16. Let A be a linear transformation in an n dimensional space. Suppose A has n distinct characteristic roots $\xi_1, \xi_2, \cdots, \xi_n$. Show that ad A acting in $\mathfrak{E}$ has the n^2 characteristic roots $\xi_i - \xi_j$, $i, j = 1, 2, \cdots, n$.

17. Let $\mathfrak{M}$ be a finite-dimensional module, $\mathfrak{M}^*$ the contragredient module. Show that if $\mathfrak{N}$ is a submodule of $\mathfrak{M}$, then $\mathfrak{N}^{\perp} = \{z^* \mid < y, z^* > = 0, y \in \mathfrak{N}\}$ is a submodule of $\mathfrak{M}^*$. Hence show that $\mathfrak{M}$ is irreducible if and only if $\mathfrak{M}^*$ is irreducible.

18. Let $\mathfrak{M}$ and $\mathfrak{M}^*$ be as in Exercise 17. Show that $\mathfrak{M} \otimes \mathfrak{M}^*$ contains a $u \neq 0$ such that $ul = 0$ for all l. Assume $\mathfrak{M}$ irreducible and suppose $\mathfrak{N}$ is any module such that $\mathfrak{M} \otimes \mathfrak{N}$ contains a $u \neq 0$ such that $ul = 0$, $l \in \mathfrak{L}$. Show that $\mathfrak{N}$ contains a submodule isomorphic to $\mathfrak{M}^*$.

19. Let $\mathfrak{A}$ be a non-associative algebra such that $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_r$ where the $\mathfrak{A}_i$ are ideals satisfying $\mathfrak{A}_i^2 = \mathfrak{A}_i$. Show that the derivation algebra $\mathfrak{D} = \mathfrak{D}_1 \oplus \mathfrak{D}_2 \oplus \cdots \oplus \mathfrak{D}_r$ where $\mathfrak{D}_i$ is an ideal in $\mathfrak{D}$ and is isomorphic to the derivation algebra of $\mathfrak{A}_i$.

20. Show that the derived algebra Φ_{nL}' of the Lie algebra Φ_{nL} is the set of matrices of trace 0. Show that the center of Φ_{nL} is the set $\Phi 1$ of multiples of 1 and that the only ideals in Φ_{nL} are Φ_{nL}' and $\Phi 1$ unless $n = 2$ and the characteristic is 2.

21. Give an example of a Lie algebra over the field C of complex numbers which is not of the form $\mathfrak{L}_C$ where $\mathfrak{L}$ is a Lie algebra over the field R of real numbers. (*Hint*: Consider the Lie algebras satisfying dim $\mathfrak{L} = 3$, dim $\mathfrak{L}' = 2$.)

22. Let $\mathfrak{B}$ be an ideal in a non-associative algebra $\mathfrak{A}$, D a derivation in $\mathfrak{A}$. Show that $\mathfrak{B} + \mathfrak{B}D$ is an ideal. Show that if $\mathfrak{A}$ is finite-dimensional associative of characteristic zero with radical $\mathfrak{R}$, then $\mathfrak{R}D \subseteq \mathfrak{R}$ for every derivation D of $\mathfrak{A}$. (This fails for characteristic p; see p. 75.) Prove the same result for Lie algebras.

23. Show that if $\mathfrak{C}$ is a commutative, associative algebra (with 1) and $\mathfrak{L}$ is a Lie algebra, then $\mathfrak{C} \otimes \mathfrak{L}$ is a Lie algebra. Give an example to show that the tensor product of an associative algebra and a Lie algebra need not be a Lie algebra and an example to show that the tensor product of two Lie algebras need not be a Lie algebra. (*Hint*: For the first of these, take the associative algebra to be Φ_2 and note that $\Phi_n \otimes \mathfrak{B} \cong \mathfrak{B}_n$ where $\mathfrak{B}$ is any non-associative algebra and $\mathfrak{B}_n$ is the algebra of $n \times n$ matrices with entries in $\mathfrak{B}$.)

