

1. INTRODUCTION

One of the main applications of the theory of automorphic forms is in the theory of L -functions. L -functions $L(z)$ are given as products $\prod_{p \in \mathcal{P}} L_p(z)$ where \mathcal{P} is the set of prime numbers and $L_p(z)$ have a form $L_p(z) = \frac{1}{r_p(p^{-z})}$ where $r_p(t)$ is a polynomial. These products are absolutely convergent for $\operatorname{Re}(z) \gg 0$. In the case when $L_p(z)$ admits the meromorphic continuations to \mathbb{C} and satisfies the appropriate functional equations one can obtain very interesting results in Number Theory.

We start with the study of the Riemann ζ -function $\zeta(z)$ given in the half-plane $\operatorname{Re}(z) > 1$ by an absolutely convergent product $\prod_{p \in \mathcal{P}} \frac{1}{1-p^{-z}}$. We will see that $\zeta(z)$ has a meromorphic continuation to \mathbb{C} with the only pole at $z = 1$ and satisfies the functional equation which related $\zeta(1-z)$ and $\zeta(z)$. These properties of the ζ -function are central for the understanding of the distribution of prime number. Analogously the existence of meromorphic continuations and functional equations for other L -functions is the main tool for solutions of many questions in the Number Theory.

Lemma 1.1. $\zeta(z) = \sum_{n>0} n^{-z}$ for all $z, \operatorname{Re}(z) > 1$.

Proof. For any $N > 0$ let $p_1 < p_2 < \dots < p_{r_N}$ be the set of prime numbers smaller than N . Consider the function $\zeta_N(z) := \prod_{p \in \mathcal{P}, p < N} \frac{1}{1-p^{-z}}$. Since $\frac{1}{1-p^{-z}} = \sum_{k \geq 0} p^{-kz}$ we see that

$$\zeta_N(z) = \sum_{k_i \geq 0 | 1 \leq i \leq r_N} \prod_{i=1}^{r_N} p_i^{k_i}$$

Since any positive integer can be decomposed uniquely as the product of power of different prime numbers we see that $\zeta_N(z) = \sum_{n \in \mathbb{Z}_N} n^{-z}$ where \mathbb{Z}_N is the set of natural numbers which are product of primes smaller than N . Since $\zeta(z) = \lim_{N \rightarrow \infty} \zeta_N(z)$ for all $z, \operatorname{Re}(z) > 1$ we see that $\zeta(z) = \sum_{n>0} n^{-z}$ for all $z, \operatorname{Re}(z) > 1$. \square

To prove the existence of a meromorphic continuation and a functional equation for the Riemann ζ -function we will start with some general results in complex analysis.

Claim 1.2. a) The function $\zeta(z)$ extends to a meromorphic function on \mathbb{C} with the only simple pole at $z = 1$ with the residue equal to 1.

b) $\zeta(z) \neq 0$ if $\operatorname{Re}(z) \geq 1$.

We will prove the first part of the Claim later and I'll indicate how to prove the second part.

I'll start the course with deducing the Prime Number Theorem from Claim 1.2. The deduction is based on the following general result in complex analysis which I present without a proof.

Definition 1.3. A Dirichlet series is a function given by a series $f(z) = \sum_{n>0} \frac{a_n}{n^z}$ which is absolutely convergent for z with $\operatorname{Re}(z) \gg 0$

Theorem 1.4. Assume that a Dirichlet series $f(z) = \sum_{n>0} \frac{a_n}{n^z}$ is convergent for $\operatorname{Re}(z) > 1$ and extends to a meromorphic function on a neighborhood of the line $\operatorname{Re}(z) = 1$ which is regular outside $z = 0$ and has a simple pole at $z = 0$ with the residue 1. Then $A_f(x) \sim x$ where $A_f(x) := \sum_{n<x} a_n$. [That is $\lim_{x \rightarrow \infty} \frac{A_f(x)}{x} = 1$].

Let $\Lambda(n)$ be the von Mangoldt function given by $\Lambda(n) = \ln(p)$ if $n = p^k$ where p is a prime number and $\Lambda(n) = 0$ if n is not a power of a prime number.

