Problem 0.1. Let $f : X \to Y$ be a morphism of algebraic varieties, $X$ is affine and irreducible and $f : X \to Y$ is a bijection. Then $\dim(X) = \dim(Y)$

Definition 0.2. A curve is an algebraic variety of dimension 1.

Remark. I always assume that curves are irreducible.

Let $X$ be an irreducible affine variety, $f : X \to C$ be a non-constant morphism to a curve, $c \in \text{Im}(f)$, $Y := f^{-1}(c)$.

Claim. Then $\dim(Y) = \dim(X) - 1$.

This is a very useful result but the proof of this result is based on some results from Commutative algebra [such as the Normalization Lemma of Noether] but the proof requires more extensive knowledge of Algebra then I assume. I'll prove a very special case of the theorem which will suffice for our needs. We start with the following general result.

Lemma 0.3. Let $X$ be an irreducible affine variety, $f : X \to C$ be a non-constant morphism to a curve, There exists a curve $Y \subset X$ such that the restriction of $f$ on $Y$ is not a constant.

Proof. It is clear (?) that we can assume that the curve $C$ is affine. The proof is by induction in the dimension of $X$. If $\dim(X) = 1$ then we can take $Y = X$. So assume that $\dim(X) > 1$. It is sufficient to show the existence of a proper closed subset $Y \subset X$ such that that the restriction of $f$ on $Y$ is not a constant morphism.

Since $\dim(X) > 1$ there exist (?) regular functions

$$g : X \to \mathbb{A}^1, r : C \to \mathbb{A}^1$$

such that $r \circ f$ and $g$ are algebraically independent. Consider the map

$$\phi : X \to \mathbb{A}^2, x \to (r \circ f(x), g(x))$$

and define $X' := \text{Im}(\phi) \subset \mathbb{A}^2$. Then $X' \subset \mathbb{A}^2$ and since $\mathbb{A}^2$ is irreducible we see that $\dim(\mathbb{A}^2 - X') < 2$. Therefore (?) there exists $b \in k$ such that the set $\{a \in k | (a, b) \notin X'\}$ is finite. Let $Y = g^{-1}(b)$. Then the restriction of $f$ on $Y$ is not a constant morphism.$\square$

Lemma 0.4. Let $f : X \to C$ be as Lemma 3 and assume that an algebraic group $H$ acts on $X$ without fixed points in such a way that fibers of $f$ are $H$-orbits. Then $\dim(H) = \dim(G) - 1$.

Proof. As follows from Problem 1 we have $\dim(f^{-1}(x)) = \dim(H)$ for all $x \in X$. Since $X$ is irreducible we see that $\dim(H) < \dim(G)$. Assume that $\dim(H) < \dim(G) - 1$. Let $Y \subset G$ be a curve as in
Lemma 3 and consider the map $a : H \times Y \to G$ given by $a(h, y) := hy$. Since $f(Y)$ is not constant the image $Im(f)$ is dense in $C$ and therefore the subset $Z := a(H \times Y)$ is dense in $X$. So

$$dim(H) + 1 = dim(H \times Y) \geq dim(Z) = dim(X)$$

So $dim(H) \geq dim(X) - 1$.

**Problem 0.5.** a) Let $f : X \to Y$ be a morphism of irreducible algebraic varieties such that $f(X)$ is dense in $Y$.

a) There exists a closed subvariety $Z$ of $X$ such that $dim(Z) = dim(Y)$ and $f(Z)$ is dense in $Y$.

b) Assume that an algebraic group $H$ acts on $X$ without fixed points in such a way that fibers of $f$ are $H$-orbits. Then $dim(H) = dim(X) - dim(Y)$.

**Lemma 0.6.** Let $X, Y$ be irreducible algebraic varieties and $p : X \to Y$ a morphism such that $p(X)$ is dense in $Y$ and there exists a non-empty open subset $U$ of $Y$ such that $p^{-1}(u)$ is finite for all $u \in U$. Then $dim(X) = dim(Y)$.