CHAPTER II

Solvable and Nilpotent Lie Algebras

The main theme of the last chapter has been the analogy between Lie algebras and groups. In this chapter we pursue another idea, namely, relations between Lie algebras and associative algebras. We consider an associative algebra $\mathfrak{A}$—usually the algebra of linear transformations in a finite dimensional vector space—and a subalgebra $\mathfrak{L}$ of $\mathfrak{A}_L$. We are interested in studying relations between the structure of $\mathfrak{L}$ and of the subalgebra $\mathfrak{L}^*$ of $\mathfrak{A}$ generated by $\mathfrak{L}$. We study this particularly for $\mathfrak{L}$ solvable or nilpotent.

The results we obtain in this way include the classical theorems of Lie and Engel on solvable Lie algebras of linear transformations and a criterion for complete reducibility of a Lie algebra of linear transformations. We introduce the notion of weight spaces and we establish a decomposition into weight spaces for the vector space of a "split" nilpotent Lie algebra of linear transformations. These results will play an important role in the structure theory of the next chapter.

1. *Weakly closed subsets of an associative algebra*

Our first results can be established for subsets of an associative algebra which are more general than Lie algebras. It is not much more difficult to treat these more general systems. Moreover, occasionally these are useful in the study of Lie algebras themselves.

DEFINITIONS 1. A subset $\mathfrak{W}$ of an associative algebra $\mathfrak{A}$ over a field Φ is called *weakly closed* if for every ordered pair (a, b), $a, b \in \mathfrak{W}$, there is defined an element $\gamma(a, b) \in \Phi$ such that $ab + \gamma(a, b)ba \in \mathfrak{W}$. We assume the mapping $(a, b) \rightarrow \gamma(a, b)$ is fixed and write $a \times b = ab + \gamma(a, b)ba$. A subset $\mathfrak{U}$ of $\mathfrak{W}$ is called a *subsystem* if $c \times d \in \mathfrak{U}$ for every $c, d \in \mathfrak{U}$ and $\mathfrak{U}$ is a *left ideal* (*ideal*) if $a \times c \in \mathfrak{U}$ ($a \times c$ and $c \times a \in \mathfrak{U}$) for every $a \in \mathfrak{W}$, $c \in \mathfrak{U}$.

Examples

(1) Any subalgebra $\mathfrak{L}$ of $\mathfrak{A}_L$ is weakly closed in $\mathfrak{A}$ relative to $\gamma(a, b) \equiv -1$.

(2) If $\mathfrak{L}(\subseteq \mathfrak{A})$ is the Lie algebra with basis (e, f, h) such that $[ef] = h$, $[eh] = 2e$, $[fh] = -2f$ then $\mathfrak{W} = \Phi e \cup \Phi f \cup \Phi h$ is a subsystem of $\mathfrak{L}$.

(3) The set of symmetric matrices is weakly closed in the algebra Φ_n of $n \times n$ matrices if we take $\gamma(a, b) = 1$.

(4) Let $\mathfrak{W} = \mathfrak{H} \cup \mathfrak{S}$ where $\mathfrak{H}$ is the set of symmetric matrices and $\mathfrak{S}$ is the set of skew matrices. Define $\gamma(a, b) = 1$ if a and b are symmetric and $\gamma(a, b) = -1$ otherwise. Then $\mathfrak{W}$ is weakly closed and $\mathfrak{H}$ is an ideal in $\mathfrak{W}$.

(5) If $\gamma(a, b) \equiv 0$ we have a multiplicative semigroup in $\mathfrak{A}$.

DEFINITION 2. If $\mathfrak{S}$ is a subset of an associative algebra $\mathfrak{A}$ (an algebra = associative algebra with 1) we denote by $\mathfrak{S}^*(\mathfrak{S}^\dagger)$ the subalgebra of $\mathfrak{A}$ (subalgebra containing 1) generated by $\mathfrak{S}$. We call $\mathfrak{S}^*(\mathfrak{S}^\dagger)$ the *enveloping associative algebra* (*enveloping algebra*) *of* $\mathfrak{S}$ (*in* $\mathfrak{A}$).

We shall now note some properties of weakly closed systems which will be needed in the proof of our main theorem on such sets.

