Problem 0.1. a) Let $R \subset E$ be a set of vectors satisfying the conditions (R1), (R2) and (R3) and $\alpha, c\alpha \in R, c \in \mathbb{R}$. Show that the only possible values of $c$ are $\pm 1/2, \pm 1, \pm 2$.

b) Let $R \subset E$ be a root system and $\{\alpha_i\}, i \in I$ be the set of simple roots corresponding to a polarization $R = R^+ \cup R^-$. We define

$$R' := \{ v \in E'| < v, \beta > \in \mathbb{Z} \}, \forall \beta \in R$$

Show that $\beta' \in R'$ for all $\beta \in R$.

$R'$ is a root system

$\{\alpha'_i\}, i \in I$ is a set of simple roots for $R'$.

c) Show that map $W \to \pm 1, w \to -l(w)$ is a group homomorphism.

d) Let $v = \sum_i k_i \alpha_i, k_i \in \mathbb{Z}_{\geq 0}$ be a non-negative combination of simple roots which is not a multiple of a root. Show the existence of $w \in W$ such that $w(v) = \sum_i k'_i \alpha_i$ where some of $\{k'_i\}$ are positive and some are negative.

A hint. Let $L = \{ x \in E | (v, x) = 0 \}$. Since $v$ is not a multiple of a root we can find a regular point $v' \in L$. Then the element $w \in W$ be such that $w(v) \in C_+$ is the solution of d).

We say that $g \in \text{Aut}(E)$ is a reflection if $g^2 = e$ and $\dim(E^g) = \dim(E) - 1$ where $E^g := \{ x \in E | g(x) = x \}$.

e) Show that any reflection $w \in W$ has a form $w = s_\beta, \beta \in R$.

f) Let $E = \mathbb{R}^8, e_1, \ldots, e_8$ the standard basis of $E$

$$\Lambda' := \mathbb{Z}(e_1 + \cdots + e_8)^+ = \mathbb{Z}^8 \Lambda \subset E$$

and $\Lambda \subset \Lambda'$ consists of elements

$$\sum_{i=1}^8 c_i e_i + c(e_1 + \cdots + e_8)$$

such that $\sum_{i=1}^8 c_i$ is even. Let $R := \{ v \in \Lambda | (v, v) = 2 \}$.

Show that $\Lambda \subset \Lambda'$ is a subgroup.

$$R = \{ \pm e_i \pm e_j \}, i \neq j \cup \{ 1/2 \sum_{i=1}^8 \epsilon_i e_i \}$$

where $\epsilon_i = \pm 1$ and $\prod_{i=1}^8 \epsilon_i = 1$. 1
$R$ is a root system and the set
\[
\{1/2[e_1 - (e_2 + \cdots + e_7) + e_8], e_1 + e_2, e_j + 1 - e_j\}, \quad 1 \leq j \leq 6
\]
is the set of simple roots of the type $E_8$.

f) Construct root systems of the types $E_6, E_7$.

Let $R \subset E$ be a root system, $\Delta = \{\alpha_i\}, i \in I$ is a set of simple roots and $\sigma \in \text{Aut} I$ such that $(\alpha_i, \alpha_j) = (\alpha_{\sigma(i)}, \alpha_{\sigma(j)}), \forall i, j \in I$ and $\Gamma \subset \text{Aut}(\Delta)$ be the group generated by $\sigma$. We also denote by $\sigma \in \text{Aut}(E)$ the linear map such that $(\alpha_i, \alpha_j) = (\sigma_i, \sigma_j)$, $\forall i, j \in I$. We assume that for any $i \in I$, $k \in \mathbb{Z}$ such that $\sigma^k(i) \neq i$ the roots $\alpha_i, \alpha_{\sigma^k(i)}$

g) Show that set
\[
\{\tilde{\alpha}_i\} := \sum_{i \in \Omega_i} \alpha_i \in E^\sigma, \tilde{i} \in \tilde{I}
\]
is a set of simple roots for a root system $\tilde{R}$ in $E^\sigma$. Moreover the vertices of the Dynkin diagram $\tilde{I}$ of $\tilde{R}$ are $\Gamma$-orbits in $I$ and $(\tilde{\alpha}_i, \tilde{\alpha}_i) = | \in \Omega_i$.

