Definition 0.1. a) A continuous map $p : X \to Y$ is *closed* if for any closed subset Z of X the image $p(Z) \subset Y$ is closed.

b) A continuous map $p : X \to Y$ is *proper* if for any algebraic variety $\underline{Z}$ the projection $p_Z : X \times Z \to Y \times Z$ is closed.

c) An algebraic variety $\underline{X}$ is *complete* the map from X to a point is proper. In other words $\underline{X}$ is complete if the map for any algebraic variety $\underline{Y}$ the projection $p_Y : X \times Y \to Y$ is closed.

Example 0.2. The variety $\mathbb{A}^1$ is not complete. to see this take $\underline{Y} = \mathbb{A}^1$ and $Z = \{(x, y)|xy = 1\} \subset X \times Y = \mathbb{A}^2$. Then Z is a closed subset of $X \times Y$ but $p_Y(Z)$ is not a closed subset of Y.

Problem 0.3. a) Let $\underline{X}$ be an algebraic variety such that the projection $p_Y : X \times Y \to Y$ is closed for all affine algebraic varieties Y. Then $\underline{X}$ is complete.

Assume that $\underline{X}$ is complete. Then

b) Any closed subset of X is complete.

c) For any complete $\underline{Y}$ the product $\underline{X} \times \underline{Y}$ is complete.

d) For any morphism $\underline{f} : \underline{X} \to \underline{Y}$ the image $f(X) \subset Y$ is closed and complete.

e) If X is connected then any regular function on X is constant.

Theorem 0.4. *$\underline{P}^n$ is a complete variety.*

Proof. Let $Z \subset P^n \times Y$ be a closed subset. We want to show that the image $p(Z) \subset Y$ is closed. As follows from the Problem 3 we can assume that $\underline{Y} = (Y, A)$ is an affine algebraic variety and we can also assume that Z is irreducible (?). If the image $p(Z) \subset Y$ is not dense we can replace A by the quotient A/I where $I = I(\overline{p(Z)})$. So we assume from now on that $Y = \overline{p(Z)}$.

Consider the graded ring $S = A[x_0, ..., x_n] = \oplus_{n=0} S_n$ where S_n is the A-module of homogeneous polynomials of degree n. Let

$$q_Y : (k^{n+1} - \{0\}) \times Y \to P^n \times Y$$

be the projection induced by the natural projection $q : k^{n+1} - \{0\} \to P^n$. For any closed subset M of $P^n \times Y$ we denote by $M^\star$ a closed subset

$$M^\star = q_Y^{-1}(M) \cup 0 \times Y \subset k^{n+1} \times Y$$

and by $I^\star(M) \subset S$ the corresponding ideal. Conversely for any homogeneous ideal $I^\star \subset S$ such that $I \cap S_0 = 0$ we denote by $\mathcal{V}(I^\star)$ the set of line $L \in P^n$ such that $f_{|L} = 0$ for all $f \in I^\star$.

Problem 0.5. a) The map $M \to I^\star(M)$ defines a bijection between closed irreducible subsets Z of $P^n \times Y$ such that $p(Z)$ is dense in Y and homogeneous prime ideals $I^\star \subset S$ such that $I \cap S_0 = 0$. Moreover the inverse map is given by $I^\star \to \mathcal{V}(I^\star)$.

b) Let $I^\star \subset S$ be a homogeneous ideal such that $I^\star \cap S_0 = \{0\}$. Then $\mathcal{V}(I^\star) = \emptyset$ iff there exists n such that $S_n \subset I$.

Now we can finish the proof the Theorem. Fix $y \in Y$. We want to show that $p_Y^{-1}(y) \cap Z \neq \emptyset$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to y. Then $p_Y^{-1}(y) \cap Z = \mathcal{V}(I^\star + \mathfrak{m}S_+)$(?). So to show that $y \in p_Y(Z)$ it is sufficient to show that $p_Y^{-1}(y) \cap Z \neq \emptyset$ or [see Problem 5 b)] that So we have to show that $S_n \not\subseteq I^\star + \mathfrak{m}S_+$. Therefore it is sufficient to prove the following result.