I. *If* W *is an element of a weakly closed system* $\mathfrak{W}$, *then* $\mathfrak{Z} \equiv \{W\}^* \cap \mathfrak{W}$ *is a subsystem of* $\mathfrak{W}$ *such that* $\mathfrak{Z}^* = \{W\}^*$.

Proof: The enveloping associative algebra $\{W\}^*$ is the algebra of polynomials in W with constant terms 0. If W_1 and W_2 are two such polynomials then $W_1 \times W_2$ is a polynomial. Hence $\mathfrak{Z} = \{W\}^* \cap \mathfrak{W}$ is a subsystem. Since $\mathfrak{Z} \ni W$, $\mathfrak{Z}^* \supseteq \{W\}^*$. Since $\mathfrak{Z} \subseteq \{W\}^*$ and the latter is a subalgebra, $\mathfrak{Z}^* \subseteq \{W\}^*$. Hence $\mathfrak{Z}^* = \{W\}^*$.

II. *If* $\mathfrak{B}$ *is a subsystem of* $\mathfrak{W}$ *and* W *is an element of* $\mathfrak{W}$ *such that* $B \times W \in \mathfrak{B}^*$ *for every* $B \in \mathfrak{B}$ *then*

$$\mathfrak{B}^* W \subseteq W\mathfrak{B}^* + \mathfrak{B}^*. \tag{1}$$

Proof: The elements of $\mathfrak{B}^*$ are linear combinations of monomials $B_1 B_2 \cdots B_r$, $B_i \in \mathfrak{B}$. If $B \in \mathfrak{B}$ then $BW = -\gamma(B, W)WB + B \times W \in W\mathfrak{B}^* + \mathfrak{B}^*$. Induction on r now shows that if $B_i \in \mathfrak{B}$ then $B_1 \cdots B_r W \in W\mathfrak{B}^* + \mathfrak{B}^*$. This proves (1).

III. *Let* $\mathfrak{B}$ *be a subsystem of* $\mathfrak{W}$ *such that* $\mathfrak{B}^*$ *is nilpotent and* $\mathfrak{B}^* \neq \mathfrak{W}^*$. *Then there exists a* $W \in \mathfrak{W}$ *such that* $W \notin \mathfrak{B}^*$ *but* $B \times W \in \mathfrak{B}^*$ *for every* B *in* $\mathfrak{B}$.

Proof: The assumption $\mathfrak{B}^* \neq \mathfrak{W}^*$ implies that there exists a $W_1 \in \mathfrak{W}$, $W_1 \notin \mathfrak{B}^*$. If $B \times W_1 \in \mathfrak{B}^*$ for all $B \in \mathfrak{B}$ then we take $W = W_1$. Otherwise we have a $W_2 = B_1 \times W_1 \in \mathfrak{W}$, $\notin \mathfrak{B}^*$. We repeat the argument with W_2 in place of W_1. Either this can be taken to be the W of the statement or we obtain $W_3 = B_2 \times W_2 = B_2 \times (B_1 \times W_1) \in \mathfrak{W}$, $\notin \mathfrak{B}^*$. This procedure leads in a finite number of steps to the required element W or else we obtain an infinite sequence $W_1, W_2, \cdots, W_i = B_{i-1} \times W_{i-1}$, where $W_i \in \mathfrak{W}$ but $W_i \notin \mathfrak{B}^*$. We shall show that this last possibility cannot occur and this will complete the proof. We note that W_k is a linear combination of products of $k - 1$ B's belonging to $\mathfrak{B}$ and W_1. Since $\mathfrak{B}^*$ is nilpotent there exists a positive integer n such that any product of n elements of $\mathfrak{B}$ is 0. Now W_{2n} is a linear combination of terms $C_1 \cdots C_j W_1 D_1 \cdots D_k$ where the C_r and $D_s \in \mathfrak{B}$ and $j + k = 2n - 1$. Either $j \geqq n$ or $k \geqq n$ so that we have $C_1 \cdots C_j W_1 D_1 \cdots D_k = 0$. Hence $W_{2n} = 0$ and $W_{2n} \in \mathfrak{B}^*$ contrary to assumption

