Definition 0.1. a) Let V be a finite-dimensional vector space and $W \subset V$ a linear subspace. We denote by $\mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$ the natural inclusion.

b) If $T \subset W$ is an affine subspace we define $\mathbb{P}(T) := \mathbb{P}(T') \subset \mathbb{P}(V)$ where $T' \subset V$ is the linear subspace parallel to T.

c) Let $W \subset V$ be a subspace of codimension 1 and $v \in V - W$. We denote by $\kappa_v : W \to \mathbb{P}(V) - \mathbb{P}(W)$ the map which associates with $w \in W$ the line through $v + w$.

d) For any affine subspace $T \subset W$ we denote by $\tilde{T}$ the closure of $\kappa_v(T)$ in $\mathbb{P}(V)$.

Problem 0.2. a) Let V, W, v be as in Definition 1 c) and $T \subset W$ be an affine subspace. Then $\tilde{T} \cap \mathbb{P}(W) = \mathbb{P}(T)$.

b) The map $\kappa_v : W \to \mathbb{P}(V) - \mathbb{P}(W)$ defines an isomorphism of algebraic varieties.

c) For any $d < dim(W)$ define the structure of an algebraic variety on the set $Gr_{af}(W, d)$ of d-dimensional affine subspaces of W and moreover κ_v defines an isomorphism of $Gr_{af}(W, d)$ with an open subset of the Grassmanian of $d + 1$–dimensional linear subspaces of V.

d) Let $R \subset \mathbb{P}(V)$ be a finite set and $f : W \to k^2$ a linear surjection map. Then the set $\Lambda_{R,f}$ of affine lines $L \subset k^2$ such that $\mathbb{P}(f^{-1}(L)) \cap R = \emptyset$ is a non-zero open subset of $Gr_{af}(k^2, 1)$.

e) Let $\rho : \mathbb{G}_m \to GL(V)$ be a representation. Then for

i) Any $x \in \mathbb{P}(V)$ the map $f : \mathbb{G}_m \to \mathbb{P}(V), a \to ax$ can be extended to a morphism $\bar{f} : \mathbb{P}^1 \to \mathbb{P}(V)$.

ii) The points $f(0), f(\infty)$ in $\mathbb{P}(V)$ are $\mathbb{G}_m$-invariant.

iii) If $f(0) = f(\infty)$ then x is a $\mathbb{G}_m$-invariant point.

f) Let T be a torus, $\rho : T \to GL(V)$ be a representation , $e_i, 1 \leq i \leq d$ a basis of V such that $\rho(t)e_i = \chi_i(t)e_i, \chi_i \in X^\star(T), t \in T$ and $\lambda \in X_\star(T)$ such that $< \chi_i, \lambda > \neq 0$ for all $i, 1 \leq i \leq d$. Then any point $x \in \mathbb{P}(V)$ such that $\rho \circ \lambda(a)x = x$ for all $a \in k^\star$ is $\rho(T)$-invariant.

[A hint]. Let $v \in V$ be a non-zero vector on the line x and $V' \subset V$ be the subspace spanned by $\rho(t)v, t \in k^\star$. Choose a basis $e_i, 1 \leq i \leq d$ of V' such that

$$\rho(t)e_i = t^{n_i} e_i, t \in k^\star$$

where $n_1 \leq n_2 \leq ... \leq n_d$.

Let V be a finite-dimensional vector space $W \subset V$ a subspace of codimension $1, X \subset \mathbb{P}(V)$ an irreducible closed subset of dimension d.

Proposition 0.3. *$dim(X \cap \mathbb{P}(W)) \geq d-1$*

Proof. I'll prove the result only in the case when $d = 1$ and $d = 2$ since we will need only these cases. Consider first the case $d = 1$. In this case we want to show that $X \cap \mathbb{P}(W) \neq \emptyset$. But if $X \cap \mathbb{P}(W) = \emptyset$ then $X \subset W$. Since X is complete we see that X is a point.

