

sentation of L by generators and relations which depend only on the root system Φ , thereby proving both the existence and the uniqueness of a semi-simple Lie algebra having Φ as root system. In this section, contrary to our general convention, *Lie algebras are allowed to be infinite dimensional.*

18.1. Relations satisfied by L

Let L be a semisimple Lie algebra, H a CSA, Φ the corresponding root system, $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a fixed base. Recall that $\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \alpha_i(h_j)$ ($h_j = h_{\alpha_j}$). Fix a standard set of generators $x_i \in L_{\alpha_i}$, $y_i \in L_{-\alpha_i}$, so that $[x_i, y_i] = h_i$.

Proposition. *With the above notation, L is generated by $\{x_i, y_i, h_i | 1 \leq i \leq \ell\}$, and these generators satisfy at least the following relations:*

$$(S1) \quad [h_i, h_j] = 0 \quad (1 \leq i, j \leq \ell).$$

$$(S2) \quad [x_i, y_i] = h_i, \quad [x_i, y_j] = 0 \text{ if } i \neq j.$$

$$(S3) \quad [h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j, \quad [h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j.$$

$$(S_{ij}^+) \quad (\text{ad } x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0 \quad (i \neq j).$$

$$(S_{ij}^-) \quad (\text{ad } y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0 \quad (i \neq j).$$

Proof. Proposition 14.2 implies that L is already generated by the x_i and y_i . Relation (S1) is clear, as is (S2), in view of the fact that $\alpha_i - \alpha_j \notin \Phi$ when $i \neq j$ (Lemma 10.1). (S3) is obvious. Consider now (S_{ij}^+) ((S_{ij}^-) will follow by symmetry). Since $i \neq j$, $\alpha_j - \alpha_i$ is not a root, and the α_i -string through α_j consists of $\alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + q\alpha_i$, where $-q = \langle \alpha_j, \alpha_i \rangle$ (see (9.4) or Proposition 8.4(e)). Since $\text{ad } x_i$ maps x_j successively into the root spaces for $\alpha_j + \alpha_i, \alpha_j + 2\alpha_i, \dots$, (S_{ij}^+) follows. \square

Notice that the relations in the proposition involve constants which depend only on the root system. Serre discovered that these form a complete set of defining relations for L (Theorem 18.3 below). As a first step toward proving Serre's Theorem, we shall examine the (possibly infinite dimensional) Lie algebra defined by (S1)–(S3) alone.

18.2. Consequences of (S1)–(S3)

Fix a root system Φ , with base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Abbreviate the Cartan integer $\langle \alpha_i, \alpha_j \rangle$ by c_{ij} . We begin with a free Lie algebra \hat{L} (see (17.4)) on 3ℓ generators $\{\hat{x}_i, \hat{y}_i, \hat{h}_i | 1 \leq i \leq \ell\}$. Let \hat{K} be the ideal in \hat{L} generated by the following elements: $[\hat{h}_i, \hat{h}_j], [\hat{x}_i, \hat{y}_j] - \delta_{ij}\hat{h}_i, [\hat{h}_i, \hat{x}_j] - c_{ji}\hat{x}_j, [\hat{h}_i, \hat{y}_j] + c_{ji}\hat{y}_j$. Set $L_o = \hat{L}/\hat{K}$, and let x_i, y_i, h_i be the respective images in L_o of the generators. (In general, $\dim L_o = \infty$.)

The trouble with L_o is that it is defined too abstractly (it might even be trivial, for all we know at this point). To study L_o concretely we attempt to construct a suitable representation of it. The construction which follows is

the prototype of one which plays a prominent role in Chapter VI, so the reader is urged to follow the argument closely.

As was noted in (17.5), there is no problem about constructing a module for \hat{L} : we need only specify a linear transformation corresponding to each of the 3ℓ generators. Let V be the tensor algebra (= free associative algebra) on a vector space with basis (v_1, \dots, v_ℓ) , but forget the product in V . To avoid cumbersome notation we abbreviate $v_{i_1} \otimes \dots \otimes v_{i_t}$ by $v_{i_1} \dots v_{i_t}$. These tensors (along with 1) form a basis of V over F . Next, define endomorphisms of V as follows:

$$\begin{cases} \hat{h}_j.1 = 0 \\ \hat{h}_j.v_{i_1} \dots v_{i_t} = -(c_{i_1 j} + \dots + c_{i_t j})v_{i_1} \dots v_{i_t} \\ \hat{y}_j.1 = v_j \\ \hat{y}_j.v_{i_1} \dots v_{i_t} = v_j v_{i_1} \dots v_{i_t} \\ \hat{x}_j.1 = 0 = \hat{x}_j.v_i \\ \hat{x}_j.v_{i_1} \dots v_{i_t} = v_{i_1}(\hat{x}_j.v_{i_2} \dots v_{i_t}) - \delta_{i_1 j}(c_{i_2 j} + \dots + c_{i_t j})v_{i_2} \dots v_{i_t} \end{cases}$$

