In this lecture we assume that $k = \mathbb{R}$ [or $\mathbb{C}$]. The results are true if k is any field of characteristic but the statements and proofs require the knowledge of basics of Algebraic Geometry.

Definition 0.1. *a) For any finite-dimensional vector space V we define $GL(V) := \{g \in End(V) | det(g) \neq 0\}$. It is clear that $GL(V)$ is a group.*

b) Let $Q : V \to \mathbb{R}$ be a positive definite quadratic form [=Euclidean structure on V]. For any $A \in End(V)$ we define

$$\|A\| := max\{Q(Av)/Q(v)\}|v \in V - \{0\}$$

c) Let W be a real vector space and X a subset of W. We say that X is is smooth at $x \in X$ if there exists an open subset B of a vector space $L, 0 \in B$ and a smooth [=∞-differentiable] map $f : B \to W$ of such that

α) $f(0) = x$

β) The differential $df_0 : L \to W$ is an imbedding and

γ) there exists an open neighborhood $U \subset W$ of x in W such that $U \cap X \subset f(B)$.

d) We say that X is a submanifold of W if every point $x \in X$ is smooth.

Problem 0.2. *a) For any $A \in End(V)$ the series $\exp(A) := \sum_{i=0} A^i/i!$ is convergent and $\exp(A)\exp(-A) = Id_V$.*

b) Any $g \in End(V)$ such that $\|g - Id_V\| < 1$ belongs to $GL(V)$. Moreover $g^{-1} = \sum_{i=0}(-1)^i A^i$ where $A := g - Id_V$.

c) For any $g \in End(V)$ such that $\|g - Id_V\| < 1$ the series $\ln(g) := -\sum_{i=1}(-A)^i/i$ is convergent and $\exp(\ln(g)) = g$

Definition 0.3. *Let $\mathfrak{g}$ be a Lie algebra. We define*

$Aut(\mathfrak{g}) := \{g \in GL(\mathfrak{g}) | [g(x), g(y)] = g([x, y], \forall x, y \in \mathfrak{g}\}.$

Lemma 0.4. *Show that for any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ and any $D \in= \mathcal{D}(\mathfrak{g}) \subset$ [see the definition 9 of Lecture 1] we have $\exp(D) \in Aut(\mathfrak{g})$.*

Proof. Since $D \in \mathcal{D}(\mathfrak{g})$ we have $D([x, y]) \equiv [Dx, y] + [x, Dy]$. Therefore

$$D^2([x, y])/2 \equiv D([Dx, y])/2+D([x, Dy])/2 = [D^2x, y]/2+[Dx, Dy]+[x, D^2y]/2$$

By induction one shows that for any $n.0$ we have

$$D^n([x, y])/n! = \sum_{i+j=n} \binom{n}{i} [D^i(x), D^j(y)]/n! = \sum_{i+j=n} [D^i(x)/i!, D^j(y)/j!]$$

So $\exp(D)[x,y] =)[\exp(D)x, \exp(D)y]$ □

Problem 0.5. *Let Q be a Euclidean structure on $\mathfrak{g}$ and $g \in Aut(\mathfrak{g})$ be such that $\|A\| < 1, A := g - Id_V$. Then $\ln(g) \in \mathcal{D}(\mathfrak{g})$*

Theorem 0.6. *$Aut(\mathfrak{g})$ is a submanifold of $End(\mathfrak{g})$.*

Proof. It is clear that $e := Id_V \in Aut(\mathfrak{g})$. We first check that $Aut(\mathfrak{g})$ is smooth at e. Let $L := \mathcal{D}(\mathfrak{g}) \subset End(\mathfrak{g})$. We take $B = L$ and define $f(D) = \exp(D), D \in L$. It is clear that the conditions α and β are satisfied. The validity of γ follows from Problem 5.

For arbitrary $g \in Aut(\mathfrak{g})$ we take $B = L := \mathcal{D}(\mathfrak{g})$ and define $f_g : B \to Aut(\mathfrak{g})$ by $f_g(D) = g\exp(D), D \in L$. □

Problem 0.7. *Let $G \subset Aut(\mathfrak{g})$ be the subgroup generated by $\exp(x), x \in \mathcal{D}(\mathfrak{g})$. Show that*

a) $Aut(\mathfrak{g})$ is closed in $GL(V)$.

b) G is an open subgroup of $Aut(\mathfrak{g})$.

c) G is closed in $Aut(\mathfrak{g})$ [and therefore in $GL(V)$].

Definition 0.8. *a) Let V be an $\mathbb{R}$-vector space, and $\mathfrak{h} \subset End(V)$ a Lie subalgebra. We denote by $G_{\mathfrak{h}}$ the subgroup of $GL(V)$ generated by $\exp(x), x \in \mathfrak{h}$ and by $\bar{G}_{\mathfrak{h}}$ the closure of $G_{\mathfrak{h}}$ in $GL(V)$.*

b) We say that a Lie subalgebra $\mathfrak{h} \subset End(V)$ is algebraic if $\bar{G}_{\mathfrak{h}} = G_{\mathfrak{h}}$ [that is if $G_{\mathfrak{h}}$ is closed in $GL(V)$].

Remark As follows from Problem 7 that the answer is positive if $\mathfrak{h} = \mathcal{D}(\mathfrak{g}) \subset End(\mathfrak{g})$.

Problem 0.9. *Show that*

a) If $\mathfrak{h} = \mathbb{R}x$ then the map $t \to exp(tx), t \in \mathbb{R}$ defines a surjection $\mathbb{R} \to G_{\mathfrak{h}}$.

b) For any $n > 1$ there exists $x \in End(\mathbb{R}^n)$ such that $dim(\bar{G}_{\mathbb{R}x}) = n - 1$. [So the Lie subalgebra $\mathfrak{h} = \mathbb{R}x$ is not algebraic].

Remark There exists a purely algebraic criterion for algebraicity of a Lie subalgebra $\mathfrak{h} \subset End(V)$. using this criterion one can show that any Lie subalgebra $\mathfrak{h} \subset End(V)$ such that $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ is algebraic.