We start with some definitions and problems from linear algebra. For simplicity I assume that $\text{char}(k) \neq 2$. Please notice that a number of definitions are in the middle of homework assignments.

Definition 0.1. Let $V, W$ be $k$-vector spaces.

a) We denote by $V^\vee := \text{Hom}(V, k)$ the dual vector space of linear functionals $\lambda : V \to k$.

b) For any linear map $T : V \to W$ we denote by $T^\vee : W^\vee \to V^\vee$ the map given by $T^\vee(\lambda)(v) := \lambda(T(v)), \lambda \in W^\vee, v \in V$. We say that $T^\vee$ is the map dual to $T$.

c) Let $B : V \times V \to k$ be a bilinear form. We say that $Q$ is non-degenerate if for any $v' \in V - \{0\}$ there exists $v'' \in V$ such that $B(v', v'') \neq 0$.

d) Let $Q : V \to k$ be a quadratic form. We associate with $Q$ a symmetric bilinear form

$$B_Q : V \times V \to k, B_Q(v, w) := Q(v + w) - Q(v) - Q(w)$$

We say that the quadratic form $Q$ is non-degenerate if the bilinear form $B_Q$ is non-degenerate.

e) We say that a skew-symmetric bilinear form $B : V \times V \to k$ is symplectic if it is non-degenerate.

Problem 0.2. a) Construct a natural linear map $V \to (V^\vee)^\vee$ and show that this map is an isomorphism if $V$ is finite-dimensional.

b) If $U, V, W$ are $k$-vector spaces and $S : U \to V, V \to W$ are linear maps then $(T \circ S)^\vee = S^\vee \circ T^\vee$.

c) Let $V$ be a finite-dimensional vector space, $B : V \times V \to k$ a non-degenerate bilinear form. Show that for any $v'' \in V - \{0\}$ there exists $v' \in V$ such that $B(v', v'') \neq 0$.

d') Is the assumption of the finite-dimensionality is important for the validity of c).

Let $k$ be a field, $V', V''$ be $k$-vector spaces. For any $k$-vector space $W$ we denote by $B(V', V''; W)$ the $k$-vector space of bilinear forms $b : V' \times V'' \to W$.

Definition 0.3. A tensor product of $V'$ and $V''$ is a pair $(V, m)$ where $V$ is a $k$-vector space and $m : V' \times V'' \to V$ a bilinear form such that for any $k$-vector space $W$ and a bilinear form $b : V' \times V'' \to W$ there exists unique linear map $T : V \to W$ such that $b(v', v'') \equiv T(m(v', v''))$. 

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Problem 0.4. Show that

a) In the case when a tensor product \((V, m)\) of vector spaces \(V'\) and \(V''\) exists it is well defined

[you have to show that if \((\tilde{V}, \tilde{m})\) is another tensor product \((V, m)\)
of \(V'\) and \(V''\) then there exists unique linear map \(S : V \to \tilde{V}\) such that

\[ S((m(v', v'')) \equiv \tilde{m}(v', v'') \]

and moreover the map \(S\) is an isomorphism].

Since the tensor product \((V, m)\) of \(V'\) and \(V''\) is well defined we can talk about the tensor product of vector spaces \(V'\) and \(V''\) which we denote by \(V' \otimes V''\) [we did not yet show that the tensor product \(V' \otimes V''\) exists] and write \(v' \otimes v'' \in V' \otimes V''\) instead of \(m(v', v'')\).

b) for any \(k\)-vector spaces \(V'\) and \(V''\) the tensor product \(V' \otimes V''\) exists

[A hint: Use the existence of bases \(e'_i, e''_j\) for vector spaces \(V', V''\).]

c) If \(V', V''\) are finite-dimensional \(k\)-vector spaces then

\[ \dim(V' \otimes V'') = \dim(V') \dim(V'') \]

d*) Let \(A\) be a commutative ring and \(M', M''\) be \(A\)-modules. Give a definition of the tensor product \(M' \otimes_A M''\) and prove the uniqueness and the existence of the tensor product \(M' \otimes_A M''\).

Definition 0.5. a) Let \(k\) be a field. A \(k\)-Lie algebra is a pair \((\mathfrak{g}, [,])\) where \(\mathfrak{g}\) is a \(k\)-vector space and \([,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) is a bilinear map such that

\[ [x, x] = 0, x \in \mathfrak{g} \] (that is \([,]\) is skew-symmetric) and

\[ [[x, y], z] + [[z, x], y] + [[y, z], x] = 0, x, y, z \in \mathfrak{g} \] (the Jacobi identity).