Then there is a (unique) extension to \hat{L} of this action by its generators, yielding a representation $\hat{\phi}: \hat{L} \rightarrow \mathfrak{gl}(V)$.

Proposition. Let $\hat{K}_0 = \text{Ker } \hat{\phi}$. Then $\hat{K} \subset \hat{K}_0$, i.e., $\hat{\phi}$ factors through L_0 , thereby making V an L_0 -module.

Proof. Notice first that \hat{h}_j acts diagonally on V (relative to the chosen basis of V), so that $\hat{\phi}(\hat{h}_i)$ and $\hat{\phi}(\hat{h}_j)$ commute, i.e., $[\hat{h}_i, \hat{h}_j] \in \hat{K}_0$. On the other hand, $\hat{\phi}(\hat{y}_j)$ is simply left multiplication by v_j . (It is only the action of \hat{x}_j which complicates matters.)

Setting $j = i_1$ in the formulas, we obtain: $\hat{x}_i.\hat{y}_j.v_{i_2} \dots v_{i_t} - \hat{y}_j.\hat{x}_i.v_{i_2} \dots v_{i_t} = -\delta_{j i_1}(c_{i_2 i_1} + \dots + c_{i_t i_1})v_{i_2} \dots v_{i_t} = \delta_{j i_1}\hat{h}_i.v_{i_2} \dots v_{i_t}$. Also, $(\hat{x}_i.\hat{y}_j - \hat{y}_j.\hat{x}_i).1 = 0 = \delta_{i_1 j}\hat{h}_i.1$. Therefore, $[\hat{x}_i.\hat{y}_j] - \delta_{i_1 j}\hat{h}_i \in \hat{K}_0$.

Next, $(\hat{h}_i.\hat{y}_j - \hat{y}_j.\hat{h}_i).1 = \hat{h}_i.v_j = -c_{j i}v_j = -c_{j i}\hat{y}_j.1$. Similarly, $(\hat{h}_i.\hat{y}_j - \hat{y}_j.\hat{h}_i).v_{i_1} \dots v_{i_t} = \hat{h}_i.v_j.v_{i_1} \dots v_{i_t} + (c_{i_1 i} + \dots + c_{i_t i})v_j.v_{i_1} \dots v_{i_t} = -c_{j i}\hat{y}_j.v_{i_1} \dots v_{i_t}$. Therefore, $[\hat{h}_i.\hat{y}_j] + c_{j i}\hat{y}_j \in \hat{K}_0$.

For the remaining step, we make a preliminary observation:

$$(*) \quad \hat{h}_i.\hat{x}_j.v_{i_1} \dots v_{i_t} = -(c_{i_1 i} + \dots + c_{i_t i} - c_{j i})\hat{x}_j.v_{i_1} \dots v_{i_t}.$$

This is proved by induction on t , starting with the case $t = 0$ (then $v_{i_1} \dots v_{i_t} = 1$, by convention), where both sides are 0. The induction hypothesis just says that $\hat{x}_j.v_{i_2} \dots v_{i_t}$ is an eigenvector for \hat{h}_i , with eigenvalue $-(c_{i_2 i} + \dots + c_{i_t i} - c_{j i})$. Multiplying this eigenvector by v_{i_1} on the left evidently produces another eigenvector for \hat{h}_i , with eigenvalue $-(c_{i_1 i} + \dots + c_{i_t i} - c_{j i})$. From these remarks and the definitions (*) follows quickly.