Theorem 1.5. Claim 1.2 implies that $\sum_{n<x} \Lambda(n) \sim x$

Proof. We start with a number of simple results. For any sequence a_n we define a function $A(x) := \sum_{n<x} a_n$

Problems 1.6. a) Let $f(z)$ be a meromorphic function on \mathbb{C} with poles at $a_i, i > 0$ of orders l_i and zeros at $b_i, i > 0$, of orders m_i . Then the logarithmic derivative $F(z) := \frac{f'(z)}{f(z)}$ is a meromorphic function on \mathbb{C} with simple poles at a_i and b_i .

b) The residue of $F(z)$ at a_i is equal to $-l_i$ and the residue of $F(z)$ at b_i is equal to m_i .

c) $F(z) = \ln'(f(z))$

d) [Partial summation]. For any sequence a_n and a continuously differentiable function $h(x), x > 1$ we have

$$\sum_{n<x} a_n h(n) = A(x)h(x) - \int_1^x A(u)h'(u)du$$

To prove the Theorem 1.5 we observe that the equality $\frac{\zeta'(z)}{\zeta(z)} = -\sum_{p \in \mathcal{P}} \frac{(1-p^{-z})'}{1-p^{-z}}$ follows immediately from the part c) of Problem 1.6. Since $\frac{1}{1-p^{-z}} = \sum_k (p^k)^{-z}$ and $(1-p^{-z})' = \ln(p)p^{-z}$ we see that $-\frac{\zeta'(z)}{\zeta(z)} = \sum_n \frac{\Lambda(n)}{n^z}$.

As follows from Claim 1.2 and Problem 1.6 b) the function $\frac{\zeta'(z)}{\zeta(z)}$ satisfies the conditions of Theorem 1.5. Therefore $\lim_{x \rightarrow \infty} \frac{\alpha(x)}{x} = 0$ where

$$\alpha(x) := \sum_{n < x} \Lambda(n) - x. \square$$

Corollary 1.7. $\lim_{x \rightarrow \infty} \frac{\pi(x) \ln(x)}{x} = 1$ where $\pi(x)$ is the number of primes smaller than x .

Proof. The difference $\sum_{n < x} \Lambda(n) - \sum_{p \in \mathcal{P}, p < x} \ln(p) = \sum_{k > 1, p \in \mathcal{P} | p^k < x} \ln(p)$ is a sum of less than \sqrt{x} term where each term is $< \ln(x)$. So $|\sum_{n < x} \Lambda(n) - \sum_{p \in \mathcal{P}, p < x} \ln(p)| < \sqrt{x} \ln(x)$. Let $\beta(x) := \sum_{p \in \mathcal{P}, p < x} \ln(p) - x = o(x)$. We see that $\beta(x) = o(x)$ [that is $\lim_{x \rightarrow \infty} \frac{\beta(x)}{x} = 0$]. If we apply the partial summation [Problem 1.6 d)] to the sequence $a_p = \ln(p), p \in \mathcal{P}$ and the function $h(x) = \frac{1}{\ln(x)}$ we find that

$$\pi(x) \equiv \sum_{p < x} a_p h(p) = \frac{x + \beta(x)}{\ln(x)} + \int_2^x (u + \beta(u)) \frac{1}{u \ln^2(u)} du$$

Since $\beta(x) = o(x)$ we see that it is sufficient for the proof of Corollary 1.7 to show that $\int_2^x \frac{1}{\ln^2(u)} du < \frac{1}{\ln^2(x)}$.

To estimate the integral we write it as a sum $I' + I''$ where $I' = \int_2^{\sqrt{x}} \frac{1}{\ln^2(u)} du$ and $I'' = \int_{\sqrt{x}}^x \frac{1}{\ln^2(u)} du$. But it is clear that $I' \leq \sqrt{x} \max_{2 < u < \sqrt{x}} \frac{1}{\ln^2(u)} < \sqrt{x}$ and $I'' < x \max_{\sqrt{x} < u < x} \frac{1}{\ln^2(u)} < \frac{1}{\ln^2(x^{1/2})}$. \square

Conjecture 1.8 (The Riemann conjecture). $\zeta(z) \neq 0$ if $\operatorname{Re}(z) > 1/2$

Remark 1.9. The Riemann conjecture is equivalent to the following result on the distribution of prime numbers.