We often say "a Lie algebra" instead of "a \(k\)-Lie algebra" and write \(\mathfrak{g}\) instead of \((\mathfrak{g}, [,])\).

b) We define \(z(\mathfrak{g}) := \{ x \in \mathfrak{g} | [x, y] = 0, \forall y \in \mathfrak{g} \}\). We say that \(z(\mathfrak{g})\) is the center of \(\mathfrak{g}\).

c) Let \((\mathfrak{g}, [,])\) be a Lie algebra and \(\mathfrak{h} \subset \mathfrak{g}\) be a subspace. We say that \(\mathfrak{h}\) is a Lie subalgebra if \([h', h''] \in \mathfrak{h}, \forall h', h'' \in \mathfrak{h}\) and say that \(\mathfrak{h}\) is an ideal if \([x, h] \in \mathfrak{h}, \forall x \in \mathfrak{g}, h \in \mathfrak{h}\).

d) Let \(\mathfrak{h}, \mathfrak{g}\) be Lie algebras. A linear map \(f : \mathfrak{h} \to \mathfrak{g}\) is called a Lie algebra homomorphism (or simply a homomorphism) if

\[ [f(x), f(y)] = f([x, y]), x, y \in \mathfrak{h}. \]
e) We say that a homomorphism $f$ is an isomorphism if $f$ is one-to-one and onto.

**Problem 0.6.**

a) Show that for any Lie algebra homomorphism $f : h \to g$ the image $\text{Im}(f) \subset g$ of a homomorphism $f$ is a Lie subalgebra and the kernel $\text{Ker}(f) \subset h$ is an ideal.

b) Let $A$ be an associative $k$-algebra. We define the map $[,] : A \times A \to A$ by $[a, b] := ab - ba, a, b \in A$. Show that $(A, [,])$ is a Lie algebra.

**Definition 0.7.**

a) In the case when $A = \text{End}_k(V)$ is the algebra of endomorphisms of a $k$-vector space $V$ we denote the Lie algebra $(\text{End}(V), [,])$ by $\mathfrak{gl}(V)$.

b) In the case when $V$ is finite-dimensional we denote by $\mathfrak{sl}_n(V) \subset \mathfrak{gl}(V)$ the subspace of endomorphisms $T \in \text{End}(V)$ such that $\text{Tr}(T) = 0$.

c) If $V = k^n$ we write $\mathfrak{gl}_n(k)$ instead of $\mathfrak{gl}(k^n)$.

d) If $Q : V \to k$ is a non-degenerate quadratic form we denote by $\mathfrak{so}_Q \subset \mathfrak{gl}(V)$ of endomorphisms $T \in \text{End}(V)$ such that $B_Q(Tv', v'') + B_Q(v', Tv'') = 0, \forall v', v'' \in V$.

e) If $B : V \times V \to k$ is a symplectic form we denote by $\mathfrak{sp}_B \subset \mathfrak{gl}(V)$ of endomorphisms $T \in \text{End}(V)$ such that $B(Tv', v'') + B(v', Tv'') = 0, \forall v', v'' \in V$.

**Problem 0.8.**

Show that $y$

a) $\mathfrak{sl}_n(k)$ is an ideal of $\mathfrak{gl}_n(k)$.

b) Let $\mathfrak{g}$ be the 3-dimensional $k$-vector space with the basis $(f, e, h)$ and $[,]$ be the bilinear skew-symmetric map such that $[e, f] = h, [h, e] = 2e, [h, f] = -2f$.

Prove that $(\mathfrak{g}, [,])$ is a Lie algebra and that it is isomorphic to the Lie algebra $\mathfrak{sl}_2(k)$.

c) Let $\mathfrak{so}_n := \mathfrak{so}_{Q_n}$ where $Q_n$ is the quadratic form on $V = k^n$ given by $Q(x_1, ..., x_n) := \sum_{i=1}^{n} x_i^2$. Show that $\mathfrak{so}_n$ is the set of skew-symmetric matrices.

d) Let $\mathfrak{sp}_{2n} := \mathfrak{sp}_{B_n}$ where $B_n$ is the symplectic form on $V = k^{2n}$ given by

$$B_n(x_1, ..., x_{2n}; y_1, ..., y_{2n}) := \sum_{i=1}^{n} (x_i y_{i+n} - x_{i+n} y_i)$$