Using (*) we calculate: $(\hat{h}_i.\hat{x}_j - \hat{x}_j.\hat{h}_i).1 = 0$, $(\hat{h}_i.\hat{x}_j - \hat{x}_j.\hat{h}_i).v_{i_1} \dots v_{i_t} = (-c_{i_1 i} + \dots + c_{i_t i} - c_{j i}) + (c_{i_1 i} + \dots + c_{i_t i})\hat{x}_j.v_{i_1} \dots v_{i_t} = c_{j i}\hat{x}_j.v_{i_1} \dots v_{i_t}$. So $[\hat{h}_i.\hat{x}_j] - c_{j i}\hat{x}_j \in \hat{K}_0$. Finally, $\hat{K} \subset \hat{K}_0$. \square

Theorem. Given a root system Φ with base $\{\alpha_1, \dots, \alpha_\ell\}$, let L_0 be the Lie

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algebra with generators $\{x_i, y_i, h_i | 1 \leq i \leq \ell\}$ and relations (S1)–(S3). Then the h_i are a basis for an ℓ -dimensional abelian subalgebra H of L_o and $L_o = Y + H + X$ (direct sum of subspaces), where Y (resp. X) is the subalgebra of L_o generated by the y_i (resp. x_i).

Proof. This proceeds in steps, using the representation $\phi: L_o \rightarrow \mathfrak{gl}(V)$ constructed above: $\phi(x) = \hat{\phi}(\hat{x})$ if x is the image in L_o of $\hat{x} \in \hat{L}$.

(1) $\sum F\hat{h}_j \cap \text{Ker } \hat{\phi} = 0$. If $\hat{h} = \sum_{j=1}^{\ell} a_j \hat{h}_j$, and $\hat{\phi}(\hat{h}) = 0$, then in particular the eigenvalues $-\sum_j a_j c_{ij}$ ($1 \leq i \leq \ell$) of $\hat{\phi}(\hat{h})$ are all 0. But the Cartan matrix (c_{ij}) of Φ is nonsingular, so this forces all $a_j = 0$, i.e., $\hat{h} = 0$.

(2) The canonical map $\hat{L} \rightarrow L_o$ sends $\sum F\hat{h}_j$ isomorphically onto $\sum Fh_j$. This follows directly from (1).

(3) The subspace $\sum F\hat{x}_j + \sum F\hat{y}_j + \sum F\hat{h}_j$ of \hat{L} maps isomorphically into L_o . Fix i . The relations (S1)–(S3) include: $[x_i, y_i] = h_i$, $[h_i, x_i] = 2x_i$, $[h_i, y_i] = -2y_i$; so $Fx_i + Fy_i + Fh_i$ is a homomorphic image of $\mathfrak{sl}(2, F)$. But the latter is simple, and $h_i \neq 0$ (step (2)), so $Fx_i + Fy_i + Fh_i$ must be isomorphic to $\mathfrak{sl}(2, F)$. Now the set $\{x_j, y_j, h_j | 1 \leq j \leq \ell\}$ is linearly independent, because its elements are nonzero and satisfy relations (S1)–(S3) (cf. the eigenvalues of the $\text{ad } h_j$). This proves (3).

(4) $H = \sum Fh_j$ is an ℓ -dimensional abelian subalgebra of L_o . This follows from (2) and relation (S1).

(5) If $[x_{i_1} \dots x_{i_t}]$ denotes $[x_{i_1}[x_{i_2} \dots [x_{i_{t-1}}x_{i_t}] \dots]]$, then $[h_j[x_{i_1} \dots x_{i_t}]] = (c_{i_1, j} + \dots + c_{i_t, j})[x_{i_1} \dots x_{i_t}]$, and similarly for the y_i in place of the x_i , $-c_{ij}$ in place of c_{ij} . For $t = 1$ this is (S3). The general case follows quickly by induction, using the Jacobi identity.

(6) If $t \geq 2$, then $[y_j[x_{i_1} \dots x_{i_t}]] \in X$, and similarly for Y . By relation (S2), $[y_j, x_i] = -\delta_{ij}h_i$, and therefore the case $t = 2$ is immediate from the Jacobi identity and (S3). An easy induction on t completes the argument.

(7) $Y + H + X$ is a subalgebra of L_o , hence coincides with L_o . That $Y + H + X$ is a subalgebra follows from (4), (5), (6). But $Y + H + X$ contains a set of generators of L_o , so it coincides with L_o .

(8) The sum $L_o = Y + H + X$ is direct. Indeed, (5) shows how to decompose L_o into eigenspaces for $\text{ad } H$; directness follows (cf. (1), (2)). \square

It is convenient to describe the decomposition $L_o = Y + H + X$ in terms of "weights" (to use the language of §20). For $\lambda \in H^*$, let $(L_o)_\lambda = \{t \in L_o | [ht] = \lambda(h)t \text{ for all } h \in H\}$. The proof of the preceding theorem shows that $H = (L_o)_0$. Moreover, the only nonzero λ for which $(L_o)_\lambda \neq 0$ are those of the form $\lambda = \sum_{i=1}^{\ell} k_i \alpha_i$ ($k_i \in \mathbf{Z}$), with all $k_i \geq 0$ (write $\lambda > 0$ in this case) or all $k_i \leq 0$ (write $\lambda < 0$). Then $X = \sum_{\lambda > 0} (L_o)_\lambda$ and $Y = \sum_{\lambda < 0} (L_o)_\lambda$.