Claim 0.6. *Let $I^\star \subset S$ be a homogeneous ideal such that $I \cap S_0 = 0$ and $S_n \subset I^\star + \mathfrak{m}S_+$ for some $n > 0$. Then $S_n \subset I^\star$.*

Proof of Claim. Consider $N := S_n/I^\star \cap S_n$. Since $S_n \subset I^\star + \mathfrak{m}S_+$ we have $\mathfrak{m}N = N$. The same arguments as in the proof of the Nakayama's lemma show the existence of $a \in A - \{0\}$ such that $aN = 0$. On the other hand since Z is irreducible the ring $S/I^\star$ is integral and [since $I^\star \cap S_0 = 0$] the map $A = S_0 \to S/I^\star$ is an imbedding. So the multiplication $\hat{a} : S/I^\star \to S/I^\star, \bar{s} \to a\bar{s}$ is an imbedding. So $N = \{0\}$ and $S_n \subset I^\star$□

Corollary 0.7. *Let $X \subset \mathbb{P}^n$ be a closed subset and $Y \subset \mathbb{P}^n$ a hyperplane such that $X \cap Y = \emptyset$. Then X is finite.*

Proof. We can assume that X is irreducible and $Y = \mathbb{P}^n - U_0$ where $U_0 \subset \mathbb{P}^n$ is the open set as in the Definition 4.16 (?). Since $X \cap Y = \emptyset$ we have $X \subset U_0$. By Problem 3 e) the restriction the regular function $x_i/x_0, 1 \leq i \leq n$ on X is constant. But this implies that X is a point.□

Problem 0.8. a) Prove the variant of the Nakayama's lemma used above.

b) Let $f : X \to Y$ be a continuous map of topological spaces. Then $f(\bar{U}) \subset \overline{f(U)}$ for any $U \subset X$ where $\bar{U}$ is the closure of U in X and $\overline{f(U)}$ is the closure of $f(U)$ in Y.

Definition 0.9. a) Let $\underline{G}$ be an affine algebraic group, $\underline{X}$ an algebraic variety. An action of $\underline{G}$ on $\underline{X}$ is a morphism

$\underline{G} \times \underline{X} \to \underline{X}, (g, x) \to gx$ such $(g'g'')(x) = g'(g''(x)), ex = x, g', g'' \in G, x \in X$.

b) for any $x \in X$ we denote by $\Omega(x)$ the subset $\{gx\} \subset X, g \in G$ and call it *the orbit of x.*

Remark. As follows from Lemma 3.4 $\Omega(x)$ is a constructive subset of X.

c) We denote by $\bar{\Omega}(x)$ the closure of $\Omega(x)$ in X.

Lemma 0.10. *a) The orbit $\Omega(x)$ is an open subset of $\bar{\Omega}(x)$.*

b) Let $x \in X$ be such that $dim(\Omega(x)) \leq dim(\Omega(y)), \forall y \in X$. Then $\Omega(x)$ is a closed subset of X.

Proof. Let G^0 be the connected component of G. Since the quotient G/G^0 is finite [see Problem 3.5] it is sufficient (?) to prove the result in the case when the group G is connected.

a) Since $\Omega(x)$ is a dense constructive subset of $\bar{\Omega}(x)$ it contains a subset U which is dense and open in $\bar{\Omega}(x)$. But then $gU\bar{\Omega}(x)$ is an open subset for all $g \in G$. Since $\Omega(x) = \cup_{g \in G} gU$ we see that $\Omega(x)$ is open in $\bar{\Omega}(x)$.□

b) Observe first that the closure $\bar{\Omega}(x) \subset X$ is G-invariant. For this consider the action map $a : G \times X \to X$ and apply the Problem 8 b) to the subset $G \times \Omega(x) \subset G \times X$.

Since, by our assumption, G is connected and therefore irreducible [see Problem 3.5] the set $\bar{\Omega}(x)$ is irreducible. Therefore

$$dim(\bar{\Omega}(x) - \Omega(x)) < dim(\Omega(x))$$

So $dim(\Omega(x)) > (\Omega(y)) \forall y \in X$ for any $y \in \bar{\Omega}(x) - \Omega(x)$. This would contradict the assumption of the Lemma. So $\bar{\Omega}(x) - \Omega(x) = \emptyset$.□

Problem 0.11. a) Introduce the structure of an algebraic variety on $\Omega(x)$ such that the imbeddings $\Omega(x) \hookrightarrow X$ is an algebraic morphism.

b) Show that the map $\phi_x : G \to \Omega(x), g \to gx$ defines an algebraic morphism $\underline{\phi}_x : \underline{G} \to \underline{\Omega}(x)$.

c) Show that the map $\phi_x : G \to \Omega(x), g \to gx$ defines a bijection $\bar{\phi}_x : G/St_x \to \Omega(x)$ where $St_x := \{g \in G | gx = x\}$.

d)*. Construct an example of an action of G on X such that the algebraic morphism $\underline{\phi}_x : \underline{G} \to \underline{\Omega}(x)$ is not an isomorphism for all $x \in X$.