**Remark.** The conclusion of the Lemma is true under the much weaker assumption. It is sufficient to know that there exists one $y \in Y$ such that the set $p^{-1}(y)$ is finite and not empty.

**Proof.** Since the image $p(X)$ is dense in $Y$ it is easy to see (?) that $dim(X) \geq dim(Y)$. Assume that $dim(X) > dim(Y)$. One can easily reduce the proof to the case when $X = (X, A)$ and $Y = (Y, B)$ are affine and the map $p : X \to Y$ is surjective. Since $p(X)$ is dense in $Y$ we see that $p^* : B \to A$ is an imbedding and we consider $B$ as a subring of $A$.

Let $F, E$ be the fields of fractions of the rings $A, B$. We want to show that $trdeg_k(B) = trdeg_k(A)$. Assume that $trdeg_k(B) < trdeg_k(A)$. Then we can find (?) $f \in A$ such that for any $b_0, \ldots, b_n \in B, b_n \neq 0$ we have $\sum_{i=0}^n b_i f^i \neq 0$. Let $Z \subset Y \times k$ be the image of the map $x \to (p(x), f(x))$. Since $Z$ is constructible we see (?) that $Z$ is dense in $Y \times k$ and therefore there exists a closed proper subset $W \in Y \times k$ such that $Z \supset (Y \times k) - W$. Since $Y$ is irreducible and $W$ is proper subset of $Y \times k$ the intersection $(U \times k) \cap W$ is a proper subset of $U \times k$. Therefore (?) there exists $u \in U$ such that the intersection $(\{u\} \times k) \cap W$ is finite. But then the fiber $p^{-1}(u) = \{u\} \times k \cap Z$ is infinite. $\square$
Lemma 0.7. a) If $P \subset G$ is a parabolic subgroup then for any action of $G$ on $X$ and a point $x \in X$ such that $P \subset St_x$ the orbit $\Omega(x) \subset X$ is closed and complete.

b) A closed subgroup $P \subset G$ is parabolic iff there exists a finite dimensional representation $\rho : G \to GL(V)$ and a line $L \subset V$ such that $P = St_L$ and the orbit $\Omega(L) \subset \mathbb{P}(V)$ is closed.

c) If $G$ is connected then $Z^0(G) \subset Z(B) \subset Z(G)$ where $Z(G)$ is the center of $G$ and $B \subset G$ is a Borel subgroup.

d) If $G$ is connected and $B$ is nilpotent then $B = G$.

Proof of a). As follows from the proof of Proposition 14 there exists a finite dimensional representation $\rho : G \to GL(V)$ and a line $L \subset V$ such that $P \supset St_L$ and the orbit $\Omega(L) \subset \mathbb{P}(V)$ is closed. We can assume (?) $\Omega(x) \subset X$ is dense. Let $Z = \Omega(L)$. Consider the $G$ orbit $Z$ of the point $(x, L) \in X \times Y$ of diagonal action of $G$ on $X \times Y$.

Claim 0.8. $Z \subset X \times Y$ is closed.

Proof of Claim. It is clear (?) that the restriction $q$ of the projection $p_Y : X \times Y \to Y$ on $Z$ is a bijection. Consider the closure $\bar{Z}$. Since $Y$ is a $G$-orbit all the fibers of restriction $\bar{q}$ of the projection $p_Y$ to $\bar{Z}$ are isomorphic. But this implies (?) that they consists of one point. Since $q : Z \to Y$ is onto we see that $\bar{Z} = Z$.□

Since $Y$ is complete the projection $p_X(Z) \subset X$ is closed and proper [see Problem 5.3].□

Proof of b). It is easy to derive (?) from the part a) and the Chevalley theorem that for any for any parabolic subgroup $P \subset G$ there exists a finite dimensional representation and a line $L \subset V$ such that $P = St_L$ and the orbit $\Omega(L) \subset \mathbb{P}(V)$ is closed. Conversely, let $\rho : G \to GL(V)$ be Choose a Borel subgroup $B'$. As follows from Proposition 5.14 there exists a point $x \in \Omega(L)$ such that $B' \subset St_x$. Let $g \in G$ be such that $x = gL$. But then $B := g^{-1}B'g \subset P = St_L$.□

Proof of c). Since $Z^0(G)$ is connected and solvable it lies in some Borel subgroup $B'$. But since $B$ and $B'$ are conjugate $Z^0(G)$ lies in $B$. But then $Z^0(G) \subset Z(B)$.