18.3. Serre's Theorem

In (18.2) we studied the structure of the Lie algebra L_o determined by (S1)–(S3) alone. Now we ask what happens when we impose the "finiteness"

conditions (S_{ij}^+) , (S_{ij}^-) of (18.1). Set $x_{ij} = (\text{ad } x_i)^{-c_{ji}+1}(x_j)$, $y_{ij} = (\text{ad } y_i)^{-c_{ji}+1}(y_j)$ ($i \neq j$). (These are elements of L_o .)

Lemma. *In the algebra L_o of (18.2), $\text{ad } x_k(y_{ij}) = 0$ ($1 \leq k \leq \ell$) for each $i \neq j$.*

Proof. Case (a): $k \neq i$. Then $[x_k y_i] = 0$ (by (S2)), so $\text{ad } x_k$ and $\text{ad } y_i$ commute. Therefore, $\text{ad } x_k(y_{ij}) = (\text{ad } y_i)^{-c_{ji}+1} \text{ad } x_k(y_j)$. If $k = j$, this reads $(\text{ad } y_i)^{-c_{ji}+1}(h_j)$. But by (S3), $\text{ad } y_i(h_j) = c_{ij}y_i$. If this is nonzero, then c_{ji} is nonzero (and negative, since $i \neq j$), so $-c_{ji}+1 \geq 2$. It follows that $(\text{ad } y_i)^{-c_{ji}+1}(h_j) = 0$. If $k \neq j$, then $[x_k y_j] = 0$ (S2), so the same assertion follows.

Case (b): $k = i$. Recall from the proof of Theorem 18.2 that $S = Fx_i + Fy_i + Fh_i$ is a subalgebra of L_o isomorphic to $\mathfrak{sl}(2, F)$. We can therefore say quite a bit about the adjoint action of S on L_o . Even though L_o is (in general) infinite dimensional, some of the reasoning used in §7 carries over directly to the present situation. In particular, since $j \neq i$, $[x_i y_j] = 0$, so that y_j is a "maximal vector" for S , of "weight" $m = -c_{ji}$ (because $[h_i y_j] = -c_{ji}y_j$). An easy induction on t shows that $\text{ad } x_i (\text{ad } y_i)^t(y_j) = t(m-t+1) (\text{ad } y_i)^{t-1}(y_j)$. Therefore the right side is 0 when $t = -c_{ji}+1$. \square

Before stating Serre's Theorem, we mention a useful construction. Call an endomorphism x of an infinite dimensional vector space V **locally nilpotent** if every element of V is killed by a sufficiently large power of x . In that case, x is nilpotent on each finite dimensional subspace W of V stable under x , so it makes sense to form $\exp(x|_W)$. It is clear that $\exp(x|_W)$ and $\exp(x|_{W'})$ agree on $W \cap W'$, so we can patch these maps together to obtain an automorphism "exp x " of V .

Theorem (Serre). *Fix a root system Φ , with base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Let L be the Lie algebra generated by 3ℓ elements $\{x_i, y_i, h_i | 1 \leq i \leq \ell\}$, subject to the relations (S1), (S2), (S3), (S_{ij}^+) , (S_{ij}^-) listed in (18.1). Then L is a (finite dimensional) semisimple algebra, with CSA spanned by the h_i and with corresponding root system Φ .*

Proof. This will be carried out in steps. By definition, $L = L_o/K$, L_o as in (18.2) and K the ideal generated by all x_{ij}, y_{ij} ($i \neq j$). To avoid notational problems, we work at first inside L_o . Let I (resp. J) be the ideal of X (resp. Y) generated by all x_{ij} (resp. y_{ij}). (So K includes I, J .)

