There two topics I want to discuss- the construction of root systems and the proof of the Serre's theorem.

The construction of root systems.As you know [Theorem 6.6] any finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ defines a [reduced] root system which is irreducible if the Lie algebra is simple. You have checked already that root systems corresponding to simple Lie algebras of types A_n, B_n, C_n, D_n are root systems of types A_n, B_n, C_n, D_n. It is also clear that for a root system R with the Dynkin diagram D [remember that vertices of $D \leftrightarrow$ simple roots] and any subset D' of the set D of simple roots there is a roots system R' with the Dynkin diagram D' [here we consider D' as a subset of D]. So the construction of a root system of the type E_8 gives as immediately constructions of a root systems of the types E_6 and E_7.

There are different ways to construct a root system of the type E_8. One can take the corresponding Cartan matrix A and check that it is positive definite. If so A defines a positive definite quadratic form on $E = \mathbb{R}^8$. Since the entries of A are integers the reflections $s_i, 1 \leq t \leq 8$ preserve the lattice Λ [a span of a basis over $\mathbb{Z}$] of linear combinations of the standard basis $\Delta = \{e_1, \dots, e_8\}$ with integral coefficients. It is easy to see now [please check] the that the subgroup $W \subset Aut(E)$ generated by reflection $s_i, 1 \leq t \leq 8$ is finite and the set $R := W\Delta \subset E$ is a root system of the type E_8.

On the other hand one can construct explicitly a root system $R \subset \mathbb{R}^8$ of the type E_8 where we consider the standard Euclidean structure on $\mathbb{R}^8$. This is done in Problem $1g$) of the homework 10.

The Serre's theorem. We have seen [Theorem 6.6] that any finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ defines a [reduced] root system. Let $\Delta = \{\alpha_i\}, i \in I$ be the corresponding system of simple roots and $A = (a_{ij}), a_{ij} =< \alpha_i^\vee, \alpha_j >$ be the corresponding Cartan matrix. Then there exist elements $e_i, h_i, f_i \in \mathfrak{g}$ such that

$[h_i, h_j] = 0, [h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j$ and $[e_i, f_j] = \delta_{ij}h_i$.

Moreover it is easy to see [Theorem 7.52] that

$$ad_{e_i}^{1-a_{ij}}(e_j) = ad_{f_i}^{1-a_{ij}}(f_j) = 0$$

Let now $A = (a_{ij})$ be Cartan matrix of a reduced root system. We denote by L_A the Lie algebra with generators $x_i, \tilde{h}_i, y_i, i \in I$ and relations

$\{S(1)\}$ $[\tilde{h}_i, \tilde{h}_j] = 0$

$\{S(2)\}$ $[x_i, y_i] = h_i$ and $[x_i, y_j] = 0$ if $i \neq j$

$\{S(3)\}$ $[\tilde{h}_i, x_j] = a_{ij}x_j, [\tilde{h}_i, y_j] = -a_{ij}y_j$

$\{S_{ij}^+\}$ $ad_{x_i}^{1-a_{ij}} x_j = 0$ and

$\{S_{ij}^-\}$ $ad_{y_i}^{1-a_{ij}} y_j = 0$

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra with A as the Cartan matrix. As follows from Theorem 7.52 there is a surjective Lie algebra homomorphism $f : L_A \to \mathfrak{g}$ such that $f(x_i) = e_i, f(\tilde{h}_i) = h_i, f(y_i) = f_i$. Serre proved that $f : L_A \to \mathfrak{g}$ is an isomorphism. Moreover he proved the following result.

Theorem 0.1. *For any Cartan matrix A of a reduced root system the Lie algebra L_A is a finite-dimensional semi-simple Lie algebra with the root system corresponding to A.*

This is an extremely important result leading to the theory of Kac-Moody algebras which is central for many areas of mathematics.

I posted the proof of the Theorem 1 which is presented in the book of Humphreys "Introduction to Lie algebras and representation theory". The proof consists in a large number of simple steps. I think that it could be helpful to outline the general idea of the proof. To simplify notations I'll write L instead of L_A and h_i instead of $\tilde{h}_i$.

Step 1.

Let L_0 be the Lie algebra with generators $x_i, h_i, y_i, i \in I$ and relations

$S(1)$ $[h_i, h_j] = 0$

$S(2)$ $[x_i, y_i] = h_i$ and $[x_i, y_j] = 0$ if $i \neq j$

$S(3)$ $[h_i, x_j] = a_{ij}x_j, [h_i, y_j] = -a_{ij}y_j$

We denote by $X, Y \subset L_0$ the Lie algebras generated by x_i and y_i and denote by $H \subset L_0$ the span of h_i.

Claim 1.1 a) $L_0 = X \oplus H \oplus Y, [H, X] \subset X, [H, Y] \subset Y, [X, Y] \subset H$.

b) The elements $x_i, h_i, y_i \in L_0, i \in I$ are linearly independent.