Consider the case $d = 2$ and assume that $dim(X \cap \mathbb{P}(W)) < d-1 = 1$. In other words assume that the set $R := X \cap \mathbb{P}(W)$ is finite. Since $dim(U) = 2$ the can find two linear functions λ, ν on W such that the restrictions of λ, ν on U are algebraically independent. Let $f : U \to k^2$ be the morphism given by $u \to (\lambda(u), \nu(u))$. Then the image $Y \subset k^2$ of f contains a non-zero open subset U' of k^2. As follows from Problem 2 c) there exists a line $L \subset k^2$ such that $L \cap U' \neq \emptyset$ and $\mathbb{P}(f^{-1}(L)) \cap R = \emptyset$. Let Y be the closure of the intersection $X \cap f^{-1}(L)$. Since $dim(Y) > 0$ we see that $Y \cap \mathbb{P}(W) \neq \emptyset$. On the other hand by the construction we have $Y \cap R = \emptyset$. But this contradicts the assumption that $X \cap \mathbb{P}(W) = R$.$\square$

Theorem 0.4. *Let $\rho : T \to GL(V)$ be a representation of a torus $T, X \subset \mathbb{P}(V)$ a closed T-invariant irreducible subset and $Y \subset X$ the subset of $\mathbb{G}_m$-invariant points. Then*

a) if $dim(X) > 0$ then $|Y| \geq 2$.

b) if $dim(X) > 1$ then $|Y| \geq 3$.

As follows from Problem 3 it is sufficient to prove the theorem in the case $T = \mathbb{G}_m$.

Proof of a). If $Y = X$ then there is nothing to proof. On the other hand if $Y \neq X$ choose any $x \in X$ which is not $\mathbb{G}_m$-invariant. By Problem 2 d) the map $f : \mathbb{G}_m \to \mathbb{P}(V), a \to \lambda(a)x$ can be extended to a morphism $\bar{f} : \mathbb{P}^1 \to \mathbb{P}(V)$ and $f(0), f(\infty)$ are distinct points of Y.$\square$

Proof of b). It is clear that we can assume that the line $L_x \subset V, x \in X$ span V. Choose a basis $e_i, 1 \leq i \leq d$ of V such that

$$\lambda(t)e_i = t^{n_i}e_i, t \in k^\star$$

where $n_1 \leq n_2 \leq ... \leq n_d$ and choose a point $x \in X$ such that that for $v \in L_x - \{0\}$ we have $v = \sum_{i=1}^d c_i e_i, c_i \in k, c_1 \neq 0$. Let $\bar{f} : \mathbb{P}^1 \to \mathbb{P}(V)$ be the morphism such that $\bar{f}(t) = \lambda(a)x$ for $a \in \mathbb{G}_m \subset \mathbb{P}^1$ and $y_0 := \bar{f}(0) \in Y$. To prove the result we have to construct two other points in Y.

Let $W \subset V$ be the span of $e_i, 2 \leq i \leq d$. Consider $Z := Y \cap \mathbb{P}(W)$. As follows from Proposition 3 we have $dim(Z) \geq 1$. It is clear that Z is $\mathbb{G}_m$-invariant. Since $\mathbb{G}_m$ is connected every irreducible component of Z is also $\mathbb{G}_m$-invariant. It follows now from the part a) that $|Z \cap Y| > 1$. Since (?) $y_0 \notin \mathbb{P}(W)$ we see that $|Y| \geq 3$.□

Problem 0.5. Let G be a connected algebraic group, $T \subset G$ a maximal torus, $P \subset G$ a proper parabolic subgroup and $G/P^T \subset G/P$ the subset of T-fixed points. Then

a) $|G/P^T| > 1$ and $|G/P^T| > 2$ if $dim(G/P) > 1$.

b) $W_G = \{e\}$ iff G is solvable.

c) If $W = \mathbb{Z}/2\mathbb{Z}$ iff $dim(\mathcal{B}) = 1$.

