

## 1. $L$ -FUNCTIONS

Let  $N$  be a positive number,  $\mathbb{Z}/N\mathbb{Z}$  the ring of residues mod  $N$  and  $G = \mathbb{Z}/N\mathbb{Z}^\star \subset \mathbb{Z}/N\mathbb{Z}$  the multiplicative group of invertible elements of the ring  $\mathbb{Z}/N\mathbb{Z}$ . We write elements of  $G \subset \mathbb{Z}/N\mathbb{Z}$  as  $\bar{a}$  where  $a \in \mathbb{Z}$  is a number prime to  $N$  and  $\bar{a} \in \mathbb{Z}/N\mathbb{Z}$  is the residue of  $a$  mod  $N$ .

**Definition 1.1.** a) A *character* of  $G$  is a function  $\tilde{\chi} : G \rightarrow \mathbb{C}^\star$  such that  $\tilde{\chi}(\bar{1}) = 1$  and  $\tilde{\chi}(gh) = \tilde{\chi}(g)\tilde{\chi}(h)$  for all  $g, h \in G$ .

b) We denote by  $G^\vee$  the set of characters of the group  $G$  and by  $\mathbf{1} \in G^\vee$  the *trivial* character  $\mathbf{1}(\bar{a}) \equiv 1$ .

c) For any character  $\tilde{\chi} \in G^\vee$  we denote by  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  the corresponding *primitive Dirichlet character* which is the function such that  $\chi(n) := \tilde{\chi}(\bar{n})$  if  $n$  is prime to  $N$  and  $\chi(n) = 0$  otherwise.

d) For any  $\tilde{\chi} \in G^\vee$  we define the Dirichlet  $L$ -function  $L_\chi(s)$  for  $s \in \mathbb{C}, \operatorname{Re}(s) > 1$  by the absolutely convergent series

$$L_\chi(s) = \sum_{n \geq 1} \chi(n)/n^s$$

**Exercise 1.2.** a)  $\sum_{\tilde{\chi} \in G^\vee} \tilde{\chi}(\bar{a}) = 0$  for any  $g \in G, g \neq \bar{1}$ .

b) For any  $g, h \in G^\vee$  we have

$$\sum_{\tilde{\chi} \in G^\vee} \tilde{\chi}^{-1}(g)\tilde{\chi}(h) = 0$$

if  $g \neq h$  and

$$\sum_{\tilde{\chi} \in G^\vee} \tilde{\chi}^{-1}(\bar{a})\tilde{\chi}(\bar{b}) = |G|$$

if  $g = h$ .

c)  $|G| = N \prod_{p \in \mathcal{P}_N} (1 - 1/p)$  where  $\mathcal{P}_N$  is the set of primes dividing  $N$ .

It is clear that  $L_1(s) = \zeta(s)$ . The understanding of the analytic behavior of functions  $L_\chi(s)$  is central for Number theory.

**1.1. The Dirichlet theorem.** Let  $N$  be a positive number and  $a$  a positive number prime to  $N$ .

**Theorem 1.3.** *For any number  $a$  prime to  $N$  the arithmetic progression  $a, a + N, a + 2N, \dots$  contains an infinite numbers of primes.*

In the case when  $N = 1$  or  $N = 2$  the result was know to Euclead but for general  $N$  the theorem was proven by Lejeune Dirichlet in 1837.

We denote by  $\mathcal{P}$  the set of prime numbers.

**Lemma 1.4.** *For any  $\tilde{\chi} \in G^\vee$ ,  $\tilde{\chi} \neq \mathbf{1}$  the series  $L_{\tilde{\chi}}(s) = \sum_{n \geq 1} \tilde{\chi}(n)/n^s$  is convergent for any  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 0$ .*

*Proof.* The result follows from Exercise 1.2 and the following result from Calculus

**Exercise 1.5.** Let  $a_n, n \geq 1$  a decreasing sequence of positive numbers such that  $\lim_{n \rightarrow \infty} a_n = 0$  and  $b_n, n \geq 1$  a sequence of complex numbers such that the sequence  $B_n := \sum_{k=1}^n b_k$  is bounded. Then the series  $\sum_{n \geq 1} a_n b_n$  is convergent.

□

The following proposition is an analogue of the Theorem 4.3 in in Lecture 1.

**Proposition 1.6.**  $L_{\tilde{\chi}}(1) \neq 0$  for any  $\tilde{\chi} \in G^\vee$ ,  $\tilde{\chi} \neq \mathbf{1}$

The proof of Proposition 1.6 is not too difficult [ see for example the book of Ram Murty “Problems in Analytic Number Theory” pp.24-29] but it would take us away from our theme.