(1) I and J are ideals of L_o . It suffices to consider J (the argument for I being analogous). On the one hand, y_{ij} is an eigenvector for $\text{ad } h_k$ ($1 \leq k \leq \ell$), with eigenvalue $-c_{jk} + (c_{ji} - 1)c_{ik}$. Since $\text{ad } h_k(Y) \subset Y$, it follows from the Jacobi identity that $\text{ad } h_k(J) \subset J$. On the other hand, the lemma above says that $\text{ad } x_k(y_{ij}) = 0$. It is clear that $\text{ad } x_k$ maps Y into $Y+H$ (cf. 18.2); combining this with the Jacobi identity and the fact that $\text{ad } h_k(J) \subset J$, we get $\text{ad } x_k(J) \subset J$. Finally, $\text{ad } L_o(J) \subset J$, again by the Jacobi identity (since the x_k, y_k generate L_o).

(2) $K = I+J$. By definition, $I+J \subset K$. But $I+J$ is an ideal of L_o (by (1)) containing all x_{ij}, y_{ij} , and K is the smallest such ideal.

(3) $L = N^- + H + N$ (direct sum of subspaces), where $N^- = Y/J$, $N = X/I$, and H is identified with its image under the canonical map $L_o \rightarrow L$. Use (2), along with the direct sum decomposition $L_o = Y + H + X$ (Theorem 18.2).

(4) $\sum Fx_i + \sum Fh_i + \sum Fy_i$ maps isomorphically into L . This follows just as in step (3) of the proof of Theorem 18.2, since H maps isomorphically into L (by (3) above). We may therefore identify x_i, y_i, h_i with elements of L (and these generate L).

(5) If $\lambda \in H^*$, let $L_\lambda = \{x \in L | [hx] = \lambda(h)x \text{ for all } h \in H\}$. Then $H = L_o$, $N = \sum_{\lambda > 0} L_\lambda$, $N^- = \sum_{\lambda < 0} L_\lambda$ (cf. remarks at end of (18.2)), and each L_λ is finite dimensional. This is clear, because of (3), (4).

(6) For $1 \leq i \leq \ell$, ad x_i and ad y_i are locally nilpotent endomorphisms of L . It suffices to consider ad x_i , for fixed i (by symmetry). Let M be the subspace of all elements of L which are killed by some power of ad x_i . If $x \in M$ (resp. $y \in M$) is killed by $(\text{ad } x_i)^r$ (resp. $(\text{ad } x_i)^s$), then $[xy]$ is killed by $(\text{ad } x_i)^{r+s}$ (cf. Lemma 15.1). So M is actually a subalgebra of L . But all $x_k \in M$ (by relations (S_{ij}^+)) and all $y_k \in M$ (by (S2), (S3)). These elements generate L , so $M = L$, as desired.

(7) $\tau_i = \exp(\text{ad } x_i) \exp(\text{ad } (-y_i)) \exp(\text{ad } x_i)$ (for $1 \leq i \leq \ell$) is a well-defined automorphism of L . This follows from (6) and the remarks just preceding the theorem.

(8) If $\lambda, \mu \in H^*$ and $\sigma\lambda = \mu$ ($\sigma \in \mathcal{W}$, the Weyl group of Φ), then $\dim L_\lambda = \dim L_\mu$. It suffices to prove this when $\sigma = \sigma_{\alpha_i}$ is a simple reflection, because these generate \mathcal{W} (Theorem 10.3(d)). The automorphism τ_i of L constructed in step (7) coincides on the finite dimensional space $L_\lambda + L_\mu$ with the ordinary product of exponentials, and we conclude (as in the last part of (7.2)) that τ_i interchanges L_λ, L_μ . In particular, $\dim L_\lambda = \dim L_\mu$.

(9) For $1 \leq i \leq \ell$, $\dim L_{\alpha_i} = 1$, while $L_{k\alpha_i} = 0$ for integers $k \neq 0, 1, -1$. This is clear for L_o , hence also for L because of (4).

(10) If $\alpha \in \Phi$, then $\dim L_\alpha = 1$, but $L_{k\alpha} = 0$ for $k \neq 0, 1, -1$. Each root is \mathcal{W} -conjugate to a simple root (Theorem 10.3(c)), so this follows from (8), (9).

(11) If $L_\lambda \neq 0$, then either $\lambda \in \Phi$ or $\lambda = 0$. Otherwise λ is an integral combination of simple roots, with coefficients of like sign (not all 0), λ not a multiple of any root because of (10). Exercise 10.10 shows that some \mathcal{W} -conjugate $\sigma\lambda$ has a strictly positive as well as a strictly negative coefficient. This means that $L_{\sigma\lambda} = 0$ (cf. (5)), contradicting the conclusion of step (8).

(12) $\dim L = \ell + \text{Card } \Phi < \infty$. In view of (5), this follows from (10), (11).