For any $\epsilon > 0$ there exists c_ϵ such that $|\sum_{n < x} \Lambda(n) - x| < c_\epsilon x^{1/2+c_\epsilon}$ which can be restated in the form

$$|\pi(x) - \operatorname{li}(x)| < c_\epsilon x^{1/2+c_\epsilon} \text{ where } \operatorname{li}(x) := \int_2^x \frac{1}{\ln(u)} du.$$

2. ANALYTIC CONTINUATION OF THE ζ -FUNCTION

To prove the existence of a meromorphic continuation and a functional equation for the Riemann ζ -function we will start with some general results in complex analysis.

Let $\phi(t) : [0, \infty] \rightarrow \mathbb{C}$ be a smooth [=infinitely differentiable] function such that $\lim_{t \rightarrow \infty} \phi(t) t^n = 0$ for all $n > 0$. For any $z \in \mathbb{C}, \operatorname{Re}(z) > 0$ we define $M_\phi(z) := \int_0^\infty \phi(t) t^{z-1} dt$.

Problem 2.1. Show that the integral is absolutely convergent and that M_ϕ is a holomorphic function on the half-plane $\operatorname{Re}(z) > 0$. Moreover if $\lim_{t \rightarrow 0} \phi(t)/t^n = 0$ then M_ϕ extends to a holomorphic function on the half-plane $\operatorname{Re}(z) > -n - 1$.

Lemma 2.2. *The function M_ϕ extends to a meromorphic function on \mathbb{C} with simple poles at $\{-n\}$, $n > 0$ and residues equal to $\phi^{(n)}(0)/n!$.*

Proof. Define $M'_\phi(z) := \int_0^1 \phi(t)t^{z-1}dt$ and $M''_\phi(z) := \int_1^\infty \phi(t)t^{z-1}dt$ for $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$. It is clear that $M''_\phi(z)$ extends to a holomorphic function on \mathbb{C} . On the other hand for any $n > 0$ we have $\phi(t) = \sum_0^n \frac{\phi^{(m)}(0)}{m!}t^m + \phi_n(t)$ where $\lim_{t \rightarrow 0} \phi_n(t)/t^n = 0$ and . therefore

$$M'_\phi(z) = \sum_0^n \frac{\phi^{(m)}(0)}{m!} \int_0^1 t^{z-1+m}dt + M'_{\phi_n}(z)$$

So Lemma 2.2 follows from the equality $\int_0^1 t^{z-1+n}dt = \frac{1}{z-1-n}$

Problem 2.3. Extend the formulation and the proof of the Lemma 2.2 to the case when the function ϕ has a pole at $t = 0$.

We define $\Gamma(z) := M_{e^{-t}} = \int_0^\infty e^{-t}t^{z-1}dt$. As follows from Lemma 2.2 $\Gamma(z)$ is a meromorphic function with only poles at $z = -n$, $n \geq 0$. It is clear that $\Gamma(1) = 1$

Lemma 2.4. $\int_0^\infty \frac{e^{-t}}{1-e^{-t}}t^{z-1}dt = \Gamma(z)\zeta(z)$

Proof.

$$\int_0^\infty \frac{e^{-t}}{1-e^{-t}}t^{z-1}dt = \sum_{k>0} \int_0^\infty e^{-tk}t^{z-1}dt = \sum_{k>0} k^{-z} \int_0^\infty e^{-t}t^{z-1}dt = \Gamma(z)\zeta(z)$$

Corollary 2.5. *The function $\Gamma(z)\zeta(z)$ extends to a meromorphic function on \mathbb{C} with a simple pole at $z = 1$ with the residue equal to 1 and at most simple poles at $z = -n$, $n \geq 0$.*

We define Bernoulli numbers B_k by $\sum_{k=0}^\infty \frac{B_k}{k!}s^k = \frac{s}{e^s-1}$.

