

sentation of L by generators and relations which depend only on the root system Φ , thereby proving both the existence and the uniqueness of a semi-simple Lie algebra having Φ as root system. In this section, contrary to our general convention, *Lie algebras are allowed to be infinite dimensional.*

18.1. Relations satisfied by L

Let L be a semisimple Lie algebra, H a CSA, Φ the corresponding root system, $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a fixed base. Recall that $\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \alpha_i(h_j)$ ($h_j = h_{\alpha_j}$). Fix a standard set of generators $x_i \in L_{\alpha_i}$, $y_i \in L_{-\alpha_i}$, so that $[x_i, y_i] = h_i$.

Proposition. *With the above notation, L is generated by $\{x_i, y_i, h_i | 1 \leq i \leq \ell\}$, and these generators satisfy at least the following relations:*

$$(S1) \quad [h_i, h_j] = 0 \quad (1 \leq i, j \leq \ell).$$

$$(S2) \quad [x_i, y_i] = h_i, \quad [x_i, y_j] = 0 \text{ if } i \neq j.$$

$$(S3) \quad [h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j, \quad [h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j.$$

$$(S_{ij}^+) \quad (\text{ad } x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0 \quad (i \neq j).$$

$$(S_{ij}^-) \quad (\text{ad } y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0 \quad (i \neq j).$$

Proof. Proposition 14.2 implies that L is already generated by the x_i and y_i . Relation (S1) is clear, as is (S2), in view of the fact that $\alpha_i - \alpha_j \notin \Phi$ when $i \neq j$ (Lemma 10.1). (S3) is obvious. Consider now (S_{ij}^+) ((S_{ij}^-) will follow by symmetry). Since $i \neq j$, $\alpha_j - \alpha_i$ is not a root, and the α_i -string through α_j consists of $\alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + q\alpha_i$, where $-q = \langle \alpha_j, \alpha_i \rangle$ (see (9.4) or Proposition 8.4(e)). Since $\text{ad } x_i$ maps x_j successively into the root spaces for $\alpha_j + \alpha_i, \alpha_j + 2\alpha_i, \dots$, (S_{ij}^+) follows. \square

Notice that the relations in the proposition involve constants which depend only on the root system. Serre discovered that these form a complete set of defining relations for L (Theorem 18.3 below). As a first step toward proving Serre's Theorem, we shall examine the (possibly infinite dimensional) Lie algebra defined by (S1)–(S3) alone.

18.2. Consequences of (S1)–(S3)

Fix a root system Φ , with base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Abbreviate the Cartan integer $\langle \alpha_i, \alpha_j \rangle$ by c_{ij} . We begin with a free Lie algebra \hat{L} (see (17.4)) on 3ℓ generators $\{\hat{x}_i, \hat{y}_i, \hat{h}_i | 1 \leq i \leq \ell\}$. Let \hat{K} be the ideal in \hat{L} generated by the following elements: $[\hat{h}_i, \hat{h}_j], [\hat{x}_i, \hat{y}_j] - \delta_{ij}\hat{h}_i, [\hat{h}_i, \hat{x}_j] - c_{ji}\hat{x}_j, [\hat{h}_i, \hat{y}_j] + c_{ji}\hat{y}_j$. Set $L_o = \hat{L}/\hat{K}$, and let x_i, y_i, h_i be the respective images in L_o of the generators. (In general, $\dim L_o = \infty$.)

The trouble with L_o is that it is defined too abstractly (it might even be trivial, for all we know at this point). To study L_o concretely we attempt to construct a suitable representation of it. The construction which follows is

the prototype of one which plays a prominent role in Chapter VI, so the reader is urged to follow the argument closely.

As was noted in (17.5), there is no problem about constructing a module for \hat{L} : we need only specify a linear transformation corresponding to each of the 3ℓ generators. Let V be the tensor algebra (= free associative algebra) on a vector space with basis (v_1, \dots, v_ℓ) , but forget the product in V . To avoid cumbersome notation we abbreviate $v_{i_1} \otimes \dots \otimes v_{i_t}$ by $v_{i_1} \dots v_{i_t}$. These tensors (along with 1) form a basis of V over F . Next, define endomorphisms of V as follows:

$$\begin{cases} \hat{h}_j.1 = 0 \\ \hat{h}_j.v_{i_1} \dots v_{i_t} = -(c_{i_1 j} + \dots + c_{i_t j})v_{i_1} \dots v_{i_t} \\ \hat{y}_j.1 = v_j \\ \hat{y}_j.v_{i_1} \dots v_{i_t} = v_j v_{i_1} \dots v_{i_t} \\ \hat{x}_j.1 = 0 = \hat{x}_j.v_i \\ \hat{x}_j.v_{i_1} \dots v_{i_t} = v_{i_1}(\hat{x}_j.v_{i_2} \dots v_{i_t}) - \delta_{i_1 j}(c_{i_2 j} + \dots + c_{i_t j})v_{i_2} \dots v_{i_t} \end{cases}$$

Then there is a (unique) extension to \hat{L} of this action by its generators, yielding a representation $\hat{\phi}: \hat{L} \rightarrow \mathfrak{gl}(V)$.