Definition 0.12. Let $\underline{G}$ be a linear algebraic group. A *Borel subgroup* $\underline{B}$ of $\underline{G}$ is a maximal connected solvable subgroup of $\underline{G}$.

Theorem 0.13. *Any two Borel subgroups of $\underline{G}$ are conjugate.*

Proof. We start the proof with the following result.

Proposition 0.14. *Let $\rho : \underline{G} \hookrightarrow \underline{GL}(V)$ be a connected solvable group and $X \subset P(V)$ a nonempty irreducible closed G-invariant subset. Then X contains a G-invariant point.*

We prove the Proposition by induction in $dim(X)$. If $dim(X) = 0$ then X is a point and there is nothing to prove. So we assume that $dim(X) > 0$. Using the induction in $dim(V)$ we reduce the proof to the case when there is no proper G-invariant subspace $W \subset V$ such that $P(W)$ contains X (?).

Let $\rho^\vee : \underline{G} \to \underline{GL}(V^\vee)$ be the dual representation given by

$$< \rho^\vee(g)(l), v >:=< l, \rho(g^{-1})(v) >, v \in V, l \in V^\vee, g \in G$$

Since $\underline{G}$ is a connected solvable group there exists [by the Lie-Kolchin theorem] a non-zero vector $l \in V^\vee$ such that the line $L^\vee$ through l is $\rho^\vee(G)$-invariant. Let

$$W := \{v \in V | < l.v >= 0\}$$

Then $W \subset V$ is a proper G-invariant subspace.Since we assume that there is no proper G-invariant subspace $W \subset V$ such that $P(W)$ contains X the intersection $Y := X \cap W$ is a proper non-empty [by Corollary 7] G-invariant subset of a $P(V)$. Since X is irreducible we have $dim(Y) < dim(X)$ and therefore [by the induction assumption] Y contains a G-invariant point $y \in Y \subset X$.$\square$

Now we can prove the Theorem. Let $B.B'$ be two Borel subgroups of G. By Theorem 2.1 we may assume that G is a closed subgroup of $GL(W)$ where W is a finite-dimensional k-vector space. Then G acts on $\mathcal{B}(W)$. Choose a G-orbit $Z \subset \mathcal{B}(W)$ of the minimal dimension. Then [see Lemma 2] Z is a closed not-empty subset of $\mathcal{B}(W)$. Using the imbedding $\kappa : \mathcal{B}(W) \to \mathbb{P}(V)$ [see the Lecture 4] we can consider Z is closed not-empty subset of $\mathbb{P}(V)$. By the Proposition 13 there exist points $z, z' \in Z$ such that $Bz = z, B'z' = z'$.

Consider the *stationary subgroup* St_z of z.

$$St_z := \{g \in G | gz = z\} \subset G$$

and denote by St_z^0 the connected component of St_z. Since B is connected and $Bz = z$ we have $B \subset St_z^0$. To show that $B = St_z^0$ consider the flag

$$b \in \mathcal{B}(W), b = W_1 \subset W_2 \subset ... \subset W$$

such that $z = \kappa(b)$. Then $gb = b$ for any $g \in St_z$. Then $gW_i = W_i, 1 \leq i \leq dim(W)$. So (?) the group St_z^0 is a subgroup of the group of upper-triangular matrices and therefore it is solvable . Since[by the definitions of Borel subgroups] B is a maximal connected solvable algebraic subgroup of G we see that $B = St_z^0$ and analogously that $B' = St_{z'}^0$.

