Problem 0.1. Let $A, B$ be associative unital $k$-algebras. Consider a 4-linear form $\mu : A \times B \times A \times B \to A \otimes B$ given by

$$\mu(a', b', a'', b'') := a'a'' \otimes b'b''$$

a) Explain why the 4-linear form $\mu$ defines a bilinear map

$$m : (A \otimes B) \times (A \otimes B) \to A \otimes B$$

(A hint). As we know the 4-linear form $\mu$ defines a linear map $\tilde{f} : A \otimes B \otimes A \otimes B \to A \otimes B$ which we can consider as a linear map $\tilde{f} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$. Now we can define $m : (A \otimes B) \times (A \otimes B) \to A \otimes B$ by $m(x, y) := \tilde{f}(x \otimes y)$.

b) Show that $m$ defines a structure of an associative unital $k$-algebra on the vector space $A \otimes B$ and the maps

$$i_A : A \to A \otimes B, i_B : B \to A \otimes B, i_A(a) := a \otimes 1_B, i_B(b) := 1_A \otimes b$$

are algebra homomorphisms.

c) Show that for any associative $k$-algebra $C$ and algebra homomorphisms $f_A : A \to C, f_B : B \to C$ such that

$$f_A(a)f_B(b) = f_B(b)f_A(a), \forall a \in A, b \in B$$

there exists unique algebra homomorphism $f : A \otimes B \to C$ such that $f \circ i_A \equiv f_A, f \circ i_B \equiv f_B$

Problem 0.2. a) Let $g$ be a Lie algebra, $\rho : g \to \text{gl}(V)$ be a representation. We denote by $\rho^\vee : g \to \text{gl}(V^\vee)$ the linear map

$$\rho^\vee(x) := -(\rho(x))^\vee x \in g$$

Show that $\rho^\vee : g \to \text{gl}(V^\vee)$ is a representation of $g$.

b) Let $\rho' : g \to \text{gl}(V'), \rho'' : g \to \text{gl}(V'')$ be representations of $g$. We denote by $\rho' \otimes^\vee \rho'' : g \to \text{gl}(V' \otimes V'')$ the linear map

$$\rho' \otimes^\vee \rho''(x) := \rho'(x) \otimes \text{Id}_{V''} + \text{Id}_{V'}, \otimes \rho''(x)$$

Show that $\rho' \otimes^\vee \rho'' : g \to \text{gl}(V' \otimes V'')$ is a representation of $g$.

Let $\text{Hom}(V', V'')$ be the space of linear maps from $V'$ to $V''$. We denote by $\text{Hom}(\rho', \rho'') : g \to \text{gl}(\text{Hom}_k(V', V''))$ the linear map

$$\text{Hom}(\rho', \rho'')(x)(T) := -T\rho'(x) + \rho''(x)T, T \in \text{Hom}(V', V'')$$

Show that $\text{Hom}(\rho', \rho'') : g \to \text{gl}(\text{Hom}(V', V''))$ is a representation of $g$.

d) Construct an isomorphism of Lie-algebra representations

$$\text{Hom}(\rho', \rho'') \cong \rho' \otimes \rho''$$
Definition 0.3. a) Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation. The representation $\rho^\vee$ is called the representation dual to $\rho$.

b) Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation. We denote by $V^g \subset V$ the subspace of $v \in V$ such that $\rho(x)v = 0$ for all $x \in \mathfrak{g}$.

c) Let $\rho' : \mathfrak{g} \rightarrow \mathfrak{gl}(V'), \rho'' : \mathfrak{g} \rightarrow \mathfrak{gl}(V'')$ be representations of $\mathfrak{g}$. The representation $\rho' \otimes \rho''(x)$ is called the tensor product of $\rho'$ and $\rho''$.

d) We denote by $\text{Hom}_\mathfrak{g}(\rho', \rho'')$ the space of linear maps $T : V' \rightarrow V''$ such that $T \rho'(x) = \rho''(x) T$. We call elements of $\text{Hom}_\mathfrak{g}(\rho', \rho'')$ morphisms of $\mathfrak{g}$-representations.

e) Let $V', V'', V$ be vector spaces and $S : V' \rightarrow V$, $T : V \rightarrow V''$ be linear maps. We say that the sequence $V' \rightarrow V \rightarrow V''$ is exact if $\text{Im}(S) = \text{Ker}(T)$. If $V', V'', V$ are representations of $\mathfrak{g}$ and $S, T$ are morphisms of $\mathfrak{g}$-representations we say that $V' \rightarrow V \rightarrow V''$ is an exact sequence of representations if $V' \rightarrow V \rightarrow V''$ is an exact sequence of vector spaces.

f) A representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is completely reducible if $V$ can be written in the form $V = \bigoplus V_i$ where $V_i$ are irreducible subrepresentations of $V$.

