**Definition 0.1.** a) A *torus* is a connected diagonalizable group [see Definition 4.4].

b) For any two tori $\underline{T}, \underline{T}'$ we denote by $Hom(\underline{T}, \underline{T}')$ the abelian group of algebraic homomorphisms from $T$ to $T'$.

c) For any torus $\underline{T}$ we define

$$X_\star(\underline{T}) := Hom(\mathbb{G}_m, \underline{T}), X^\star(\underline{T}) := Hom(\underline{T}, \mathbb{G}_m)$$

and denote by $<,>$ the pairing

$$X^\star(\underline{T}) \times X_\star(\underline{T}) \to Hom(\mathbb{G}_m, \mathbb{G}_m), (\chi, \phi) \to \chi \circ \phi$$

**Problem 0.2.** a) The map

$$a : \mathbb{Z} \to Hom(\mathbb{G}_m, \mathbb{G}_m), a(n)(x) := x^n, x \in k^\star$$

is a bijection. We will identify the group $Hom(\mathbb{G}_m, \mathbb{G}_m)$ with $\mathbb{Z}$.

b) For any torus $\underline{T}$ the groups $X_\star(\underline{T}), X^\star(\underline{T})$ are finitely generated free abelian group and the pairing

$$<,>: X^\star(\underline{T}) \times X_\star(\underline{T}) \to Hom(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$$

is perfect [that is the induced homomorphism $X^\star(\underline{T}) \to Hom(X_\star(\underline{T}), \mathbb{Z})$ is an isomorphism].

c) Any representation of $\in T$ is a direct sum of characters [one-dimensional representations].

d) Let $S \subset T$ be a torus. There exists a nonempty open subset $U$ in $S$ such that $Z_G(S) = Z_G(s)$ for all $s \in U$. [A hint. Consider first the case when $G = GL_n$].

**Lemma 0.3** (Rigidity of tori)**.** .

*Let $T, T'$ be tori, $X$ an irreducible algebraic variety and $f : T \times X \to T'$ a morphism such that for any $x \in X$ the map $f_x : T \to T', f_x(t) := f(t, x)$ is a group homomorphism. Then the homomorphism $f_x$ does not depend on $x \in X$.*

**Proof**. It is sufficient to show that for any $\chi \in X(T')$ the homomorphism $\chi \circ f_x : \underline{T} \to \mathbb{G}_m$ does not depend on $x \in X$. So we can assume that $\underline{T}' = \mathbb{G}_m$ and consider $f$ as a regular function on $X \times T$. As follows from Lemma 4.2 we can write $f$ as a finite sum

$$f(t, x) = \sum_{\chi \in X(T)} \chi(t) f_\chi(x), f_\chi \in k[X]$$

Since $f_x$ is a character of $T$ it follows from Problem 4.5 that for any $x \in X$ there exists $\chi_x \in X(T)$ such that $f_\chi(x) = 0$ if $\chi \neq \chi_x$ and

$f_{\chi_x}(x) = 1$. For any $\chi \in X(T)$ we define $X_\chi := \{x \in X | \chi_x = \chi\}$. It is clear (?) that $X_\chi \subset X$ is closed and also open. Since $X$ is irreducible we see that either $X_\chi = X$ or $X$ is empty. Since $X = \cup_{\chi \in X(T)} X_\chi$ we see that $f(t,x) \equiv \chi(t)$ for some $\chi \in X(T)$.□

**Definition 0.4.** Let $G$ be a connected algebraic group. A *Cartan subgroup* $C$ of $G$ is a group of the form $C = Z_G^0(T)$ where $T$ is a maximal torus of $G$.

**Remark**. As follows from Corollary 6.17 all Cartan subgroups of $G$ are conjugate.

**Lemma 0.5.** *Let $G$ be a connected algebraic group and $C = Z_G^0(T)$ a Cartan subgroup of $G$. Then*

*a) $C$ is nilpotent.*

*b) $C = N_G^0(C)$*

**Proof of a).** Let $B$ be a Borel subgroup of $C = N_G^0(T)$ where $T$ is a maximal torus of $G$. It follows from Theorem 6.14 that $B$ is nilpotent. But then by Lemma 5.19 $C = B$.□

**Proof of b).** As follows Corollary 6.17 $T$ is the unique maximal torus of $C$. Therefore the group from $N_G(C)$ normalizes $T$. and we obtain the homomorphism $f : N_G(C) \to Aut(T)$. But it follows from Lemma 3 that any homomorphism from a connected group to the group of automorphism of a torus is trivial. So $N_G^0(C) \subset Z_G^0(T) = C$.□