2. *Nil weakly closed sets*

The main result we shall obtain on weakly closed systems is the following

THEOREM 1. *Let* $\mathfrak{W}$ *be a weakly closed subset of the associative algebra* $\mathfrak{E}$ *of linear transformations of a finite-dimensional vector space* $\mathfrak{M}$ *over* Φ. *Assume every* $W \in \mathfrak{W}$ *is associative nilpotent, that is,* $W^k = 0$ *for some positive integer* k. *Then the enveloping associative algebra* $\mathfrak{W}^*$ *of* $\mathfrak{W}$ *is nilpotent.*

Proof: We shall prove the result by induction on $\dim \mathfrak{M}$. The result is clear if $\dim \mathfrak{M} = 0$ or if $\mathfrak{W} = \{0\}$. Hence we assume $\dim \mathfrak{M} > 0$, $\mathfrak{W} \neq \{0\}$. Let Ω be the collection of subsystems $\mathfrak{B}$ of $\mathfrak{W}$ such that $\mathfrak{B}^*$ is nilpotent and let $\mathfrak{B}$ be an element of Ω such that $\dim \mathfrak{B}^*$ is maximal (for the elements of Ω). We shall show that $\mathfrak{B}^* = \mathfrak{W}^*$ and this will imply the theorem. We note first that $\mathfrak{B}^* \neq 0$. Thus let W be a non-zero element of $\mathfrak{W}$. Then by I, $\mathfrak{Z} = \mathfrak{W} \cap \{W\}^*$ is a subsystem and $\mathfrak{Z}^* = \{W\}^*$. Since $\{W\}^*$ is the set of polynomials in W with constant term 0, $\{W\}^*$ is nilpotent. Hence $\mathfrak{Z} \in \Omega$. Since $\mathfrak{Z} \neq 0$ it follows that $\mathfrak{B}^* \neq 0$. This implies that the subspace $\mathfrak{N}$ spanned by all the vectors xB^*, $x \in \mathfrak{M}$, $B^* \in \mathfrak{B}^*$, is not 0. Also $\mathfrak{N} \neq \mathfrak{M}$. For otherwise, any $x = \sum x_i B_i^*$, $x_i \in \mathfrak{M}$,

$B_i^* \in \mathfrak{B}^*$. If we use similar expressions for the x_i we obtain $x = \sum y_j B_j^* C_j^*$, B_j^*, C_j^* in $\mathfrak{B}^*$. A repetition of this process gives $x = \sum z_k B_{k1}^* B_{k2}^* \cdots B_{kr}^*$, B_{kj}^* in $\mathfrak{B}^*$. Since $\mathfrak{B}^*$ is nilpotent this implies $x = 0$, and $\mathfrak{M} = 0$ contrary to assumption. Hence we have $\mathfrak{M} \supset \mathfrak{N} \supset 0$. Let $\mathfrak{C}$ be the subset of $\mathfrak{W}$ of elements C such that $\mathfrak{N}C \subseteq \mathfrak{N}$. Then it is clear that $\mathfrak{C}$ is a subsystem containing $\mathfrak{B}$. Moreover $\mathfrak{C}$ induces weakly closed systems of nilpotent linear transformations in $\mathfrak{N}$ and in the factor space $\mathfrak{M}/\mathfrak{N}$. Since $\dim \mathfrak{N}$, $\dim \mathfrak{M}/\mathfrak{N} < \dim \mathfrak{M}$ we may assume that the induced systems have nilpotent enveloping associative algebras. The nilpotency of the algebra in $\mathfrak{M}/\mathfrak{N}$ implies that there exists a positive integer p such that if x is any element of $\mathfrak{M}$ and $C_1, C_2, \cdots, C_p$ are any C_i in $\mathfrak{C}$ then $xC_1C_2 \cdots C_p \in \mathfrak{N}$. Also there exists an integer q such that if $D_1, D_2, \cdots, D_q \in \mathfrak{C}$ and $y \in \mathfrak{N}$ then $yD_1 \cdots D_q = 0$. This implies that if $C_1, \cdots, C_{p+q} \in \mathfrak{C}$ then $C_1C_2 \cdots C_{p+q} = 0$. Thus $\mathfrak{C}^*$ is nilpotent and $\mathfrak{C} \in \Omega$. We can now conclude that $\mathfrak{W}^* = \mathfrak{B}^*$. Otherwise, by III., there exists $W \in \mathfrak{W}$, $\notin \mathfrak{B}^*$ such that $B \times W \in \mathfrak{B}^*$ for all B in $\mathfrak{B}$. By II., we have for any $x \in \mathfrak{M}$, $B^* \in \mathfrak{B}^*$, $xB^*W = x(WB_1^* + B_2^*)$, $B_i^* \in \mathfrak{B}^*$. Hence $\mathfrak{N}W \subseteq \mathfrak{N}$ so $W \in \mathfrak{C}$. Since $W \notin \mathfrak{B}^*$, $\dim \mathfrak{C}^* > \dim \mathfrak{B}^*$ and since $\mathfrak{C} \in \Omega$ this contradicts the choice of $\mathfrak{B}$. Hence $\mathfrak{B}^* = \mathfrak{W}^*$ is nilpotent.