Problem 0.4. Show that

a) $\text{Hom}(\rho', \rho'') = (\text{Hom}(\rho', \rho''))^g$

b) Let $\{0\} \rightarrow V' \rightarrow V \rightarrow V''$ be an exact sequence of representations of $\mathfrak{g}$-representations. Then the sequence

$\{0\} \rightarrow V'^g \rightarrow V^g \rightarrow V''^g$

is exact.

c) Give an example of an exact sequence

$\{0\} \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow \{0\}$

of $\mathfrak{g}$-representations such that the sequence

$\{0\} \rightarrow V'^g \rightarrow V^g \rightarrow V''^g \rightarrow \{0\}$

is not exact.

d) Show that for any exact sequence

$\{0\} \rightarrow V' \rightarrow V \rightarrow V''$
of \( g \)-representations such that \((\rho,V)\) is completely reducible then the sequence
\[
\{0\} \to V^g \to V^n \to V^{n^g} \to \{0\}
\]
each.

Let \( H_n := \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \) as a vector space and define
\[
[,] : H_n \times H_n \to H_n
\]
by
\[
[(x'_1, ..., x'_n; y'_1, ..., y'_n, t'), (x''_1, ..., x''_n; y''_1, ..., y''_n, t'')] = (0, ..., 0; 0, ..., 0, t)
\]
where \( t := \sum_{i=1}^n x'_i y''_i - x''_i y'_i \) where \( t \) is preimages of \( B \) form.

b) Let \( g \) be Lie algebra, \( a \subset g \) an ideal. Is it always possible to find a subalgebra \( h \subset g \) such that \( g = h \oplus a \) as a vector space? Let \( h, a \) be Lie algebras and \( \phi : h \to D(a) \) a Lie algebra homomorphism. Let \( g = h \oplus a \) as a vector space and \([,] : g \times g \to g \) is given by
\[
[(h', a'), (h'', a'')] = ([h', h''] + \phi(a')(h'') - \phi(a'')(h'), [a', a''])
\]
i) Show that \((g, [,])\) is a Lie algebra.

A hint for a solution of \( g^* \). Let \( Z = \{0\} \times \{0\} \times \mathbb{C} \) be the center of the Lie algebra \( D(H_n) \). Since for \( d \in D(H_n) \) we have \( d(Z) \subset Z \) we obtain a Lie algebra homomorphism \( \phi : D(H_n) \to \text{End}(Z) = \mathbb{C} \).

Let \( D_n := \text{Ker}\phi \). Let \( V := D(H_n)/Z = \mathbb{C}^n \times \mathbb{C}^n \). For any \( d \in D(H_n) \) we denote by \( \bar{d} \in \text{End}(V) \) the induced linear trasformation of the quotient space \( V := D(H_n)/Z \). The map \( v \to (v, 0) \) defines an imbedding \( r : V \hookrightarrow H_n \) of vector spaces. Let \( H = \{d \in D_n | d(r(V)) \subset r(V)\} \).

The bracket \([,] : H_n \times H_n \to H_n \) defines a skew-symmetric bilinear form \( B : V \times V \to Z = \mathbb{C} \) by \( B(v', v'') := [h', h''] \) where \( h', h'' \in H_n \) are preimages of \( v', v'' \in V \). It is easy to see [please check] that for any \( d \in D_n \) we have \( \bar{d} \in sp_B \). Show that

a) the homomorphism \( f : D_n \to sp_B, d \to \bar{d} \) is onto.

b) \( \text{Ker}(f) = \text{Im}d_{H_n} = v \)

c) \( H \) is a Lie subalgebra of \( D \) and the restriction of \( f \) on \( H \) defines an isomorphism \( f : H \to sp_B \).
d) $\mathcal{D}_n = sp_B \ltimes V$

e) $\mathcal{D}(H_n) = gsp_B \ltimes V$ where $gsp_B = \{A \in \text{End}(V) | \exists c \in k \text{ such that } B(Av', v'') + B(v', Av'') = c(v', v''), \forall v', v'' \in V\}.$

Remark. If $n = 1$ then $sp_B = sl_2$

**Definition 0.5.** The Lie algebra $\mathfrak{g}$ described in i) is called the semi-direct product of Lie algebras $\mathfrak{h}$ and $\mathfrak{a}$. We will denote it by $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{h}$