Lemma 0.6. *Let G be a connected algebraic group, and $T \subset G$ be maximal torus G. Then is generated by Borel subgroups containing T.*

Proof. We prove the Lemma by induction in $dim(G)$. Let P be the subgroup of G generated by Borel subgroups contains T. Then (?) P is a closed subgroup of G containing a Borel subgroup. Therefore P is a parabolic subgroup. If $P \neq G$ then there exists $x \in N_G(T)$ such that $xPx^{-1} \neq P$ [see Problem 5 a)]. By inductive assumptions xPx^{-1} is generated by Borel subgroups containing T. So $xPx^{-1} \subset P$.□

Problem 0.7. a) Let G be a connected algebraic group, Then there exists the maximal normal invariant solvable connected subgroup $R(G) \subset G$ and it is closed.

b)There exists the maximal connected normal invariant unipotent connected subgroup $R_u(G) \subset G$ and it is closed.

c) $R(G)$ is equal to the connected component of the intersection of all the Borel subgroups of G.

Definition 0.8. Let G be a connected algebraic group,

a) The subgroup $R(G) \subset G$ is called *the radical* of G.

b) The subgroup $R_u(G) \subset G$ is called *the unipotent radical* of G.

c) G is *reductive* if $R_u(G) = (e)$.

d) G is *semisimple* if $R(G) = (e)$.

e) The *rank* $r(G)$ of G is the dimension of a maximal torus of G.

f) The *semisimple rank* $sr(G)$ of G is the dimension of a maximal torus of $G/R(G)$.

Problem 0.9. a) The groups $GL_n, n > 0$ are reductive.

b) The groups $SL_n, n > 1, Sp(2n)n > 0$ and $SO(n), n > 2$ are semisimple.

c) $r(SL_n) = n - 1, r(Sp(2n)) = n, r(SO(n)) = [n/2]$.

d) Construct an isomorphism $PGL_2 \to SO(3)$ where $PGL_2 := GL_2/Z(GL_2)$ where $Z(GL_2) = \mathbb{G}_m$ is the center of the group GL_2.

e) If we write elements of the group GL_2 as matrices $\kappa = \begin{pmatrix} a_{11}(\kappa) & a_{12}(\kappa) \\ a_{21}(\kappa) & a_{22}(\kappa) \end{pmatrix} \in GL(2, k)$ we see that the functions

$$\phi_{i,j;i',j'}(\kappa) := a_{ij}(\kappa)a_{i'j'}(\kappa)/det(\kappa)$$

are regular functions of the group PGL_2. Prove that the ring $k[PGL_2]$ is generated by these functions.

Claim 0.10. *If Let G is a connected algebraic semisimple group of semisimple rank 1 then $dim(\mathcal{B}) = 1$ and $|\mathcal{B}^T| = 2$.*

Proof of Claim. It is sufficient to prove the result for the group $G/R(G)$. So we can assume that G is semisimple. Let T be a maximal torus of G and $\mathcal{B}$ be the variety of Borel subgroups of G. Since $r(G) = 1$ we have (?) $dim(T) = 1$. So T is isomorphic to $\mathbb{G}_m$ and therefore $|W_G| \leq |Aut(\mathbb{G}_m)| = 2$ and it follows from Problem 5 that $dim(\mathcal{B}) = 1$ and $|\mathcal{B}^T| = 2$.□

Fix a point $y \in \mathcal{B} - \mathcal{B}^T$ and consider the morphism

$$f : \mathbb{G}_m \to \mathcal{B} - \mathcal{B}^T, f(a) := ay, a \in k^\star$$

As follows from Problem 2 the morphism f extends to a morphism $\bar{f} : \mathbb{P}^1 \to \mathcal{B}$. We write $x_0 := \bar{f}(0), x_\infty := \bar{f}(\infty)$ and denote by

$$f_0 : \mathbb{P}^1 - \{\infty\} \to \mathcal{B} - x_\infty, f_\infty : \mathbb{P}^1 - \{0\} \to \mathcal{B} - x_0$$

the restrictions of $\bar{f}$ on $\mathbb{P}^1 - \{0\}$ and $\mathbb{P}^1 - \{\infty\}$.

