In this lecture we assume that $k = \mathbb{R}$ [or $\mathbb{C}$]. The results are true if $k$ is any field of characteristic but the statements and proofs require the knowledge of basics of Algebraic Geometry.

**Definition 0.1.** a) For any finite-dimensional vector space $V$ we define $GL(V) := \{g \in End(V) | \det(g) \neq 0\}$. It is clear that $GL(V)$ is a group.

b) Let $Q : V \to \mathbb{R}$ be a positive definite quadratic form [=Euclidean structure on $V$]. For any $A \in End(V)$ we define $\|A\| := \max \{Q(Av)/Q(v)\} | v \in V - \{0\}$

c) Let $W$ be a real vector space and $X$ a subset of $W$. We say that $X$ is smooth at $x \in X$ if there exists an open subset $B$ of a vector space $L, 0 \in B$ and a smooth [=∞-differentiable] map $f : B \to W$ of such that

a) $f(0) = x$

b) The differential $df_0 : L \to W$ is an imbedding and

c) For any $x \in X$ there exists an open neighborhood $U \subset W$ of $x$ in $W$ such that $U \cap X \subset f(B)$.

d) We say that $X$ is a submanifold of $W$ if every point $x \in X$ is smooth.

**Problem 0.2.** a) For any $A \in End(V)$ the series $\exp(A) := \sum_{i=0}^{\infty} A^i/i!$ is convergent and $\exp(A) \exp(-A) = I_d_V$.

b) Any $g \in End(V)$ such that $\|g - I_d_V\| < 1$ belongs to $GL(V)$. Moreover $g^{-1} = \sum_{i=0}^{\infty} (-1)^i A^i$ where $A := g - I_d_V$.

c) For any $g \in End(V)$ such that $\|g - I_d_V\| < 1$ the series $\ln(g) := -\sum_{i=1}^{\infty} (-A)^i/i$ is convergent and $\exp(\ln(g)) = g$

**Definition 0.3.** Let $\mathfrak{g}$ be a Lie algebra. We define $\text{Aut}(\mathfrak{g}) := \{g \in GL(\mathfrak{g})| [g(x), g(y)] = g([x, y]), \forall x, y \in \mathfrak{g}\}$.

**Lemma 0.4.** Show that for any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ and any $D \in \mathcal{D}(\mathfrak{g}) \subset [see the definition 9 of Lecture 1]$ we have $\exp(D) \in \text{Aut}(\mathfrak{g})$.

**Proof.** Since $D \in \mathcal{D}(\mathfrak{g})$ we have $D([x, y]) \equiv [Dx, y] + [x, Dy]$. Therefore $D^2([x, y])/2 \equiv D([Dx, y])/2 + D([x, Dy])/2 = [D^2x, y]/2 + [Dx, Dy] + [x, D^2y]/2$

By induction one shows that for any $n, 0$ we have

$D^n([x, y])/n! = \sum_{i+j=n} \binom{n}{i} [D^i(x), D^j(y)]/n! = \sum_{i+j=n} [D^i(x)/i!, D^j(y)/j!]$
So \( \exp(D)[x, y] = \exp(D)x, \exp(D)y \) \( \square \)

**Problem 0.5.** Let \( Q \) be a Euclidean structure on \( g \) and \( g \in \text{Aut}(g) \) be such that \( \|A\| < 1, A := g - \text{Id}_V \). Then \( \ln(g) \in \mathcal{D}(g) \)

**Theorem 0.6.** \( \text{Aut}(g) \) is a submanifold of \( \text{End}(g) \).

**Proof.** It is clear that \( e := \text{Id}_V \in \text{Aut}(g) \). We first check that \( \text{Aut}(g) \) is smooth at \( e \). Let \( L := \mathcal{D}(g) \subset \text{End}(g) \). We take \( B = L \) and define \( f(D) = \exp(D), D \in L \). It is clear that the conditions \( \alpha \) and \( \beta \) are satisfied. The validity of \( \gamma \) follows from Problem 5.

For arbitrary \( g \in \text{Aut}(g) \) we take \( B = L := \mathcal{D}(g) \) and define \( f_g : B \to \text{Aut}(g) \) by \( f_g(D) = g \exp(D), D \in L \). \( \square \)

**Problem 0.7.** Let \( G \subset \text{Aut}(g) \) be the subgroup generated by \( \exp(x), x \in \mathcal{D}(g) \). Show that

a) \( \text{Aut}(g) \) is closed in \( \text{GL}(V) \).

b) \( G \) is an open subgroup of \( \text{Aut}(g) \).

c) \( G \) is closed in \( \text{Aut}(g) \) [and therefore in \( \text{GL}(V) \)].

**Definition 0.8.** a) Let \( V \) be an \( \mathbb{R} \)-vector space, and \( \mathfrak{h} \subset \text{End}(V) \) a Lie subalgebra. We denote by \( G_{\mathfrak{h}} \) the subgroup of \( \text{GL}(V) \) generated by \( \exp(x), x \in \mathfrak{h} \) and by \( \bar{G}_{\mathfrak{h}} \) the closure of \( G_{\mathfrak{h}} \) in \( \text{GL}(V) \).

b) We say that a Lie subalgebra \( \mathfrak{h} \subset \text{End}(V) \) is algebraic if \( \bar{G}_{\mathfrak{h}} = G_{\mathfrak{h}} \) [that is if \( G_{\mathfrak{h}} \) is closed in \( \text{GL}(V) \)].

**Remark** As follows from Problem 7 that the answer is positive if \( \mathfrak{h} = \mathcal{D}(g) \subset \text{End}(g) \).

**Problem 0.9.** Show that

a) If \( \mathfrak{h} = \mathbb{R}x \) then the map \( t \to \exp(tx), t \in \mathbb{R} \) defines a surjection \( \mathbb{R} \to G_{\mathfrak{h}} \).

b) For any \( n > 1 \) there exists \( x \in \text{End}(\mathbb{R}^n) \) such that \( \dim(\bar{G}_{\mathbb{R}x}) = n - 1 \). [So the Lie subalgebra \( \mathfrak{h} = \mathbb{R}x \) is not algebraic].

**Remark** There exists a purely algebraic criterion for algebraicity of a Lie subalgebra \( \mathfrak{h} \subset \text{End}(V) \). using this criterion one can show that any Lie subalgebra \( \mathfrak{h} \subset \text{End}(V) \) such that \( [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h} \) is algebraic.