Let \( R \subset E \) be a reduced root system, \( R = R^+ \cup R^- \) a polarization, \( \Sigma = \{ \alpha_i \}, i \in I \) the simple roots, \( C^+ \subset E \) the positive Weyl chamber, \( W \) the Weyl group. You know that \( W \) is generated by simple reflections \( s_i, i \in I \) which satisfy the relations [see the problem 7.11]

\[
s_i^2 = e, (s_is_j)^{m_{ij}} = e, i, j \in I
\]

Let \( \tilde{W} \) be the group generated by elements \( \tilde{s}_i, i \in I \) and relations

\[
\tilde{s}_i^2 = e, (\tilde{s}_i\tilde{s}_j)^{m_{ij}} = e, i, j \in I
\]

It is clear that there exists unique group homomorphism \( p : \tilde{W} \to W \) such that \( p(\tilde{s}_i) = s_i, i \in I \).

**Theorem 0.1.** The map \( p : \tilde{W} \to W \) is an isomorphism.

**Proof.** Since the group \( W \) is generated by \( s_i, i \in I \) the homomorphism \( p \) is surjective. So we have to show that \( \operatorname{Ker}(p) = \{ e \} \). The proof will use the following result from Topology.

**Lemma 0.2.** Let \( X \subset \mathbb{R}^d \) be a finite union of linear [or affine] subspaces of dimension \( < d - 2 \). Then \( \pi_1(\mathbb{R}^d - X) = \{ e \} \).

Let \( Z := \cup_{\beta \in R^+} L_\beta, E^0 := E - Z \) be the set of regular elements of \( E \). For any pair \( \beta \neq \beta' \in R^+ \) we define \( Y_{\beta\beta'} := L_\beta \cap L_{\beta'} \), write \( Y := \cup_{\beta \neq \beta'} Y_{\beta\beta'} \) and define \( X := \cup_{\beta, \beta', \beta''} L_\beta \cap L_{\beta'} \cap L_{\beta''} \) where \( \beta, \beta', \beta'' \in R^+ \) runs through all distinct triples. We write \( Y_{\beta\beta'}^0 = Y_{\beta\beta'} - (Y_{\beta\beta'} \cap X) \). It is clear that \( Y - X \) is a disjoint union of \( Y_{\beta\beta'}^0 \).

Let \( \gamma : [0,1] \to E - Y \) be a continuous map [a curve] such that \( f(0) \in C_+ \) and \( f(1) \in E^0 \). Then we can define \( \tilde{w}_\gamma \in \tilde{W} \) as follows. Let \( 0 < a_1 < \cdots < a_r < 1 \) be points such that \( f(a_j) \in Z \). Since \( f(a_j) \in Z - X \) there exists unique \( \beta_j \in R^+ \) such that \( f(a_j) \in L_{\beta_j} \). As in Lemma 7.31 we obtain a sequence of simple roots \( \alpha_{i_1}, \ldots, \alpha_{i_r} \) such that

\[
\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), 1 \leq j \leq r
\]

We define \( \tilde{w}_\gamma := \tilde{s}_{i_1} \cdots \tilde{s}_{i_r} \in \tilde{W} \).

Conversely given an elements \( \tilde{w} \in \tilde{W} \) and a representation \([\tilde{w}]\) of \( \tilde{w} \) as a product \( \tilde{w} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_l} \in \tilde{W} \) we define a curve \( \gamma([\tilde{w}]) \subset E - Y \) as the union of intervals connecting \( w_r(t) \) with \( w_{r+1}t, 1 \leq r \leq l \) where \( t \in C^+ \) is a regular element. [Please check that \( \gamma([\tilde{w}]) \subset E - Y \). It is easy to see that \( \tilde{w}_\gamma([\tilde{w}]) = \tilde{w} \).