Since[by the construction] Z is a G-orbit there exists $g \in G$ such that $gz = z'$. But then $gSt_z^0g^{-1} = St_{z'}^0$. So $gB^{-1} = B'$.□

Definition 0.15. Let V be a finite-dimensional k vector space.

a) Given a non-degenerate quadratic form Q on V we define

$$O_Q := \{g \in GL(V)|Q(gv) = Q(v)\forall v \in V\}$$

b) Given a non-degenerate symmetric bilinear form $F(v', v")$ on V we define

$$O_F := \{g \in GL(V)|F(gv', gv'') = Q(v', v'')\forall v', v'' \in V\}$$

c) Given a non-degenerate skew-symmetric bilinear form $F(v', v")$ on V we define

$$Sp_F := \{g \in GL(V)|F(gv', gv'') = Q(v', v'')\forall v', v'' \in V\}$$

d) If $F(v', v")$ is a non-degenerate symmetric bilinear form on V we define a quadratic form $Q(F)$ on V by $Q(F)(V) := F(v, v)$.

e) Let $\underline{G}$ be a connected affine algebraic group, $\underline{P} \subset \underline{G}$ an algebraic subgroup. We say that $\underline{P}$ is a *parabolic* subgroup of G if it contains a Borel subgroup.

Problem 0.16. a) Show that O_Q, O_F, Sp_F are closed subgroups of $GL(V)$.

b)* When $O_F = O_{Q(F)}$?.

c) For any non-degenerate bilinear form $F(v', v")$ on V which is either symmetric or skew-symmetric there exists subspace W of V such that $dim(W) = [dim(V)/2]$ and $F(w', w") = 0$ for all $w', w" \in W$ [here $[a]$ is the integral part of a] .

d) Formulate and prove the analog of c) for quadratic forms.

e) Describe Borel subgroups of groups $GL(n, k)$.

f) Describe Borel subgroups of groups O_Q, O_F, Sp_F

A hint. Use the results of c) and d).

g) Describe all subgroups of $GL(n, k)$ which contain the group T_n of upper-triangular matrices [that is describe all parabolic subgroups of $GL(n, k)$].

h) Let V be a finite-dimensional k vector space, $W \subset V, d := dim(W)$. For any $g \in GL(V)$ we denote by $\Lambda^d(g) \in GL(\Lambda^d(V))$ the induced automorphism of the space $\Lambda^d(V)$. Let $L_W \subset \Lambda^d(V)$ be a line as in Definition 4.18. Show that $g(W) = W$ iff $\Lambda^d(g)(L_W) = L_W$.

Theorem 0.17. *(Chevalley) Let $\underline{G} = (G, A)$ be an affine algebraic group, $H \subset G$ a closed subgroup. Then there exists a finitely -dimensional representation $\rho : \underline{G} \to \underline{GL}(V)$ and a line $L \subset V$ such that*

$$H = \{h \in G | \rho(g)L = L\}$$

Proof. Let $I := \mathcal{I}(H)$. Since the ring $k[G]$ is Noetherian the ideal I is finitely generated. As follows from Lemma 2.2 there exists a finite-dimensional subspace $V \subset A$ such that $r(g)(V) = V, \forall g \in G$ and the intersection $W := I \cap V$ generates I as an ideal. It is clear then that $H = \{h \in G | r(g)(W) = W\}$. Therefore H is the stabilizer of the line $\phi_m(W) \subset \Lambda^m(V), m = dim(W)$.□

Remark. Let's write $\underline{G/H}(\rho, L) := \underline{\Omega}(L)$. Then [see Problem 11] we have an algebraic morphism $\underline{\phi}_L : \underline{G} \to \underline{G/H}(\rho, L)$. If $char(k) = 0$ then *quotient* $\underline{\phi}_L : \underline{G} \to \underline{G/H}(\rho, L)$ does not depend on a choice of a representation $\rho : \underline{G} \to \underline{GL(V)}$ and a line $L \subset V$. But his is not true if $char(k) > 0$.