h) Construct the root system $F_4$ from the root system $E_6$ and the root system $G_2$ from the root system $D_4$

Remark. Let $(E, R)$ be the root system of the type $E_8$ and $\Lambda \subset E$ be the lattice [the span of a basis] generated by simple roots. Then $Q(\lambda, \lambda) \in 2\mathbb{Z}, \forall \lambda \in \Lambda$ where $\Lambda \subset E$ and $\det(A) = 1$ where $Q(v, v') := (v, v')$

One can show that such pairs $(Q, \Lambda), Q : E \times E \to \mathbb{R}, \Lambda \subset E$ exists only if $\text{dim}(E) \equiv 0(\text{mod}8)$. In the case when $\text{dim}(E) = 8$ any such pair is isomorphic to the one coming from $E_8$, in in the case when $\text{dim}(E) = 16$ any such pair is isomorphic either to the one coming from $E_8 \oplus E_8$ or to the one coming from $D_{16}$, in the case when $\text{dim}(E) = 24$ there are 24 classes of such pairs and in the case when $\text{dim}(E) = 32$ there are more then $10^7$ such pairs.

Definition 0.2. Let $V$ be a [not necessarily finite-dimensional] vector space over a field $k$ of characteristic zero. We say that a linear operator $T : V \to V$ is locally nilpotent if for any $v \in V$ there exists $N > 0$ such that $A^N v = 0$

Remark. If $T : V \to V$ is locally nilpotent then we can define $\exp(T) := \sum_{n \geq 0} T^n / n!$. 
Problem 0.3. Let $\mathfrak{g}$ be a Lie algebra with generators $x_i, i \in I$ and $D : \mathfrak{g} \to \mathfrak{g}$ a differentiation.

a) If there exists $N > 0$ such that $D^N x_i = 0, \forall i \in I$ then $D$ is locally nilpotent.

b) If $D$ is locally nilpotent then $\exp(D)$ is an automorphism of $\mathfrak{g}$.

c) Let $\rho : \mathfrak{sl}_2(k) \to \text{End}(V)$ be a representation, $v \in V$ be such that $\hat{e}v = 0, \hat{h}v = nv, n \geq 0$. Then $\hat{f}^{n+1}v = 0$. If $\dim(V) < \infty$ then $\hat{f}^{n+1}v = 0$.

d) Assume that the representation $\rho : \mathfrak{sl}_2(k) \to \text{End}(V)$ is such that $\hat{e}, \hat{f}$ are locally nilpotent. Then for any vector $v \in V$ there exists a finite-dimensional $\mathfrak{sl}_2(k)$-invariant subspace $V' \subset V$ containing $v$.

e) Let $\rho : \mathfrak{sl}_2(k) \to \text{End}(V)$ be as in d), $s := \exp(e) \exp(-f) \exp(e) \in \text{Aut}(V), v \in V$ by such that $hv = nv, n \in \mathbb{Z}$. Then $\hat{h}sv = -nsv$.

a)

Problem 0.4. a) Let $V, W$ be $k$-vector spaces with filtrations

\[ V_0 \subset \cdots \subset V_n \subset \cdots \subset V, W_0 \subset \cdots \subset W_n \subset \cdots \subset W \]

such that $V = \cup_n V_n, W = \cup_n W_n$. Let $T : V \to W$ be a linear map such that $T(V_n) \subset W_n$ and such that the induced maps $T_n : V_n/V_{n-1} \to W_n/W_{n-1}$ are bijections. Then $T : V \to W$ is a bijection.

b) Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}', \mathfrak{g}'' \subset \mathfrak{g}$ Lie subalgebras such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ as a vector space. Then the natural [product] map $U(\mathfrak{g}') \otimes U(\mathfrak{g}'') \to U(\mathfrak{g})$ a bijection.