Proposition. Let $\hat{K}_0 = \text{Ker } \hat{\phi}$. Then $\hat{K} \subset \hat{K}_0$, i.e., $\hat{\phi}$ factors through L_0 , thereby making V an L_0 -module.

Proof. Notice first that \hat{h}_j acts diagonally on V (relative to the chosen basis of V), so that $\hat{\phi}(\hat{h}_i)$ and $\hat{\phi}(\hat{h}_j)$ commute, i.e., $[\hat{h}_i, \hat{h}_j] \in \hat{K}_0$. On the other hand, $\hat{\phi}(\hat{y}_j)$ is simply left multiplication by v_j . (It is only the action of \hat{x}_j which complicates matters.)

Setting $j = i_1$ in the formulas, we obtain: $\hat{x}_i.\hat{y}_j.v_{i_2} \dots v_{i_t} - \hat{y}_j.\hat{x}_i.v_{i_2} \dots v_{i_t} = -\delta_{j i_1}(c_{i_2 i_1} + \dots + c_{i_t i_1})v_{i_2} \dots v_{i_t} = \delta_{j i_1}\hat{h}_i.v_{i_2} \dots v_{i_t}$. Also, $(\hat{x}_i.\hat{y}_j - \hat{y}_j.\hat{x}_i).1 = 0 = \delta_{i_1 j}\hat{h}_i.1$. Therefore, $[\hat{x}_i.\hat{y}_j] - \delta_{i_1 j}\hat{h}_i \in \hat{K}_0$.

Next, $(\hat{h}_i.\hat{y}_j - \hat{y}_j.\hat{h}_i).1 = \hat{h}_i.v_j = -c_{j i}v_j = -c_{j i}\hat{y}_j.1$. Similarly, $(\hat{h}_i.\hat{y}_j - \hat{y}_j.\hat{h}_i).v_{i_1} \dots v_{i_t} = \hat{h}_i.v_j.v_{i_1} \dots v_{i_t} + (c_{i_1 i} + \dots + c_{i_t i})v_j.v_{i_1} \dots v_{i_t} = -c_{j i}\hat{y}_j.v_{i_1} \dots v_{i_t}$. Therefore, $[\hat{h}_i.\hat{y}_j] + c_{j i}\hat{y}_j \in \hat{K}_0$.

For the remaining step, we make a preliminary observation:

$$(*) \quad \hat{h}_i.\hat{x}_j.v_{i_1} \dots v_{i_t} = -(c_{i_1 i} + \dots + c_{i_t i} - c_{j i})\hat{x}_j.v_{i_1} \dots v_{i_t}.$$

This is proved by induction on t , starting with the case $t = 0$ (then $v_{i_1} \dots v_{i_t} = 1$, by convention), where both sides are 0. The induction hypothesis just says that $\hat{x}_j.v_{i_2} \dots v_{i_t}$ is an eigenvector for \hat{h}_i , with eigenvalue $-(c_{i_2 i} + \dots + c_{i_t i} - c_{j i})$. Multiplying this eigenvector by v_{i_1} on the left evidently produces another eigenvector for \hat{h}_i , with eigenvalue $-(c_{i_1 i} + \dots + c_{i_t i} - c_{j i})$. From these remarks and the definitions (*) follows quickly.

Using (*) we calculate: $(\hat{h}_i.\hat{x}_j - \hat{x}_j.\hat{h}_i).1 = 0$, $(\hat{h}_i.\hat{x}_j - \hat{x}_j.\hat{h}_i).v_{i_1} \dots v_{i_t} = (-c_{i_1 i} + \dots + c_{i_t i} - c_{j i}) + (c_{i_1 i} + \dots + c_{i_t i})\hat{x}_j.v_{i_1} \dots v_{i_t} = c_{j i}\hat{x}_j.v_{i_1} \dots v_{i_t}$. So $[\hat{h}_i.\hat{x}_j] - c_{j i}\hat{x}_j \in \hat{K}_0$. Finally, $\hat{K} \subset \hat{K}_0$. \square

Theorem. Given a root system Φ with base $\{\alpha_1, \dots, \alpha_\ell\}$, let L_0 be the Lie

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algebra with generators $\{x_i, y_i, h_i | 1 \leq i \leq \ell\}$ and relations (S1)–(S3). Then the h_i are a basis for an ℓ -dimensional abelian subalgebra H of L_o and $L_o = Y + H + X$ (direct sum of subspaces), where Y (resp. X) is the subalgebra of L_o generated by the y_i (resp. x_i).