Problem 0.11. Let G be a connected algebraic semisimple group of semisimple rank $1, Y := \mathcal{B} - \mathcal{B}^T$. Then

a) The action of the torus $T = \mathbb{G}_m$ on Y is transitive.

b) Fix $y \in Y$ and define $f : \mathbb{G}_m \to Y$ by $f(a) = ay$. Show that existence of $r > 0$ such that $f^\star(k[Y]) = k[a^\pm r]$ (in other words $f^\star(k[Y])$ is the span of characters $\chi_{nr} : a \to a^{nr}, n \in \mathbb{Z}$.

[A hint] Use Lemma 4.3.

c) The field $k(\mathcal{B})$ is isomorphic to the field $k(t)$ of rational functions.

d) Let $Aut_k(k(t)$ be the group of automorphisms of the field $k(t)$ which act trivially on k. For any 2×2-matrix $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,k)$ we define

$$\alpha_\kappa \in Aut_k(k(t), \alpha_\kappa(r(t) := r(\frac{at+b}{ct+d})$$

Show that the map $\tilde{\tau} : GL(2,k) \to Aut(k(t), \kappa \to \alpha_\kappa$ which induces a homomorphism $\tau : PGL_2 \to Aut(k(t)$.

e) Prove the surjectivity of $\tau : PGL_2 \to Aut(k(t))$.

f) Let $\tilde{V} \subset (\mathbb{P}^1)^3$ be the subset of distinct triples. For any point $v_0 =\in \tilde{V}$ the map $PGL_2 \to \tilde{V}, \kappa \to \kappa(v_0)$ defines an isomorphism of of affine algebraic varieties.

We see that the action of $\underline{G}$ on $\mathcal{B}$ induces a group homomorphism $f : G \to PGL_2(k)$.

Theorem 0.12. *The group homomorphism $f : G \to PGL_2(k)$ is algebraic.*

Proof. As follows from Problem 9 e) the functions

$$\phi_{i,j;i',j'}(\kappa) := a_{ij}a_{i'j'}/det(\kappa), \kappa = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(k)$$

generate the ring $k[PGL_2]$. So it is sufficient to show that functions

$$f^\star(\phi_{i,j;i',j'}) : G \to k, f^\star(\phi_{i,j;i',j'})(g) := \phi_{i,j;i',j'}(f(g))$$

are regular. Since f is a homeomorphism and G is connected it is sufficient to show the existence of a non-empty open subset $U \subset G$ such that the restriction of $f^\star(\phi_{i,j;i',j'})$ on U are regular.

We denote by $(g,x) \to gx$ the natural action of the group G on $\mathcal{B}$. Let T be a maximal torus of G. As follows from Problem 11 there exists a regular function t on $\mathcal{B} - \mathcal{B}^T$ such that

$$k(\mathcal{B}) = k(t), t(ax) = a^r t(x), a \in k^\star, x \in \mathcal{B} - \mathcal{B}^T$$

$$t(gx) = \frac{a_{11}t + a_{12}}{a_{21}t + a_{22}}, f(g) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We denote by $V \subset \mathcal{B} - \mathcal{B}^{T^3}$ the subset of distinct triples, choose a point $v_0 = (x_1, x_2, x_3) \in V$ and define

$$U := \{g \in G | gx_i \in \mathcal{B} - \mathcal{B}^T, i = 1, 2, 3$$

$U \subset G$ is an open subset containing $\{e\}$ and the map $\tau : U \to V, g \to (gx_1, gx_2, gx_3)$ is regular.

Consider

$$U' := \{\gamma \in PGL_2 | \gamma x_i \in \mathcal{B} - \mathcal{B}^T, i = 1, 2, 3$$

As follows from Problem 11 f) the map $\theta : U' \to V, \kappa \to \kappa(v_0)$ defines an isomorphism of of affine algebraic varieties. But it is clear that $f_U = \theta^{-1} \circ \tau.square$