To finish the proof of c) we have to show that any $z \in Z(B)$ belongs to $Z(G)$. Fix $z \in Z(B)$ and consider the morphism $f : \overline{G/B} \to G, g \to gzg^{-1}$. Since $z \in Z(B)$, $f$ factors through a morphism $\overline{f} : \overline{G/B} \to G$. Since $\overline{G/B}$ is complete and $G$ is affine and $f(e) = z$ we see that $f \equiv z$.□
Proof of d). The proof is by induction in $\text{dim}(B)$. If $B = \{e\}$ then $G = G/B$ is complete and affine. So $G = \{e\}$. If $B \neq \{e\}$ then $Z(B) \neq \{e\}$. Since $Z(B) \subset Z(G)$ we can replace $G$ by $G/Z(B)$, $B$ by $B/Z(B)$ and apply the inductive assumption. □

Definition 0.9. a) We denote the algebraic variety $G/H(\rho, L)$ by $G/H$ and call the natural morphism $\phi : G \to G/H$ the canonical projection.

b) We define the stabilizer $St_{G/H} \subset G \times G/H$ of the action of $G$ on $G/H$ by $St_{G/H} := \{ (g, x) | gx = x \}$ and write $X_{G/H} := \text{proj}_G(St_{G/H}) \subset G$.

c) For any $h \in H$ we define $Y_h := \{ y \in G/H | y^{-1}hy \in H \}$.

Problem 0.10. Show that

a) The stabilizer $St_{G/H}$ is a closed subset of $G \times G/H$.

b) $\text{dim}(St_{G/H}) = \text{dim}(G)$

[A hint] Use the result of Problem 5.

c) If $H \subset G$ is a parabolic subgroup then the image $X$ of $St_{G/H}$ under the projection $G/H \times G \to G$ is closed.

Lemma 0.11. Assume that there exists an open dense subset $U$ of $H$ such that the sets $Y_u, u \in U$ are finite. Then $X := \bigcup y \in G g^{-1}Hg$ is dense in $G$.

Proof. Let $V := \bigcup y \in G u \in U g^{-1}ug \subset St_{G/H}$ and $\pi : V \to G$ be the restriction of the projection $p_G : G/H \times G \to G$ on $V, Z := \pi G(V)$.

Since $G$ is irreducible it is sufficient to show that $\text{dim}(Z) = \text{dim}(G)$.

Since for any $v = g^{-1}ug$ the fiber $\pi^{-1}(\pi(v)) = Y_u$ are finite it follows from Lemma 6 that $\text{dim}(Z) = \text{dim}(V) = \text{dim}(G) = \text{dim}(St_{G/H}) = \text{dim}(G)$. □

Solvable groups

Let $T_n$ be the group of upper-triangular $n \times n$-matrices and $U_n \subset T_n$ the subgroup of unipotent upper-triangular matrices.

Problem 0.12. a) Let $G$ be a closed connected subgroup of $T_2$. Then either $G = (e)$ or $G = U_2$ or $G$ is conjugated to $D_2$ or $G = T_2$.

b) For any closed connected solvable group $G$ the subset $G_u \subset G$ of unipotent elements is a closed normal subgroup of $G$.

Lemma 0.13. a) Let $G$ be a commutative connected affine algebraic group and $G_s, G_u$ be the subsets of semisimple and unipotent elements.

Then $G_s, G_u$ are closed subgroups of $G$ and $G = G_sG_u$. 

b) Let $G$ be a solvable connected affine algebraic group such that all $g \in G$ are semisimple. Then $G$ is diagonalizable. [that is $G$ is isomorphic to a subgroup of the group $D_n$ of diagonal matrices].