We start the proof of the Dirichlet theorem with the following observation which follows immediately from the uniqueness of the decomposition to prime factors [see the proof of Lemma 1.1 in in Lecture 1].

**Exercise 1.7.** For any  $s > 1$  and  $\tilde{\chi} \in G^\vee$  we have

$$L_{\tilde{\chi}}(s) = \prod_{p \notin \mathcal{P}_N} (1 - \tilde{\chi}(p)/p^s)$$

Consider now the function  $\ln(L_{\tilde{\chi}}(s))$ ,  $\tilde{\chi} \in G^\vee$ . For  $s > 1$  we have

$$\begin{aligned} -\ln(L_{\tilde{\chi}}(s)) &= \sum_{p \notin \mathcal{P}_N} -\ln(1 - \tilde{\chi}(p)/p^s) = \sum_{p \notin \mathcal{P}_N} (\tilde{\chi}(p)/p^s + 1/2 \tilde{\chi}^2(p)/p^{2s} + \dots) = \\ &= \sum_{p \in \mathcal{P}_N} (\tilde{\chi}(p)/p^s) + F_{\tilde{\chi}}(s) \end{aligned}$$

**Exercise 1.8.** The series for  $F_{\tilde{\chi}}(s)$  is absolutely convergent for  $s > 1/2$ .

**Corollary 1.9.** *For any  $\chi \neq \mathbf{1} \in G^\vee$  the sum  $\sum_{p \in \mathcal{P}} (\chi(p)/p^s)$  is convergent for  $s > 1/2$ .*

Let  $\mathcal{P}^a \subset \mathcal{P}$  be the set of primes congruent to  $a \bmod N$ . To show that the set  $\mathcal{P}^a$  is infinite it is sufficient to show that

$$\lim_{s \rightarrow 1^+} (s-1) f_a(s) \neq 0$$

where the function  $f_{\tilde{a}}(s)$  is defined for  $s > 1$  by the absolutely convergent series  $f_{\tilde{a}}(s) := \sum_{p \in \mathcal{P}^a} p^{-s}$ . But it follows from Exercise 1.2 b) that

$$f_{\tilde{a}}(s) = \frac{1}{|G|} \sum_{\tilde{\chi} \in G^\vee} \chi^{-1}(a) L_\chi(s)$$

for any  $s, \operatorname{Re}(s) > 1$ . Therefore

$$\lim_{s \rightarrow 1^+} (s-1) f_{\tilde{a}}(s) = \frac{1}{|G|} \sum_{\tilde{\chi} \in G^\vee} \lim_{s \rightarrow 1^+} (s-1) \chi^{-1}(a) L_\chi(s)$$

Since, by Corollary 1.9, for any non-trivial character  $\tilde{\chi} \in G^\vee$  the  $\sum_{p \in \mathcal{P}} (\chi(p)/p^s)$  is absolutely convergent for all  $s > 1/2$   $\lim_{s \rightarrow 1^+} (s-1) \tilde{\chi}^{-1}(a) L_\chi(s) = 0$  for all  $\chi \neq 1$ . It follows now from Claim 1.2 in Lecture 1 that

$$\lim_{s \rightarrow 1^+} (s-1) f_{\tilde{a}}(s) = 1/N \neq 0. \square$$

**Exercise 1.10.** a) Fill details in the proof of Theorem 1.1.1 in “Bump”.

b) Prove exercises 1.1.1-1.1.3 in “Bump”

**1.2. The functional equation.** The formulation and a proof of the functional equation for  $L$ -functions is in “Bump”. The proof uses some properties of the Fourier transform which I formulate now. As before we denote by  $\mathcal{S}(\mathbb{R})$  the space of smooth functions  $f(x)$  such that  $\lim_{x \rightarrow \pm\infty} x^a f^{(b)}(x) = 0$  for all  $a, b \geq 0$ . For any  $f \in \mathcal{S}(\mathbb{R})$  we define the Fourier transform  $\hat{f}(y), y \in \mathbb{R}$  by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx$$

**Exercise 1.11.** Let  $\mathcal{S}(\mathbb{R})$  be the space of functions as in section 3 of the Lecture 1 and  $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  be the Fourier transform. Then

$$\widehat{x f} = \frac{i}{2\pi} \frac{\partial}{\partial y} (\hat{f}), \widehat{\frac{\partial}{\partial x} f} = 2\pi i y (\hat{f})$$

for any  $f \in \mathcal{S}(\mathbb{R})$