The proof of a) is not difficult and could be left as an exercise. The proof of b) is based on a construction of an explicit representation $\rho : L_0 \to End(V)$ of the Lie algebra L_0.

To construct such a representation we observe [guess] that the relations $S(1), S(2), S(3)$ do not contain any relations between x_i of between y_i. So [we could expect that] X is the free Lie algebra on the vector space $M^+ = \oplus_{i\in I} kx_i$ and Y is the free Lie algebra on the vector space $M^- = \oplus_{i\in I} ky_i$. As follows from the PBV theorem and the part a) the natural map [the product] $U(Y) \otimes U(H) \otimes U(X) \to U(L_0)$ is a bijection [see the Problem 4 b) in homework 10] and the composition $U(Y) \to U(L_0) \to V := U(L_0)/U(L_0)(H \oplus X)$ is also bijection. Since [we expect that] Y is the free Lie algebra on the vector space $M^- = \oplus_{i\in I} ky_i$ we can identify $V = U(Y)$ with $T(M^-)$. So $V = \oplus_{n\geq 0} V_n := T^n(M^-)$ where V_n has a basis consisting of $v_{i_1,\dots,v_n} = v_{i_1} \otimes \dots \otimes v_{i_n}$ [$V_0 = k$]. For any $l \in L_0$ we write $\hat{l} \in End(V)$ instead of $\rho(l)$. By the construction

$$\hat{y}_j v_{i_1,\dots,v_n} = v_{j,i_1,\dots,v_n}, \hat{x}_i 1 = \hat{h}_i 1 = 0$$

These equalities and the relations $S(1), S(2), S(3)$ uniquely define uniquely the operators $\hat{x}_i, \hat{h}_i \in End(V)$.

Now we can turn the table around, define V as the space with a basis $v_{i_1,\dots,v_n}, n \geq 0$ and write *expected* formulas from operators $\hat{x}_i, \hat{h}_i, \hat{y}_i \in End(V)$. One checks the relations $S(1), S(2), S(3)$ [it is easy] and obtains a representations $\rho : L_0 \to End(V)$.

For any $\lambda \in H^\vee$ we define $(L_0)_\lambda := \{l \in L_0 | ad(h)l = \lambda(h)l, \forall h \in H\}$. We define $\Phi := \{\lambda \in H^\vee | (L_0)_\lambda \neq \{0\}\}$.

Claim 1.2 a) $L_0 = \oplus_{\lambda\in\Phi}(L_0)_\lambda$.

b) $(L_0)_0 = H$.

c) If $\lambda \in \Phi - \{0\}$ then $\lambda = \sum_{i\in I} k_i\alpha_i, k_i \in \mathbb{Z}$ and either all k_i are non-negative or all k_i are non-positive.

d) $dim((L_0)_{\pm\alpha_i}) = 1$ and $dim((L_0)_{k\alpha_i}) = 0$ if $|k| > 1$.

The proof of Claim 1.2 is easy.

Claim 1.3 The elements $[y_k S^+_{ij}] = [x_k, S^-_{ij}] = 0, \forall i, j, k \in I$.

The proof of Claim 1.3 is also not difficult-[it follows from Problem 3 c) in the homework. [The non-trivial part was to have a right guess].

Step 2.

Let $I \subset X, J \subset Y$ be ideals in X and Y generated by the relations S^+_{ij} and S^-_{ij} correspondingly and $K \subset L_0$ be the ideal generated by the relations S^+_{ij} and S^-_{ij}.

Claim 2.1 a) $K = I \oplus J$.

b) Elements $x_i, h_i, y_i \in L, i \in I$ are linearly independent.

The proof of Claim 2.1 is not difficult to derive from Claim 1.3.

Claim 2.2 The operators $ad_{x_i}, ad_{y_i} \in End(L), i \in I$ are locally nilpotent [see the definition in the homework 10].

The proof of Claim 2.2 follows immediately from the problem 3 a) in the homework 10.

Now comes the central construction. Since $ad_{x_i}, ad_{y_i} \in End(L), i \in I$ are locally nilpotent are locally nilpotent we can define

$$\tau_i := \exp(x_i)\exp(-y_i)\exp(x_i) \in Aut(L)$$

Claim 2.2 $\tau_i((L_0)_\lambda) = (L_0)_{s_i(\lambda)}$.

The proof of Claim 2.3 follows immediately from the problem 3 e) in the homework 10.

Claim 2.3 If $\lambda \neq 0$ and $(L_0)_\lambda \neq \{0\}$ then $\lambda \in R$ and $dim((L_0)_\lambda) = 1$.

The proof of Claim 2.3 follows immediately from Claim 1.2 c) and d) and the problem 1d) in the homework 10.