**Proof of a)**. By the Levi-Kolchin Theorem we can assume that $G \subset T_n$. It is clear (?) that $G_u = G \cap U_n$. So $G_u \subset G$ is closed.

Since $G$ be a commutative it is clear (?) that the subset $G_s \subset G$ is a subgroup. Moreover we can (?) choose a basis in $k^n$ such that $G_s \subset D_n$ where $D_n \subset GL_n(k)$ is the subgroup of diagonal matrices. But then $\bar{G}_s \subset G \cap D_n = G_s$. □

**Proof of b)**. As before we assume that $G \subset T_n$. Then the commutator $[G, G]$ lies in the subgroup $U_n$ of unipotent upper-triangular matrices. Since all elements of $G$ are semisimple we see that $G$ is commutative. So we can choose a basis in $k^n$ such that $G_s \subset D_n$. □

**Theorem 0.14.** a) There exists a torus $T \subset G$ such that the map $T \times G_u \rightarrow G, (t, u) \rightarrow tu$ is one-to-one and onto,

b) If $T' \subset G$ is any maximal torus in $G$ then there exists $u \in G_u$ such that $uT'u^{-1} \subset T$.

**Remark.** One can show that the map $T \times G_u \rightarrow G, (t, u) \rightarrow tu$ defines an isomorphism of algebraic varieties.

**Proof.** We will prove the Theorem by induction in $\dim(G)$. So we assume that the result is known for all connected solvable groups $H$ such that $\dim(H) < \dim(G)$.

Since $G$ is a solvable connected affine algebraic group we can assume that $G$ is a subgroup of the group $T_n$. Let $\Lambda$ be the set of pairs

$\Lambda := \{(i, j)\}, 1 \leq i < j \leq n, \Lambda^* := \Lambda \cup \infty$

We define an order on $\Lambda^*$ by saying that $(i, j) < (p, q)$ if either

$j - i < q - p$ or $j - i = q - p$ and $i < p$

and say that $\infty > (i, j)$ for $1 \leq i, j \leq n$.

For any pair $(i, j), 1 \leq i, j \leq n$ we can consider the $(i, j)$ matrix coefficient as a function

$a_{ij} : T_n \rightarrow k^1, a_{ij}(X) := x_{i,j}$ for $X = (x_{p,q}), 1 \leq p, q \leq n$

By the definition $a_{ij} = 0$ is $i > j$. For any subset $X \subset T_n$ we define $\Lambda(X) \subset \Lambda$ by

$\Lambda(X) = \{\lambda \in \Lambda | a_{\lambda}(X) \neq 0\}$
and define $\tilde{\lambda}(X) \in \Lambda$ by $\tilde{\lambda}(X) := \min_{\lambda \in \Lambda(X)} \lambda$. So $\tilde{\lambda}(X) = \infty$ iff $X \subset D_n$. We define $\lambda(X) := \max_{r \in T_0} \tilde{\lambda}(rXr^{-1})$. It is clear that it is sufficient to prove Theorem in the case when $\lambda(G) = \tilde{\lambda}(G)$. So we assume from now on that $\lambda(G) = \tilde{\lambda}(G)$.

Let $(i, j) = \lambda(G)$. Consider a map $\phi : G \rightarrow T_2$ given by

$$\phi(g) := \begin{pmatrix} a_{ii}(g) & a_{ij}(g) \\ 0 & a_{jj}(g) \end{pmatrix}$$

It is easy to see that the map $\phi : G \rightarrow T_2$ is homomorphism of algebraic groups.