(13) L is semisimple. Let A be an abelian ideal of L ; we have to show that $A = 0$. Since ad H stabilizes A , $A = (A \cap H) + \sum_{\alpha \in \Phi} (A \cap L_\alpha)$ (because $L = H + \sum_{\alpha \in \Phi} L_\alpha$). If $L_\alpha \subset A$, then $[L_{-\alpha}L_\alpha] \subset A$, whence $L_{-\alpha} \subset A$ and A contains a copy of the simple algebra $\mathfrak{sl}(2, F)$ (cf. step (4)). This is absurd, so instead $A = A \cap H \subset H$, whence $[L_\alpha A] = 0$ ($\alpha \in \Phi$) and $A \subset \bigcap_{\alpha \in \Phi} \text{Ker } \alpha = 0$ (the α_i span H^*).

(14) H is a CSA of L , Φ the root system. H is abelian (hence nilpotent) and

self-normalizing (because of the direct sum decomposition $L = H + \sum_{\alpha \in \Phi} L_{\alpha}$), i.e., H is a CSA. Then Φ is obviously the corresponding set of roots. \square

18.4. Application: Existence and uniqueness theorems

Finally, our efforts are rewarded.

Theorem. (a) Let Φ be a root system. Then there exists a semisimple Lie algebra having Φ as its root system.

(b) Let L, L' be semisimple Lie algebras, with respective CSA's H, H' and root systems Φ, Φ' . Let an isomorphism $\Phi \rightarrow \Phi'$ be given, sending a given base Δ to a base Δ' , and denote by $\pi: H \rightarrow H'$ the associated isomorphism (as in (14.2)). For each $\alpha \in \Delta$ ($\alpha' \in \Delta'$) select arbitrary nonzero $x_{\alpha} \in L_{\alpha}$ ($x'_{\alpha'} \in L'_{\alpha'}$). Then there exists a unique isomorphism $\pi: L \rightarrow L'$ extending $\pi: H \rightarrow H'$ and sending x_{α} to $x'_{\alpha'}$ ($\alpha \in \Delta$).

Proof. (a) This follows directly from Theorem 18.3. (b) It is clearly enough to prove this in the special case when L is the Lie algebra constructed in Theorem 18.3, taking x_{α}, y_{α} , and $h_{\alpha} = [x_{\alpha}, y_{\alpha}]$ to be the generators specified there ($\alpha \in \Delta$). Set $h'_{\alpha'} = \pi(h_{\alpha})$, and choose $y'_{\alpha'}$ (uniquely) satisfying $[x'_{\alpha'}, y'_{\alpha'}] = h'_{\alpha'}$ for each $\alpha' \in \Delta'$. Since $\Phi \cong \Phi'$, the chosen elements in L' satisfy the Serre relations (18.1). Thus Theorem 18.3 provides a unique homomorphism $\pi: L \rightarrow L'$ sending $x_{\alpha}, y_{\alpha}, h_{\alpha}$ ($\alpha \in \Delta$) to $x'_{\alpha'}, y'_{\alpha'}, h'_{\alpha'}$ (respectively). This extends the given isomorphism $\pi: H \rightarrow H'$. Since $\dim L = \dim H + \text{Card } \Phi = \dim H' + \text{Card } \Phi' = \dim L'$, we conclude that π is an isomorphism. \square

Exercises

- Using the representation of L_o on V (Proposition 18.2), prove that the algebras X, Y described in Theorem 18.2 are (respectively) free Lie algebras on the sets of x_i, y_i .
- When $\text{rank } \Phi = 1$, the relations $(S_{ij}^+), (S_{ij}^-)$ are vacuous, so $L_o = L \cong \mathfrak{sl}(2, F)$. By suitably modifying the basis of V in (18.2), show that V is isomorphic to the module $Z(0)$ constructed in Exercise 7.7.
- Prove that the ideal K of L_o in (18.3) lies in every ideal of L_o having finite codimension (i.e., L is the largest finite dimensional quotient of L_o).
- Prove that each inclusion of Dynkin diagrams (e.g., $E_6 \subset E_7 \subset E_8$) induces a natural inclusion of the corresponding semisimple Lie algebras.

Notes

In essence, the proof of Theorem 18.2 is due (independently) to Chevalley and Harish-Chandra (cf. Harish-Chandra [1]), with simplifications by Jacobson (cf. Jacobson [1]). Serre's Theorem, along with the applications to uniqueness and existence, appears in Serre [2]. See also Varadarajan [1].