If $\mathfrak{W}$ is a set of nil triangular matrices, that is, triangular matrices with 0 diagonal elements, then $\mathfrak{W}^*$ is contained in the associative algebra of nil triangular matrices. The latter is nilpotent; hence $\mathfrak{W}^*$ is nilpotent. The following result is therefore essentially just a slightly different formulation of Theorem 1.

Theorem 1′. *Let $\mathfrak{M}$ be as in Theorem 1. Then there exists a basis for $\mathfrak{M}$ such that the matrices of all the $W \in \mathfrak{W}$ have nil triangular form relative to this basis.*

Proof: We may assume $\mathfrak{M} \neq 0$. Then the proof of Theorem 1 shows that $\mathfrak{M} \supset \mathfrak{M}\mathfrak{W}^*$ where $\mathfrak{M}\mathfrak{W}^*$ is the space spanned by the vectors xW^*, $x \in \mathfrak{M}$, $W^* \in \mathfrak{W}^*$. In general, if $\mathfrak{N}$ is a subspace and $\mathfrak{S}$ is a set of linear transformations, then we shall write $\mathfrak{N}\mathfrak{S}$ for the subspace spanned by all the vectors yS, $y \in \mathfrak{N}$, $S \in \mathfrak{S}$. Then it is immediate that $\mathfrak{M}(\mathfrak{W}^*)^2 = (\mathfrak{M}\mathfrak{W}^*)\mathfrak{W}^*$. Also if $\mathfrak{M}\mathfrak{W}^* \neq 0$ the argument used before shows that $\mathfrak{M}\mathfrak{W}^* \supset \mathfrak{M}(\mathfrak{W}^*)^2$. Hence we have a chain $\mathfrak{M} \supset \mathfrak{M}\mathfrak{W}^* \supset \mathfrak{M}(\mathfrak{W}^*)^2 \supset \mathfrak{M}(\mathfrak{W}^*)^3 \supset \cdots \supset \mathfrak{M}(\mathfrak{W}^*)^{N-1} \supset \mathfrak{M}(\mathfrak{W}^*)^N = 0$, if $(\mathfrak{W}^*)^N = 0$ and $(\mathfrak{W}^*)^{N-1} \neq 0$. We now choose a basis $(e_1, \cdots, e_n)$ for $\mathfrak{M}$ such that $(e_1, \cdots e_{n_1})$ is a basis for $\mathfrak{M}(\mathfrak{W}^*)^{N-1}$, $(e_1, \cdots, e_{n_1}, \cdots, e_{n_1+n_2})$ is a basis for $\mathfrak{M}(\mathfrak{W}^*)^{N-2}, \cdots, (e_1, \cdots, e_{n_1+n_2+\cdots+n_k})$ is a basis for $\mathfrak{M}(\mathfrak{W}^*)^{N-k}$. Then the relations $\mathfrak{M}(\mathfrak{W}^*)^{N-k}\mathfrak{W} \subseteq \mathfrak{M}(\mathfrak{W}^*)^{N-k+1}$ imply that the matrix of any $W \in \mathfrak{W}$ has the form

$$\left[\begin{array}{ccccc} \boxed{0} & n_1 & & & \\ n_1 & \boxed{0} & n_2 & & \\ * & n_2 & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ * & \cdot & \cdot & * & \boxed{0}\ n_N \\ & & & & n_N \end{array}\right].$$