**Lemma 0.15.** $\text{Im}(\phi) \supset U_2$

**Proof of Lemma.** If $U_2$ does not lie in $\bar{G}$ then it follows from Problem 7 that there exists $\bar{r} \in T_2$ such that $\bar{r}\bar{G}\bar{r}^{-1} \subset D_2$. Choose a preimage $r \in G$ of $\bar{r}$. Then (?) If we have $a_{ij}(rg\bar{r}^{-1}) = 0$ for all $g \in G$ and therefore $\tilde{\lambda}(rGr^{-1}) < (i, j)$. But this contradicts the assumption that $\lambda(G) = \tilde{\lambda}(G)$. So $\text{Im}(\phi) \supset U_2$. □

Set $H := \phi^{-1}(D_2) \subset G$ and define

$$f : \bar{G} \rightarrow \mathbb{A}^1, f(g) := a_{ii}^{-1}(g)a_{ij}(g)$$

Since $f(hg) = f(g), h \in H, g \in G$ it follows from Lemma 4 that $\text{dim}(H) = \text{dim}(G) - 1$ (?). Let $\bar{H}^0 \subset \bar{H}$ be the connected component of $\bar{H}$ containing $e$.

**Lemma 0.16.** $H^0G_u = G$

**Proof.** Since $G_u$ is a normal subgroup of $G$ the set $H^0G_u$ is a subgroup of $G$. Since $G_u \nsubseteq H^0$ and $\text{dim}(\bar{H}^0) = \text{dim}(\bar{G}) - 1$ we have $\text{dim}(H^0G_u) = \text{dim}(\bar{G})$. Since $\bar{G}$ is connected we $H^0G_u = G$. □

Now we can prove the part a) of the Theorem. By the construction $\bar{H}^0$ is a solvable connected affine algebraic group and $\text{dim}(\bar{H}^0) < \text{dim}(\bar{G})$. By the inductive assumptions there exists a torus $\bar{T} \subset \bar{H}$ such that the map the map $T \times H^0_u \rightarrow H^0, (t, u) \rightarrow tu$ is one-to-one and onto. Therefore the map $T \times G_u \rightarrow G, (t, u) \rightarrow tu$ is onto. Since $T \cap G_u = (e)$ the part a) of Theorem is proven. It is clear (?) that $T$ is a maximal torus in $G$.

Now we prove the part b). Let $\bar{T}' \subset \bar{G}$ be a maximal torus. Consider $S' := \phi(\bar{T}') \subset T_2$. As follows from Problem 7 there exists $\bar{u} \in U_2$ such that $\bar{u}S'\bar{u}^{-1} \subset D_2$. By Lemma 10 there exists $u \in G_u$ such that $\bar{u} = \phi(u)$. Then $uT'u^{-1} \subset H$. The Theorem follows now from the inductive assumptions. □
Corollary 0.17. Let $G$ be a connected affine algebraic group. $T, T' \subset G$ be maximal tori. Then there exists $g \in G$ such that $gT'g^{-1} = T$.

Proof. Let $B \subset T, B' \subset T' \subset G$ be maximal connected solvable subgroups containing $T$ and $T'$. By the Borel’s theorem there exists $g \in G$ such that $gBg^{-1} = B$. Then $gT'g^{-1} \subset B$. But [by the part b) of Theorem] the tori $T, gT'g^{-1} \subset B$ are conjugate in $B$. □

Let $U$ be a connected unipotent normal subgroup of an algebraic group $G$ and $s \in G$ a semisimple element. Define

$$\gamma_s(u) := usu^{-1}s^{-1}, u \in U, M := \text{Im}(\gamma_s), C := Z_U(s)$$

Problem 0.18. If $x \in Z(U), y \in U$ then $\gamma_s(xy) = \gamma_s(x)\gamma_s(y)$

Theorem 0.19. The product morphism $\tau : C \times M \to U$ is bijective.

Proof. To prove the injectivity of $\tau$ it is it sufficient to show that $C \cap M = \emptyset$. Choose any $c \in C \cap M$. Since $c \in M$ we have $c = usu^{-1}s^{-1}, u \in U$. Then $cs = usu^{-1}$. Since $c \in C$ the product $cs$ is the Jordan decomposition of the semisimple element $usu^{-1}$. It follows from the uniqueness of Jordan decomposition that $c = e$. □

To prove the surjectivity of $\tau$ we consider first the case when $U$ is commutative. Then [see Problem 18] $\tau$ and $\gamma_s$ are group homomorphisms and $C = \text{Ker}(\gamma_s))$.