**Proposition 0.6.** *Let $G$ be a connected algebraic group and $S \subset G$ be a torus. Then*

*a) The centralizer $Z_G(S)$ is connected.*

*b) If $B$ is a Borel subgroup of $G$ containing $S$ then $B \cap Z_G(S)$ is a Borel subgroup of $Z_G(S)$.*

*c) Any Borel subgroup of $Z_G(S)$ has a form $B \cap Z_G(S)$ where $B$ is a Borel subgroup of $G$.*

**Proof of a).**

**Claim 0.7.** *For any $x \in Z_G(S)$ there exists a Borel subgroup containing $x$ and $S$.*

**Proof of Claim.** Let $\mathcal{B} := G/B, X := \{gB | xgB = gB\} \subset \mathcal{B}$. Then (?) $X$ is closed in $\mathcal{B}$ and therefore complete. So (?) there is a point $x = gB$ fixed by $S$. But then $gBg^{-1}$ contains both $x$ and $S$.□

Now the part a) follows from Corollary 6.16.

**Proof of b).** Let $B$ be a Borel subgroup of $G$ containing $S$. Then $B \cap Z_G(S)$ is a connected solvable subgroup of a connected group $Z_G(S)$. To show that $B \cap Z_G(S)$ is a Borel subgroup of $Z_G(S)$ we have to show that the quotient $Z_G(S)/B \cap Z_G(S)$ is complete. Since the action of $Z_G(S)$ on $\mathcal{B}$ induces a bijection from $Z_G(S)/B \cap Z_G(S)$ to the orbit $Y := Z_G(S)B \subset \mathcal{B}$ it is sufficient (?) to show that the orbit $Y \subset \mathcal{B}$ is equal to the closure $\bar{Y} \subset \mathcal{B}$.

**Claim 0.8.** *$\bar{y}S\bar{y}g^{-1} \subset SB_u$ for any $\bar{y} \in \bar{Y}$.*

**Proof of Claim.** For any $y \in Y$ we have $ySy^{-1} \subset B$. By continuity we have $\bar{y}S\bar{y}^{-1} \subset B$. Consider the morphism $\bar{Y} \times S \to B/B_u$ sending $s$ to $ysy^{-1}B_u$. Since $\bar{Y}$ is irreducible it follows from Lemma 3 (?) that $\bar{y}s\bar{y}^{-1} \subset sB_u$ for all $s \in S, \bar{y} \in \bar{Y}$.$\square$

It is clear then that for any $\bar{y} \in \bar{Y}$ $\bar{y}S\bar{y}^{-1} \subset SB_u$ is a maximal torus on $SB_u$ and therefore there exists $u \in B_u$ such that $\bar{y}S\bar{y}^{-1} = uSu$. So $\bar{y} \in Z_G(S)B\square$

The part c) follows from the conjugacy of Borel subgroups in $Z_G(S)B$.$\square$

**Theorem 0.9.** *Let $G$ be a connected algebraic group. Then*

*a) The union of Cartan subgroups contains the dense open subset of $G$.*

*b) Every element of $G$ lies in a Borel subgroup.*

*c) Every semi-simple element of $G$ lies in a maximal torus.*

*d) $N_G(B) = B$ for any Borel subgroup $B$ of $G$.*

**Proof of a).** Let $T \subset G$ be a maximal torus, $U \subset T$ an open subset as in Problem 2.

**Claim 0.10.** *For any $t \in U, u \in C_u$ the only conjugate of $C$ containing $tu$ is $C$.*

**Proof of Claim**. Since $t \times u$ is the Jordan decomposition of $tu$ any conjugate of $C$ containing $tu$ contains $t$. So it is sufficient to prove Claim in the case $u = e$.

Assume that $t \in xCx^{-1}, x \in G$. Then $t$ lies in the unique maximal torus $xTx^{-1}$ of $xCx^{-1}$. So we have $x^{-1}tx \in T$. Since $Z_G(t) = Z_G(T)$ we have

$$C = Z_G^0(T) \subset Z_G^0(x^{-1}tx) = x^{-1}Z_G^0(t)x = x^{-1}Z_G^0(T)x = C\square$$

Since the set $UC_u$ is open and dense in $C$ the part a) follows from Lemma 5.24.

**Proof of b).** Since $C$ is nilpotent [Lemma 5] and connected it lies in some Borel subgroup $B$. Therefore the union

$$\cup_{g\in G} gBg^{-1} \supset \cup_{g\in G} gCg^{-1}$$

is dense in $G$. But them b) follows from Problem 5.23 c).

