Since for $d \in \mathcal{D}(\mathcal{H}_n)$ we have $d(Z) \subset Z$ [why] we obtain a Lie algebra homomorphism $\phi : \mathcal{D}(\mathcal{H}_n) \to End(Z) = \mathbb{C}$. Let $\mathcal{D}_n := Ker\phi$.

Let $V := \mathcal{D}(\mathcal{H}_n)/Z[= \mathbb{C}^n \times \mathbb{C}^n]$. For any $d \in \mathcal{D}(\mathcal{H}_n)$ we denote by $\bar{d} \in End(V)$ the induced linear trasfornation of the quotient space $V := \mathcal{D}(\mathcal{H}_n)/Z$. The map $v \to (v, 0)$ defines an imbedding $r : V \hookrightarrow H_n$ of vector spaces. Let $H = \{d \in \mathcal{D}_n | d(r(V)) \subset r(V)\}$.

The bracket $[,] : \mathcal{H}_n \times \mathcal{H}_n \to \mathcal{H}_n$ defines a skew-symmetric bilinear form $B : V \times V \to Z[= \mathbb{C}]$ by $B(v', v'') := [h', h'']$ where $h', h'' \in \mathcal{H}_n$ are preimages of $v', v'' \in V$. It is easy to see [please check] that for any $d \in \mathcal{D}_n$ we have $\bar{d} \in sp_B$. Show that

a) the homomorphism $f : \mathcal{D}_n \to sp_B, d \to \bar{d}$ is onto.

b) $Ker(f) = Imad_{H_n} = v$

c) H is a Lie subalgebra of $\mathcal{D}$ and the restriction of f on H defines an isomorphism $f : H \to sp_B$.

d) $\mathcal{D}_n = sp_B \ltimes V$.

e) $\mathcal{D}(\mathcal{H}_n) = gsp_B \ltimes V$ where $gsp_B = \{A \in End(V) | \exists c \in k$ such that $B(Av', v'') + B(v', Av'') = c(v', v''), \forall v', v'' \in V$.

Remark. If $n = 1$ then $sp_B = sl_2$

Definition 0.5. *The Lie algebra $\mathfrak{g}$ described in i) is called the semi-direct product of Lie algebras $\mathfrak{h}$ and $\mathfrak{a}$. We will denote it by $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{h}$*