Theorem 1 can be generalized to a theorem about ideals in weakly closed sets. For this purpose we shall derive another general property of enveloping associative algebras of weakly closed sets:

IV. *Let $\mathfrak{W}$ be a weakly closed subset of an associative algebra and let $\mathfrak{B}$ be an ideal in $\mathfrak{W}$, $\mathfrak{B}^*$, $\mathfrak{W}^*$ the enveloping associative algebras of $\mathfrak{B}$ and $\mathfrak{W}$, respectively. Then*

$$\begin{aligned} &\mathfrak{W}^*(\mathfrak{B}^*)^k \subseteq (\mathfrak{B}^*)^k\mathfrak{W}^* + (\mathfrak{B}^*)^k\,, \\ &(\mathfrak{B}^*)^k\mathfrak{W}^* \subseteq \mathfrak{W}^*(\mathfrak{B}^*)^k + (\mathfrak{B}^*)^k\,, \end{aligned} \tag{2}$$

$$(\mathfrak{W}^*\mathfrak{B}^*)^k \subseteq \mathfrak{W}^*(\mathfrak{B}^*)^k\,, \qquad (\mathfrak{B}^*\mathfrak{W}^*)^k \subseteq (\mathfrak{B}^*)^k\mathfrak{W}^*\,. \tag{3}$$

Proof: By II. of §1, the condition that $B \times W \in \mathfrak{B}$ for $B \in \mathfrak{B}$, $W \in \mathfrak{W}$, implies that $\mathfrak{B}^*W \subseteq W\mathfrak{B}^* + \mathfrak{B}^*$. It follows that if $W_1, W_2, \cdots, W_r \in \mathfrak{W}$, then $\mathfrak{B}^*W_1W_2 \cdots W_r \subseteq \mathfrak{W}^*\mathfrak{B}^* + \mathfrak{B}^*$. Hence $\mathfrak{B}^*\mathfrak{W}^* \subseteq \mathfrak{W}^*\mathfrak{B}^* + \mathfrak{B}^*$. Induction now proves $(\mathfrak{B}^*)^k\mathfrak{W}^* \subseteq \mathfrak{W}^*(\mathfrak{B}^*)^k + (\mathfrak{B}^*)^k$. Similarly the condition that $\mathfrak{B}$ is a left ideal implies that $\mathfrak{W}^*(\mathfrak{B}^*)^k \subseteq (\mathfrak{B}^*)^k\mathfrak{W}^* + (\mathfrak{B}^*)^k$. Hence (2) holds. Now (3) is clear for $k = 1$ and if it holds for some k, then

$$\begin{aligned} (\mathfrak{W}^*\mathfrak{B}^*)^{k+1} &= (\mathfrak{W}^*\mathfrak{B}^*)^k\mathfrak{W}^*\mathfrak{B}^* \subseteq \mathfrak{W}^*(\mathfrak{B}^*)^k\mathfrak{W}^*\mathfrak{B}^* \\ &\subseteq \mathfrak{W}^*(\mathfrak{W}^*\mathfrak{B}^{*k} + \mathfrak{B}^{*k})\mathfrak{B}^* \subseteq \mathfrak{W}^*(\mathfrak{B}^*)^{k+1}\,. \end{aligned}$$

Similarly, the second part of (3) holds.

We can now prove

Theorem 2. *Let $\mathfrak{W}$ be a weakly closed set of linear transformations acting in a finite-dimensional vector space and let $\mathfrak{B}$ be an ideal in $\mathfrak{W}$ such that every element of $\mathfrak{B}$ is nilpotent. Then $\mathfrak{B}^*$(hence $\mathfrak{B}$) is contained in the radical $\mathfrak{R}$ of $\mathfrak{W}^*$.*

Proof: $\mathfrak{W}^*\mathfrak{B}^* + \mathfrak{B}^*$ is an ideal since $\mathfrak{W}^*(\mathfrak{W}^*\mathfrak{B}^* + \mathfrak{B}^*) \subseteq \mathfrak{W}^*\mathfrak{B}^*$ and