The part c) follows now from Corollary 6.16.□

**Proof of d).** We prove the equality $N_G(B) = B$ by the induction in $dim(G)$. Choose a maximal torus $T$ of $B$. Since $xTx^{-1}$ is a maximal torus of $B$ for any $x \in N_G(B)$ there exists $b \in B$ such that $xTx^{-1} = bTb^{-1}$. So it is sufficient to prove that any $x \in N_G(B) \cap N_G(T)$ belongs to $B$.

Consider the group homomorphism $\phi : T \to T, \phi(t) := xtx^{-1}t^{-1}$. We consider separately the case when $\phi$ is onto and the case when $\phi$ is not surjective.

a) Assume that $\phi$ is not surjective. Then $dim(Ker(\phi)) > 0$. Therefore $S := Ker^0(\phi)$ is a non-trivial subtorus in $G$ and $x \in Z_G(S)$. If $G \neq Z_G(S)$ we can by Proposition (?) apply the inductive assumption to $B \cap Z_G(S) \subset Z_G(S)$. On the other hand it $G = Z_G(S)$ we can apply the inductive assumption to $B/S \subset G/S$.

b) Assume that $\phi$ is surjective. Let $\rho : G \to GL(V)$ be a representation and $L \subset V$ a line such that $N_G(B) = St_G(L)$. Since $BL = L$ and the group $B_u$ doe snot have non-trivial unipotent characters we have $B_u l = l$ for all $l \in L$. We also have $t'l = l$ for any $t' \in T$. To see this we write $t' = \phi(t) = xtx^{-1}t^{-1}$ and observe that both $t$ and $x$ preserve $L$. So $Bl = l$ for all $l \in L$. Therefore the map $f : G \to V, g \to gl$ factorizes through the map $\bar{f} : G/B \to V$. Since $G/B$ is complete the maps $\bar{f}$ and $f$ are constant. So $gl = l$ for all $g \in G$ and therefore $N_G(B) = G$.□

**Problem 0.11.** a) Let $G$ be a connected algebraic group. Then $N_G(P) = P$ for any parabolic subgroup $P$ of $G$.

b) For any $g \in GL_n$ the centralizer $Z_{GL_n}(g)$ is connected.

c) For any semisimple $s \in Sp(2n)$ the centralizer $Z_{Sp(2n)}(s)$ is connected.

d) Find a unipotent element $u \in SL_2$ such that the centralizer $Z_{SL_2}(u)$ is disconnected.

e) Find a semisimple $s \in SO(3)$ such that the centralizer $Z_{SO(3)}(s)$ is disconnected.

**Definition 0.12.** Often one defines an object $X$ of a category $\mathcal{C}$ in two steps:

i) One introduces some set $A$ of of choices and for any $a \in A$ defines an object $X^a \in Ob(\mathcal{C})$ whose construction depends on a choice of $a \in A$.

ii) For any pair $a, a' \in A$ one defines an isomorphism $\alpha_{a,a'} : X^{a'} \to X^a$ in such a way that for any triple $a, a', a'' \in A$ one has the equality

$$\alpha_{a,a'} \circ \alpha_{a',a''} = \alpha_{a,a''}$$

**Problem 0.13.** a) Let $X$ be a smooth $n$-dimensional manifold, $x \in X, \mathcal{C}$ be the category of $\mathbb{R}$-vector spaces and $A$ be the set of charts $a : U \hookrightarrow \mathbb{R}^n, a(0) = x$ where $U$ is an open neighborhood of $x$ in $X$. For any $a \in A$ we define $X^a := \mathbb{R}^n$ and for any pair $a' : U' \hookrightarrow \mathbb{R}^n, a'' : U'' \hookrightarrow \mathbb{R}^n$ we define $\alpha_{a,a'} \in Aut(\mathbb{R}^n)$ as the differential of the map $a \circ a'^{-1} : V \to \mathbb{R}^n$ at 0 where $V \subset \mathbb{R}^n$ is a [sufficiently small] neighborhood of 0. Show that the isomorphisms $\alpha_{a,a'} \in Aut(\mathbb{R}^n)$ satisfy the condition ii). The resulting object of $\mathcal{C}$ [a vector space] is denoted $T_X(x)$ and is called *the tangent space to $X$ at $x$*.

b) Let $X$ be an irreducible algebraic variety, $\mathbb{C}$ be the category of fields and $A$ be the set of open non-empty open affine subsets $U_a \subset X$. We define $A_a$ as the ring of regular functions on $U_a$ and $k_a$ as the field of fractions of $A_a$. Construct isomorphisms $\alpha_{a,a'} : k_{a'} \to k_a$ for any pair of non-empty open affine subsets $U_a, U_{a'} \subset X$ of and check the condition ii). The resulting field is denoted $k(X)$ and is called *the field of rational functions* on $X$.