Problems 2.6. a) Show that

the function $\zeta(z) = \frac{\Gamma(z)\zeta(z)}{\Gamma(z)}$ extends to a meromorphic function on \mathbb{C} which is regular at $z = -n$, $n \geq 0$ and has a simple pole at $z = 1$ with the residue equal to 1.

b) $\zeta(1-k) = -\frac{B_k}{k}$ for all $k > 0$.

c) $\zeta(1-2k) = 0$ for all $k > 0$.

We will show that the meromorphic function $\Gamma(z)$ is never equal to 0. Therefore $\zeta(z) = \frac{\Gamma(z)\zeta(z)}{\Gamma(z)}$ extends to a meromorphic function on \mathbb{C} and the only singularity of $\zeta(z)$ is a simple pole at $z = 1$ with the residue equal to 1.

Problems 2.7. Show [using the integration by parts] that

- a) $\Gamma(z + 1) = z\Gamma(z)$
- b) $\int_0^n (1 - t/n)^n t^{z-1} dt = \frac{n!}{z(z+1)\dots(z+n)} n^z$
- c) $0 \leq e^{-t} - (1 - t/n)^n \leq \frac{t^2}{n} e^{-t}$
- d) $\Gamma(n) = (n - 1)!$ for $n > 1$

Lemma 2.8. $\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!}{z(z+1)\dots(z+n)} n^z$ for $\operatorname{Re}(z) > 0$.

Proof. Let $P_n(z) := \int_0^n (1 - t/n)^n t^{z-1} dt$. As follows from the part b) of the previous Lemma we have to show that $\lim_{n \rightarrow \infty} P_n(z) = \Gamma(z)$. But

$$\Gamma(z) - P_n(z) = \int_0^n (e^{-t} - (1 - t/n)^n) t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dt$$

The result follows now from the Problem 2.7 c). \square

Corollary 2.9. Let $\gamma := \lim_{n \rightarrow \infty} \sum_{m=1}^n 1/m - \ln(n)$. Then

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} [(1 + z/n) e^{-z/n}]$$

Proof. As follows from Lemma 2.8 we have $\frac{1}{\Gamma(z)} = \lim_{n \rightarrow \infty} \frac{z(z+1)\dots(z+n)}{n!} n^{-z}$. Using the equality

$$n^z = e^{z \ln(n)} = e^{z[\ln(n) - \sum_{m=1}^n 1/m]} \prod_{m=1}^n e^{z/m}$$

Problem 2.10. The product $\prod_{n=1}^{\infty} [(1 + z/n) e^{-z/n}]$ is absolutely convergent for all $z \in \mathbb{C}$.

We see that $\Gamma(z) \neq 0$ for all $z \in \mathbb{C}$.

3. POISSON SUMMATION FORMULA

The proof of the functional equation for $\zeta(z)$ and many other L -functions is based on the Poisson summation formula. I start with a reminder on the Fourier series and Fourier transform.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth periodic function [that is $f(x + 1) \equiv f(x)$]. We define $\hat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x} dx$.

Problems 3.1. a) For any smooth periodic function f and any $r > 0$ we have $\lim_{n \rightarrow \infty} \hat{f}(n)n^r = 0$.

[A hint. Use the integration by parts].

b) For any smooth periodic function f the series $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi i k x} dx$ is uniformly convergent.

Let

$$S_n(f)(x) := \sum_{k=-n}^n \hat{f}(k)e^{2\pi i k x} dx$$

Lemma 3.2. $f(x) = \lim_{n \rightarrow \infty} S_n(f)(x)$

Proof. I show that $\sum_{-\infty}^{\infty} \hat{f}(n)$ converges to $f(0)$ and leave for you to prove the general case. If $f \equiv 1$ then $\hat{f}(n) = 0$ for $n \neq 0$, $\hat{f}(0) = 1$ and so the claim is true. Therefore it is sufficient to prove the lemma for the function $\tilde{f}(x) := f(x) - f(0)$. In other words we can assume that $f(0) = 0$.