Proof. This proceeds in steps, using the representation $\phi: L_o \rightarrow \mathfrak{gl}(V)$ constructed above: $\phi(x) = \hat{\phi}(\hat{x})$ if x is the image in L_o of $\hat{x} \in \hat{L}$.

(1) $\sum F\hat{h}_j \cap \text{Ker } \hat{\phi} = 0$. If $\hat{h} = \sum_{j=1}^{\ell} a_j \hat{h}_j$, and $\hat{\phi}(\hat{h}) = 0$, then in particular the eigenvalues $-\sum_j a_j c_{ij}$ ($1 \leq i \leq \ell$) of $\hat{\phi}(\hat{h})$ are all 0. But the Cartan matrix (c_{ij}) of Φ is nonsingular, so this forces all $a_j = 0$, i.e., $\hat{h} = 0$.

(2) The canonical map $\hat{L} \rightarrow L_o$ sends $\sum F\hat{h}_j$ isomorphically onto $\sum Fh_j$. This follows directly from (1).

(3) The subspace $\sum F\hat{x}_j + \sum F\hat{y}_j + \sum F\hat{h}_j$ of \hat{L} maps isomorphically into L_o . Fix i . The relations (S1)–(S3) include: $[x_i, y_i] = h_i$, $[h_i, x_i] = 2x_i$, $[h_i, y_i] = -2y_i$; so $Fx_i + Fy_i + Fh_i$ is a homomorphic image of $\mathfrak{sl}(2, F)$. But the latter is simple, and $h_i \neq 0$ (step (2)), so $Fx_i + Fy_i + Fh_i$ must be isomorphic to $\mathfrak{sl}(2, F)$. Now the set $\{x_j, y_j, h_j | 1 \leq j \leq \ell\}$ is linearly independent, because its elements are nonzero and satisfy relations (S1)–(S3) (cf. the eigenvalues of the $\text{ad } h_j$). This proves (3).

(4) $H = \sum Fh_j$ is an ℓ -dimensional abelian subalgebra of L_o . This follows from (2) and relation (S1).

(5) If $[x_{i_1} \dots x_{i_t}]$ denotes $[x_{i_1}[x_{i_2} \dots [x_{i_{t-1}}x_{i_t}] \dots]]$, then $[h_j[x_{i_1} \dots x_{i_t}]] = (c_{i_1, j} + \dots + c_{i_t, j})[x_{i_1} \dots x_{i_t}]$, and similarly for the y_i in place of the x_i , $-c_{ij}$ in place of c_{ij} . For $t = 1$ this is (S3). The general case follows quickly by induction, using the Jacobi identity.

(6) If $t \geq 2$, then $[y_j[x_{i_1} \dots x_{i_t}]] \in X$, and similarly for Y . By relation (S2), $[y_j, x_i] = -\delta_{ij}h_i$, and therefore the case $t = 2$ is immediate from the Jacobi identity and (S3). An easy induction on t completes the argument.

(7) $Y + H + X$ is a subalgebra of L_o , hence coincides with L_o . That $Y + H + X$ is a subalgebra follows from (4), (5), (6). But $Y + H + X$ contains a set of generators of L_o , so it coincides with L_o .

(8) The sum $L_o = Y + H + X$ is direct. Indeed, (5) shows how to decompose L_o into eigenspaces for $\text{ad } H$; directness follows (cf. (1), (2)). \square

It is convenient to describe the decomposition $L_o = Y + H + X$ in terms of "weights" (to use the language of §20). For $\lambda \in H^*$, let $(L_o)_\lambda = \{t \in L_o | [ht] = \lambda(h)t \text{ for all } h \in H\}$. The proof of the preceding theorem shows that $H = (L_o)_0$. Moreover, the only nonzero λ for which $(L_o)_\lambda \neq 0$ are those of the form $\lambda = \sum_{i=1}^{\ell} k_i \alpha_i$ ($k_i \in \mathbf{Z}$), with all $k_i \geq 0$ (write $\lambda > 0$ in this case) or all $k_i \leq 0$ (write $\lambda < 0$). Then $X = \sum_{\lambda > 0} (L_o)_\lambda$ and $Y = \sum_{\lambda < 0} (L_o)_\lambda$.

18.3. Serre's Theorem

In (18.2) we studied the structure of the Lie algebra L_o determined by (S1)–(S3) alone. Now we ask what happens when we impose the "finiteness"