**Definition 0.14.** a) Let $\underline{G}$ be a connected affine algebraic group. We define the *Cartan torus* $\mathbb{T}$ of $G$ as follows. We take $A$ to be the set of Borel subgroups of $G$, for any $B \in A$ we define $\mathbb{T}^B := B/B_u$. Given a pair $B, B' \in A$ we choose $g \in G$ such that $B = gB'g^{-1}$ and define $\alpha_{B,B'} : \mathbb{T}^{B'} \to \mathbb{T}^B$ as the isomorphism induced by the isomorphism $A_g : B' \to B, A_g(b') := gb'g^{-1}$.

b) As in a) we take $A$ to be the set of Borel subgroups of $G$, For any $B \in A$ we define an algebraic variety $\mathcal{B}^B$ with a transitive action of $G$ by $\mathcal{B}^B := G/B$. Given a pair $B, B' \in A$ we choose $g \in G$ such that $B = gB'g^{-1}$ and define an isomorphism

$$\tilde{\alpha}_{B,B'} : G/B \to G/B', \tilde{\alpha}_{B,B'}(xB) := xB'g^{-1} = x'g^{-1}B$$

We denote the corresponding algebraic homogeneous $G$-variety by $\underline{\mathcal{B}}$. It is clear that points of $\underline{\mathcal{B}}$ are Borel subgroups of $G$ and the group $G$ acts on points of $\underline{\mathcal{B}}$ [= Borel subgroups of $G$] by conjugation.

c) Fix a Borel subgroup $B$ of $G$, a maximal torus $T \subset B$. The composition of the imbedding $T \hookrightarrow B$ and the natural projection $B \to B/B_u = \mathbb{T}$ defines an isomorphism $\kappa_T : T \to \mathbb{T}$ and therefore and imbedding $N_G(T)/Z_G(T) \to Aut(\mathbb{T})$. The image $W_G^B \subset Aut(\mathbb{T})$ is called the *Weyl group* of $G$.

**Problem 0.15.** a) Show that the isomorphisms $\alpha_{B,B'} : \mathbb{T}^{B'} \to \mathbb{T}^B$ and $\tilde{\alpha}_{B,B'} : G/B \to G/B'$ do not depend on a choice of $g \in G$ such that $B = gB'g^{-1}$ and

$$\alpha_{B,B'} \circ \alpha_{B',B''} = \alpha_{B,B''}, \tilde{\alpha}_{B,B'} \circ \tilde{\alpha}_{B',B''} = \tilde{\alpha}_{B,B''}$$

b) Show that the subgroup $W_G^B \subset Aut(\mathbb{T})$ does not depend on a choice of a Borel subgroup $B$. We will denote this group by $W_G$.

c) For any pair $B, B' \in \mathcal{B}$ we choose maximal tori $T \subset B, T' \subset B'$ and choose $g \in G$ such that $B = gB'g^{-1}, T = gT'g^{-1}$. Then automorphisms

$$w_{B',B} \in W_G, w_{B',B} := \kappa_T \circ A_g \circ \kappa_{T'}^{-1}$$

does not depend on a choice of tori $T, T'$ and a choice of $g$.

d) The map $\mathcal{B} \times \mathcal{B} \to W_G, (B, B') \to w_{B',B}$ defines a bijection between $G$-orbits on $\mathcal{B} \times \mathcal{B}$ and the group $W_G$.

e) Show that $W_{GL_n} = S_n$ where $S_n$ is the symmetric group

f) Describe the group by $W_G$ for $G = Sp(2n)$ and $G = SO(n)$.

[A hint]. In the case $G = SO(n)$ consider separately the cases of even and odd $n$.

g) For any $w \in S_n = W_{GL_n}$ we denote by $l(w)$ the number of pairs $1 \leq i < j \leq n$ such that $w(i) > w(j)$. We fix a Borel subgroup $B$ of $GL_n$ and define

$$X_w := \{B' \in \mathcal{B} | w_{B',B} = w\}$$

Then $l(\sigma) = dim(X_w)$.

h) Fix a maximal tori $T$ of $G$. The group $N_G(T)$ acts naturally on the subset $\mathcal{B}^T \subset \mathcal{B}$ of $T$-fixed points. Show that this action is simply transitive.