Since

$$\sum_{k=-n}^n e^{-2\pi i k x} = \frac{e^{(2n+1)\pi i x} - e^{-(2n+1)\pi i x}}{e^{\pi i x} - e^{-\pi i x}}$$

we have

$$S_n(f)(x) = \int_0^1 f(x) \frac{e^{(2n+1)\pi i x} - e^{-(2n+1)\pi i x}}{e^{\pi i x} - e^{-\pi i x}} dx = \int_0^1 g(x) (e^{(2n+1)\pi i x} - e^{-(2n+1)\pi i x})$$

where $g(x) := \frac{f(x)}{e^{\pi i x} - e^{-\pi i x}}$.

Since $f(0) = 0$ the function $g(x)$ is smooth and it follows from Problem 3.1 that $\lim_{n \rightarrow \infty} \int_0^1 g(x) e^{(2n+1)\pi i x} dx = 0$. \square

We denote by $\mathcal{S}(\mathbb{R})$ the space of smooth functions $f(x)$ such that $\lim_{x \rightarrow \pm\infty} x^a f^{(b)}(x) = 0$ for all $a, b \geq 0$. For any $f \in \mathcal{S}(\mathbb{R})$ we define the Fourier transform $\hat{f}(\lambda)$, $\lambda \in \mathbb{R}$ by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx$$

Problems 3.3. a) For any $f \in \mathcal{S}(\mathbb{R})$ the integral defining \hat{f} is absolutely convergent and $\hat{f} \in \mathcal{S}(\mathbb{R})$.

[A hint. Use the integration by parts]

b) $\int_{-\infty}^{\infty} e^{2\pi i x y} \hat{f}(y) dy = f(x) dx$.

[A hint. The proof is analogous of the proof of Lemma 4.1].

c) The function $F(x) := \sum_{n \in \mathbb{Z}} f(x+n)$ is well defined [the series is absolutely convergent] and $F(x)$ is a smooth periodic function.

d) $\hat{f}(x) = f(-x)$.

Proposition 3.4 (Poisson summation formula). *For any $f \in \mathcal{S}(\mathbb{R})$ we have $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$.*

Proof. Let $F(x) := \sum_{n \in \mathbb{Z}} f(x+n)$. Since $F(x)$ is a smooth periodic function we know that $F(0) = \sum_{n \in \mathbb{Z}} \hat{F}(n)$. On the other hand [by the definition] $\hat{F}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$. \square

Lemma 3.5. *Let $q_a(x) := e^{-a\pi x^2}$. Then $\hat{q}_a(y) = \frac{1}{\sqrt{a}} q_{1/a}(y)$.*

Proof. By the definition

$$\begin{aligned} \hat{q}_a(y) &= \int_{-\infty}^{\infty} e^{-a\pi x^2} e^{-2\pi ixy} dx = \int_{-\infty}^{\infty} e^{-\pi ax^2 - 2\pi ixy} dx = \\ &= e^{-\pi y^2/a} \int_{-\infty}^{\infty} e^{-a\pi(x + \frac{iy}{a})^2} dx = c_a(y) q_{1/a}(y) \end{aligned}$$

where $c_a(y) = \int_{-\infty}^{\infty} e^{-a\pi(x + \frac{iy}{a})^2} dx$. As follows from the Cauchy's theorem $c_a(y)$ does not depend on y and is equal to $\int_{-\infty}^{\infty} e^{-a\pi x^2} dx$. After a change of variables we see that $c_a(y) \equiv \frac{c}{\sqrt{a}}$ where $c = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$. The Lemma follows immediately from the following result.

Claim 3.6. $c = 1$

Proof of Claim.

To find c consider first

$$c^2 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx \times \int_{-\infty}^{\infty} e^{-\pi y^2} dy = \int \int e^{-\pi(x^2 + y^2)}$$

Using the polar coordinates we can write $c^2 = \int_{0 < \phi < 2\pi} \int_0^{\infty} r e^{-\pi r^2} dr$. After the change $t = \pi r^2$ of variables we find $c^2 = 2\pi \int_0^{\infty} 1/2\pi e^{-t} dt = 1$. Since c is positive we have $c = 1$. \square

Problem 3.7. For any $a > 0$ the series $\theta(a) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 a}$ is absolutely convergent and $\omega(t) := \theta(a) - 1 \in \mathcal{S}(\mathbb{R})$.