Problem 0.18. a) Let $\underline{X} = (X, A), \underline{Y} = (B, Y)$ be irreducible affine algebraic varieties $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism such that the map $f : X \to Y$ is a bijection. Then $f^\star : B \to A$ is an imbedding. Moreover $\underline{f} : \underline{X} \to \underline{Y}$ is an isomorphism if $chark = 0$ and there exists $r > 0$ such that $a^{p^r} \in B$ for all $a \in A$ where $p = chark$.

b)* Let $\underline{f}_i : \underline{X}_i \to \underline{X}_{i-1}, i > 0$ be a sequence of morphism of irreducible algebraic varieties such that all the maps $f_i : X_i \to X_{i-1}$ are bijections, $\underline{g}_i : \underline{Y} \to \underline{X}_i$ a sequence of morphisms such that $\underline{f}_i \circ \underline{g}_i = \underline{g}_{i-1}, i > 0$ and $g_0 : Y \to X_0$ is surjective. Then there exists i_0 such that all the morphisms $\underline{f}_i : \underline{X}_i \to \underline{X}_{i-1}, i > i_0$ are isomorphisms.

c) Show the existence of a representation $\rho_0 : \underline{G} \to \underline{GL}(V_0)$ and a line $L_0 \subset V_0, St_G(L_0) = H$ such that for any representation $\rho : \underline{G} \to \underline{GL}(V)$ and a line $L \subset V$ there exists a morphism

$\underline{f} : \underline{G/H}(\rho_0, L_0) \to \underline{G/H}(\rho, L)$ such that $\underline{f} \circ \underline{\phi}_{L_0} = \underline{\phi}_L$.

Lemma 0.19. *Let $\underline{G} = (G, A)$ be an affine algebraic group and $H \subset G$ a closed normal subgroup. Then there exists a finitely -dimensional representation $\tau : \underline{G} \to \underline{GL}(W)$ such that $H = Ker(\tau)$.*

Proof. Let $\rho : \underline{G} \to \underline{GL}(V), L \subset V$ be as in the Chevalley theorem and let $\underline{\chi}_0 : \underline{H} \to \mathbb{G}_m$ the character such that

$hl = \chi_0(h)l$ for $l \in L, h \in H$

Remark Let $x \in \mathbb{P}(V)$ be the point corresponding to the line L. One can show that existence of a finitely -dimensional representation $\rho : \underline{G} \to \underline{GL}(V)$ and a line $L \subset V$ such that $H = \{h \in G | gL = L\}$ such that the map $\underline{\phi_x} : \underline{G} \to \underline{\Omega}(x)$ is the *quotient* $\underline{G}/\underline{H}$. In other words for any action of $\underline{G}$ on $\underline{Y}$ and a point $y \in Y$ such that $H \subset St_y(G)$ there exists a $\underline{G}$-equivariant morphism $\underline{f} : \underline{\Omega}(x) \to \underline{\Omega}(y)$ such that $f(x) = y$.

For any character $\underline{\chi} : \underline{H} \to \mathbb{G}_m$ of $\underline{H}$ we define

$V_\chi := \{v \in V | hv = \chi(h)v\}$ for all $h \in H$

Since H is normal subgroup of G we have

$\rho(g)(V_\chi) = V_{\chi^g}$ where $\chi^g(h) := \chi(g^{-1}hg)$

Therefore the subspace $\oplus_{\chi \in X(G)} V_\chi \subset V$ is G-invariant and we can replace V by $\oplus_\chi V_\chi \subset V$. From now we assume that $V = \oplus_\chi V_\chi$.

Let $W := \{A \in End(V) | A(V_\chi)$ for all $\chi \in X(G)\}$ and $\rho : G \to Aut(W)$ be given by $\rho(g)(A) := gAg^{-1}$. I claim that $H = Ker(\tau)$.

It is clear (?) that $H \subset Ker(\tau)$. Conversely fix any $g \in Ker(\tau)$. Since $\rho(g)$ commutes with all $A \in W$ it is clear (?) that $\rho(g)(V_\chi) = V_\chi$ and the restriction of $\rho(g)$ on V_χ is a scalar for all $\chi \in X(H)$. Therefore $\rho(g)(L) = L$. So $g \in H$.$\Box$

Problem 0.20. a) Show that τ defines a morphism of algebraic groups from G onto the image $Im(\tau) \subset GL(V)$.

b)* Show the existence of a representation $\tau_0 : \underline{G} \to \underline{GL}(V_0, Ker(\tau_0) = H$ such that for any representation $\tau_0 : \underline{G} \to \underline{GL}(V), Ker(\tau) = H$ there exists a morphism of algebraic groups $\underline{f} : Im(\tau_0) \to Im(\tau)$ such that $f \circ \tau_0 = \tau$.

Definition 0.21. We denote the algebraic group $Im(\tau_0)$ by $\underline{G}/\underline{H}$.

Lemma 0.22. *Let $H \subset G$ be a closed subgroup.*