**Example.1** Assume that \( \dim(E) = 2 \) and \( \gamma \) is a simple loop around 0 such that \( f(0) \in C_+ \). Then \( \tilde{w}_\gamma := (s_is_j)^{m_{ij}} = e. \)
2) For an arbitrary root system choose a pair of distinct simple root $\alpha_i, \alpha_j$ and consider a simple loop $\gamma$ in $E - Y$ around $Y^0_{\alpha_i\alpha_j}$ with the beginning in $C^+$. Then $\hat{w}_\gamma = e$.

**Claim.** If $\gamma$ is a loop $[f(0) = f(1)]$ then $\hat{w}_\gamma = e$.

The Claim implies the Theorem.

**Proof.** Let $\hat{w} \in \hat{W}$ be such that $p(\hat{w}) = e$. Choose a represtnation $[\hat{w}]$ of $\hat{w}$ as a product $\hat{w} = \hat{s}_{i_1} \cdots \hat{s}_{i_l} \in \hat{W}$ and consider the curve $\gamma([\hat{w}])$.

Since $p(\hat{w}) = e$ it is a loop. Therefore $\hat{w}_{\gamma([\hat{w}])} = e$. But $\hat{w}_{\gamma([\hat{w}])} = \hat{w}$. □

**Proof of the Claim.** Let $S^1$ be the circle obtained from the interval $[0, 1]$ by gluing together 0 and 1. We can consider $\gamma$ as a continuous map $f : S^1 \to E - Y$. By Lemma 2, $\pi_1(E - X) = \{e\}$ and therefore the loop $\gamma$ is contractible in $E - X$. So there exists a continuous family $f_a : S^1 \to E - X$ of loops such that $f_0 \equiv f, f_a(0) = f_a(1) \equiv t$ and $f_1 \equiv t$.

We can assume that there is a finite set $A \subset (0, 1), A = \{a_1 < \ldots a_N\}$ such that $Im(f_a) \subset E - Y$ for $a \in [0, 1] - A$ and for any $q, 1 \leq q \leq N$ and $f_a(x) \in E - Y, \forall a \in [0, 1], x \neq 1/2$. For any $q, 1 \leq q \leq N$ we write $v_q := f_a_q(1/2) \in Y$. [Students who took topology please check the validity of this claim].

Since $v_q \in Y - X$ there exist unique pair $\beta_q \neq \beta_q' \in R^+$ such that $v_q \in Y_{\beta_q\beta_q'}$. For any $q, 1 \leq q < N$ we choose $b_q \in (a_q, a_{q+1})$ and define $b_0 = 0, b_{N+1} = 1$. We define $\gamma_q := f_{b_q}$ and $\hat{w}_q := \hat{w}_{\gamma_q}$. By the construction $\hat{w}_0 = \hat{w}$ and $\hat{w}_1 = e$. So it is sufficient to show that $\hat{w}_q = \hat{w}_{\gamma_q}$ for all $q, 0 \leq q \leq N$.

The loop $\gamma_{q+1} \subset X - Y$ is obtained from the $e$ loop $\gamma_q$ by crossing $Y_{\beta_q\beta_q'}$ which is a linear subspace of codimension 2 $[\text{dim}(Y_{\beta_q\beta_q'}) = r - 2]$. So we can think that $\gamma_q$ consists of a curve $\gamma_q^{+}$ from $t$ to a point $s$ near $Y_{\beta_q\beta_q'}$ and a curve $\gamma_q^{-}$ from $s$ to $t$ while the loop $\gamma_{q+1}$ is obtained from $\gamma_q$ by insertion of a small loop $\gamma'$ which starts in $s$ and goes around $Y_{\beta_q\beta_q'}$.

But it is easy to see [please check that]

$$w_{\gamma_q}^{-1} w_{\gamma_{q+1}}^{-1} = w_{\gamma_q}^{-1} = w_{\gamma_q}^{-1}$$

Here $\gamma'$ is a loop around $Y_{\alpha_q\alpha_q'}$ [as in the Example 2] where $(\alpha_q, \alpha_q') = w_{\gamma_q}^{-1}(\beta_q, \beta_q')$. Since we know [see Example 2] that $\hat{w}_\gamma = e$ we see that $\hat{W} = e$. □