Corollary 3.8. $\theta(1/a) = \sqrt{a} \theta(a)$

Proof. Apply the Poisson summation formula to $f = q_{1/a}$ and use the equality $\hat{q}_{1/a} = \sqrt{a} q_a$. \square

4. BASIC PROPERTIES OF THE ζ -FUNCTION

Theorem 4.1 (Riemann). *The function $\xi(z) := \Gamma(z/2)\pi^{-z/2}\zeta(z)$ defined for $\operatorname{Re}(z) > 1$ has a meromorphic continuation to \mathbb{C} with simple poles at $z = 1$ and $z = -1$ and it satisfies the functional equation $\xi(z) = \xi(1 - z)$*

Proof. By the definition of $\Gamma(z)$ we see that

$$\int_{-\infty}^{\infty} t^{z/2-1} e^{-n^2 t} dt = \frac{\Gamma(z/2)}{n^z \pi^{1/2}}$$

Summing over $n \in \mathbb{N}$ we see that

$$\xi(z) = \int_0^{\infty} t^{z/2-1} \left(\sum_{n=1}^{\infty} e^{-\pi n^2 t} \right) dt = \int_0^{\infty} t^{z/2-1} \omega(t) dt$$

So

$$\xi(z) = \int_0^1 t^{z/2-1} \omega(t) dt + \int_1^{\infty} t^{z/2-1} \omega(t) dt = \int_1^{\infty} t^{-z/2-1} \omega(1/t) dt + \int_1^{\infty} t^{z/2-1} \omega(t) dt$$

Since $\omega(t) := \theta(t) - 1$ it follows from Corollary 4.9 that

$$\omega(1/t) = -\frac{1}{2} - \frac{\sqrt{t}}{2} + \sqrt{t} \omega(t)$$

So we see that

$$\xi(z) = \frac{1}{z(z-1)} + \int_1^{\infty} (t^{z/2-1} + t^{-z/2-1}) \omega(t) dt$$

Since $\omega(t) \in \mathcal{S}(\mathbb{R})$ the integral is absolutely convergent for all $z \in \mathbb{C}$. So $\xi(z)$ has a meromorphic continuation to \mathbb{C} with simple poles at $z = 1$ and $z = -1$. The functional equation $\xi(z) = \xi(1 - z)$ follows from the invariance of the integrand under transformation $z \rightarrow 1 - z$. \square

Problem 4.2. Calculate $\zeta(2k)$ for $k > 0$.

A hint. Use the results of Problem 2.6

Theorem 4.3. $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R} - \{0\}$.

Proof. To show that an assumption $\zeta(1 + it) = 0, t \in \mathbb{R} - \{0\}$ leads a contradiction we start with a series of simple results.

Problems 4.4. a) $3 + 4 \cos(\alpha) + \cos(2\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$.

b) $\ln(\zeta(\sigma + it)) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{p^{-k\sigma}}{k} e^{itk \ln(p)}$ for any $\sigma > 1$.

c) $3 \ln(\zeta(\sigma) + 4 \operatorname{Re}(\ln(\zeta(\sigma + it))) + \operatorname{Re}(\ln(\zeta(\sigma + 2it))) \geq 0$ for any $\sigma > 1$.

$$\text{d) } |(\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it))| \geq 1$$

We see that $|(\zeta^3(1+c)\zeta^4(1+c+it)\zeta(1+c+2it))| \geq 1$ for all $c > 0$. On the other hand since the function ζ is regular at $z = 1 + it$ and $\zeta(1+it) = 0$ there exists $A > 0$ such that $|\zeta(1+c+it)| < Ac$ for small $c > 0$. Since the function ζ is regular at $z = 1+2it$ and has a simple pole at $z = 1$ there exists $B > 0$ such that $|(\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it))| < Bc^2$ for $c \rightarrow 0$. The contradiction [since $|(\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it))| \geq 1$ for any $\sigma > 1$]. \square