Definition 0.1. *Let $V_1, V_2, ..., V_n, W$ be k-vector spaces.*

a) We say that a map $B : V_1 \times V_2 \times ... \times V_n \to W$ is an n-linear form if for any $j, 1 \leq j \leq n$ and any vectors $v_i^0 \in V_i, 1 \leq i \leq n, i \neq j$ the map from V_j to W given by $v_j \to B(v_1^0, ..., v_{j-1}^0, v_j, v_{j+1}^0, v_n^0)$ is linear.

b) The tensor product of $V_1, V_2, ..., V_n$ is the universal n-linear form on $V_1, V_2, ..., V_n$. In other words it is a pair (V, m) where V is a vector space, $m : V_1 \times V_2 \times ... \times V_n \to V$ an n-linear form such that for any n-linear form $B : V_1 \times V_2 \times ... \times V_n \to W$ there exists unique linear map $S : V \to W$ such that $B = m \circ S$

As in the case Problem 4 of the first homework one can show the existence and uniqueness [up to a unique isomorphism] of the the universal n-linear form $m : V_1 \times V_2 \times ... \times V_n \to V_1 \otimes V_2 \otimes ... \otimes V_n$. We will write $v_1 \otimes ... \otimes v_n \in V_1 \otimes ... \otimes V_n$ instead of $m(v_1, ..., v_n)$. One can easily check [please do] that for any $j, 1 \leq j \leq n$ there exists unique linear isomorphism

$$\alpha : (V_1 \otimes ... \otimes V_j) \otimes (V_{j+1} \otimes ... \otimes V_n) \to V_1 \otimes V_2 \otimes ... \otimes V_n$$

such that

$$(v_1 \otimes ... \otimes v_j) \otimes (v_{j+1} \otimes ... \otimes v_n) \to v_1 \otimes ... \otimes v_n$$

Definition 0.2. *Let V be a k-vector space.*

a) We define $T(V) := \oplus_{n \geq 0} T^n(V)$ where $T^0(V) := k$ and

$$T^n(V) := V \otimes .. \otimes V = V \otimes^n$$

for $n > 0$.

b) We define a map $T(V) \times T(V) \to T(V)$ by

$$xy := \alpha(x \otimes y) \in T^{m+n}(V), x \in T^n(V), y \in T^m(V)$$

c) Let $v_i, i \in I$ be a basis of V. For any $n > 0$ and any $M = (i_1, ..., i_n) \in I^n$ we write $v_M := v_{i_1} \otimes ... \otimes v_{i_n} \in T^n(V)$.

d) We denote by i the natural imbedding $V = T^1(V) \hookrightarrow T(V)$.

Remark 0.3. *a) The map $(x, y) \to xy$ defines a structure of an associative algebra on $T(V)$ with the unit $1 \in k = T^0(V) \subset T(V)$.*

b) If $v_i, i \in I$ is a basis of V then $\{v_M\}, M \in I^m$ is a basis of $T^m(V)$ and $v_M v_N = v_{(M,N)}$ for any $M \in I^m, N \in I^n$

Lemma 0.4. *For any unital associative algebra A and a linear map $f : V \to A$ there exists unique homomorphism $\tilde{f} : T(V) \to A$ of unital algebras such that $f = \phi \circ i$.*

Proof. Since $i(V)$ generates A as an algebra the uniqueness of $\tilde{f}$ is obvious. To prove the existence we observe that $\{v_M\}, M \in I^n$ is a basis of $T^n(V)$ and therefore there exists unique linear map $f_n : T^n(V) \to A$ such that $f_n(v_M) = f(v_{i_1})...f(v_{i_n})$ for $M = (i_1, ..., i_n) \in I^n, n > 0$. This way we obtain a linear map $\tilde{f} : T(V) \to A$. It is easy to see that $\tilde{f}$ is a homomorphism of unital algebras. $\square$