Describe the subset $\mathfrak{sp}_B \subset \mathfrak{gl}_{2n}(k)$. 
e) Show that for \( \text{so}_Q \subset \text{gl}(V) \) is a Lie subalgebra for any quadratic form \( Q \).

f) Show that for \( \text{sp}_B \subset \text{gl}(V) \) is a Lie subalgebra for any symplectic form \( B \).

g) Find all the ideals and subalgebras of the Lie algebra \( \text{sl}_2(k) \).

h)* Classify all the Lie algebras of dimension \( \leq 3 \) over an algebraically closed field \( k \).

[ that is construct a set of \( k \)-Lie algebras \( \mathfrak{g}_i, 1 \leq i \leq N \) such that the Lie algebras \( \mathfrak{g}_i, \mathfrak{g}_j \) are not isomorphic for \( i \neq j \) and any \( k \)-Lie algebra of dimension \( \leq 3 \) is isomorphic to \( \mathfrak{g}_i \) for some \( i, 1 \leq i \leq N \).]

i) Show that Lie algebras \( \text{so}_3 \) and \( \text{sl}_2 \) are isomorphic iff \( \text{if and only if} \) there exists \( i, j \in k \) such that \( i^2 + j^2 = -1 \).

**Definition 0.9.** Let \( \mathfrak{g} \) be a \( k \)-Lie algebra. We denote by \( \mathcal{D}(\mathfrak{g}) \) the set of \( k \)-linear maps \( D : \mathfrak{g} \rightarrow \mathfrak{g} \) such that \( D([x, y]) = [x, Dy] + [Dx, y] \) for all \( x, y \in \mathfrak{g} \). We call elements of the set \( \mathcal{D}(\mathfrak{g}) \) differentiations of \( \mathfrak{g} \).

**Problem 0.10.** Show that

a) For any \( D', D'' \in \mathcal{D}(\mathfrak{g}) \) the map

\[
[D', D''] : \mathfrak{g} \rightarrow \mathfrak{g}, [D', D''](x) := D' \circ D''(x) - D'' \circ D'(x)
\]

belongs to \( \mathcal{D}(\mathfrak{g}) \).

b) The map \( (D', D'') \mapsto [D', D''] \) defines a Lie algebra structure on the vector space \( \mathcal{D}(\mathfrak{g}) \). We call it the Lie algebra of differentiations of \( \mathfrak{g} \) and denote by \( \text{Diff}(\mathfrak{g}) \).

c) For any \( x \in \mathfrak{g} \) the map

\[
\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y]
\]

belongs to \( \text{Diff}(\mathfrak{g}) \) and the map \( \text{ad}_x : \mathfrak{h} \rightarrow \text{Diff}(\mathfrak{g}) \) is a Lie algebra homomorphism.

d) The map \( \text{ad}_{\text{sl}_2(k)} \) is an isomorphism iff \( \text{if and only if} \) \( \text{char}(k) \neq 2 \)

**Definition 0.11.** Let \( \mathfrak{g} \) be a \( k \)-Lie algebra.

a) A representation of \( \mathfrak{g} \) on a \( k \)-vector space \( V \) is a Lie algebra homomorphism \( \rho : \mathfrak{g} \rightarrow \text{End}(V) \).

b) A subspace \( W \subset V \) is \( \rho \) - invariant if \( \rho(x)w \subset W \) for all \( x \in \mathfrak{g}, w \in W \).
We often say "invariant" instead of "ρ -invariant".

c) A representation ρ : g → End(V) is irreducible if for any ρ -invariant subspace W ⊂ V either W = V or W = {0}

Let V_n be the space of homogeneous polynomial P(u, v) of degree n over the field k. Consider a k-linear map ρ_n : sl_2(k) → End(V_n) given by

\[
\begin{align*}
\rho_n(f)(P) &= u\partial/\partial v(P), \\
\rho_n(e)(P) &= v\partial/\partial u(P) \\
\rho_n(h)(P) &= -u\partial/\partial u(P) + v\partial/\partial v(P)
\end{align*}
\]

**Problem 0.12.** Show that

a) the map ρ_n is a representation of the Lie algebra sl_2(k).

b) the representation ρ_n is irreducible if char(k) = 0.

c) If p := char(k) ≠ 0 then the representation ρ_n is irreducible if and only if p > n.