Since $\gamma_s : U \to M$ is a group surjective homomorphism we have $\dim(U) = \dim(C) + \dim(M)$. It follows from the injectivity of $\tau$ that $\dim(\text{Im}(\tau)) = \dim(U)$. Since $U$ is connected we see that $\tau$ is onto. □

We prove the general case by induction in $\dim(U)$. Let $V := Z(U)^0$. Then $V$ is a connected normal subgroup of $G, \dim(V) > 0$. If $V = U$ then $U$ is commutative and the result is already known. So we assume that $V \subsetneq U$. Define

$$G' := G/V, U' := U/V, s' := \pi(s) \in G'$$

where $\pi : G \to G'$ be the natural projection and denote by

$$\tau' : C' \times M' \to U', \tau_V : C_V \times M_V \to V$$

the maps corresponding to the triples $(G', s')$ and $(G, V, s)$. By the inductive assumptions we know that that maps $\tau'$ and $\tau_V$ are bijections.

Claim 0.20. For any $c' \in C'$ there exists $c \in C$ such that $c' = \pi(c)$.

Proof of Claim. Choose any $\tilde{c} \in U$ such that $c' = \pi(\tilde{c})$. Then we have $s\tilde{c}s^{-1} = \tilde{c}v, v \in V$. We want to find $y \in V$ such that $s(\tilde{c}y)s^{-1} =$
\( \tilde{c}y \). For any \( y \in V \) we have
\[
s(\tilde{c}y)s^{-1} = s\tilde{c}ss^{-1}sys^{-1} = \tilde{c}vsys^{-1} = \tilde{c}y^{-1}vsys^{-1} = (\tilde{c}y)vsys^{-1}y^{-1}
\]
since \( V \) is commutative. So we want to find \( y \in V \) such that \( vsys^{-1}y^{-1} = e \). As follows from the surjectivity of \( \tau \) we can find \( c \in U \) such that \( \pi(c) = c' \) and \( scs^{-1} = cz, z \in C_V \). But the \( z \in C \cap M = \{e\} \). □

To prove the surjectivity of \( \tau \) we have to show the existence of a decomposition \( x = cm, c \in C, m \in M \) for any \( x \in U \). Let \( x' = \pi(x) \in U' \). As follows from the surjectivity of \( \tau' \) we can write \( x' = c'u', c' \in C', m' \in M' \). Let \( c \in C \) be a preimage of \( c' \) as in the Claim. We have
\[
x = c\tilde{v}usu^{-1}s^{-1}, u \in U, c \in C, \tilde{v} \in V
\]
As follows from the surjectivity of \( \tau \) we have
\[
x = cc'vsu^{-1}s^{-1}usu^{-1}s^{-1}, u \in U, c, c' \in C, v \in V
\]
Since \( v \in Z(U) \) we have [see Problem 18] \( x = (cc')\gamma_s(uv) \). □

**Problem 0.21.** The restriction of \( \gamma_s \) on \( M \) is bijective.

**A hint.** Use the following result. Let \( f : X \rightarrow Y \) be a morphism of algebraic varieties such that \( f : X \rightarrow Y \) is a bijection. Then \( f : X \rightarrow Y \) is a homeomorphism.

**Corollary 0.22.** For any connected solvable group \( G \) and a semisimple \( s \in G \) the centralizer \( Z_G(s) \) is connected.

**Proof.** As follows from Theorem 9 we have a decomposition \( G = TG_u \) where \( T \) is a maximal torus of \( G \) containing \( s \). Then \( Z_G(s) = TZ_G(s) \). So it is sufficient to show that the group \( Z_{G_u}(s) \) is connected. As follows from Theorem 14 we have a bijection \( \tau : Z_{G_u}(s) \times M \rightarrow G_u \). So \( Z_{G_u}(s) \) is connected. □