Some notations.

Let $\underline{\mathbb{G}_a} := (K, K[x], +)$ be the additive group,

$\underline{\mathbb{G}_m} := (K^*, K[x, x^{-1}], \times)$ be the multiplicative group,

$T_n(K) := \{(a_{ij}) \in GL_n(K) | a_{ij} = 0, \forall 1 \le j < i \le n\}$,

$D_n(K) := \{(a_{ij}) \in GL_n(K) | a_{ij} = 0, \forall 1 \le j \ne i \le n\}$,

$U_n(K) := \{(a_{ij}) \in T_n(K) | a_{ii} = 1, \forall 1 \le i \le n\}$,

$SL_n(K) := \{g \in GL_n(K) | det(g) = 1\}$

Problem 1 Assume that $char(K) = 0$. Show that for any unipotent element $u \in GL(n, K)$ there exists unique homomorphism $\phi : \underline{\mathbb{G}_a} \to \underline{GL(n)}$ such that $\phi(1) = u$.

Problem 2 a) Assume that $char(K) = p \ne 0$. Construct a surjective homomorphism $f : \underline{\mathbb{G}_a} \to \underline{\mathbb{G}_a}$ such that $f(1) = 0$.

b*) Describe the ring R of automorphisms of $\underline{\mathbb{G}_a}$.

c) Describe the ring of automorphisms of $\underline{\mathbb{G}_m}$.

Now we will do some more of linear algebra.

Lemma 3.1. [Schur] Let V be a finite-dimensional K-vector space, $G \subset Aut(V)$ a group which is acting irreducibly on V. If $f : V \to V$ is a linear map such that $f \circ g = g \circ f$ for all $g \in G$ then $f = cId$ for some $c \in K$.

Proof. Let μ be an eigenvalue of f. The $Ker(f - \mu Id)$ is a non-zero G-invariant subspace of V. Since G is acting irreducibly on Vwe see that $V = Ker(f - \mu Id)$. So $f = \mu Id$.□ Let V be finite-dimensional K-vector space with an action of a group G. Then G acts on V^n.

Corollary. Let V be finite-dimensional K-vector space, ρ an irreducible representation of a group G on V and $f : V^n \to V$ is a linear map such that $f \circ \rho(g) = \rho(g) \circ f$ for all $g \in G$. Then there exist $c_i \in K, 1 \ge i \ge n$ such that $f(v_1, ..., v_n) = c_1 v_1 + ... + c_n v_n$.

Lemma 3.2. For any irreducible representation of G on V the representation of G on V^n is completely reducible. [that is for any G-invariant subspace $W \subset V^n$ there exists a G-invariant subspace $W' \subset V^n$ such that $V^n = W \oplus W'$].

Proof. The proof is by induction in n. If $n = 1$ there is nothing to prove [since V is irreducible]. Assume that the result is known for V^{n-1}.

For any $i, 1 \leq i \leq n$ we denote by $V_i \subset V^n$ the subspace of vectors $(v_1, ..., v_n)$ such that $v_j = 0$ for $j \neq i$ and denote by $V^{(i)} \subset V^n$ the subspace of vectors $(v_1, ..., v_n)$ such that $v_j = 0$ for $j = i$. So $V_i, V^{(i)}$ are G-invariant subspaces of V^n, the representation of G on V_i is equivalent to it's representation on V, the representation on $V^{(i)}$ is equivalent to representation on V^{n-1} and $V^n = V_i \oplus V^{(i)}$. We denote by $p_i : V^n \to V^{(i)}$ the projection along V_i.

Let $W \subset V^n$ be a G-invariant subspace. If $W = V^n$ there is nothing to prove. So assume that $W \not\subseteq V^n$. Since $V_i, 1 \leq i \leq n$ span V there exists $i, 1 \leq i \leq n$ such that $V_i \not\subseteq W$. Then $W \cap V_i$ is a proper G-invariant subspace of V_i. Since the representation of G on V_i is irreducible we have $W \cap V_i = \{0\}$ and the projection p_i defines an isomorphism $W \to p_i(W) \subset^{(i)}$.

By the inductive assumption there exists $W'' \subset V^{(i)}$ such that $V^{(i)} = p_i(W) \oplus W''$. But this implies that $V^n = W \oplus W'$ where $W' := V_i \oplus W'' \square$

Proposition 3.1. [Burnside] Let $\rho : G \subset Aut(V)$ an irreducible representation of a group G V be finite-dimensional vector space V. Then the space $End_k(V)$ is spanned by $\rho(g), g \in G$.

Proof. Let $e_1, ..., e_d$ be a basis of V and $L \subset V^d$ be the subspace spanned by $(\rho(g)e_1, ..., \rho(g)e_d), g \in G$. We start with the following result.

Claim. $L = V^d$.

Proof of the Claim. As follows from Lemma 3.2 there exists a G-invariant subspace $W \subset V^d$ such that $V^d = L \oplus W$. Let $p : V^d \to L$ be the projection along W and $q_i := p_i \circ p$. As follows from Corollary 1 for any $i, 1 \leq i \leq d$ there exists $c_i^j \in K, 1 \leq j \leq d$ such that $q_i(v_1, ..., v_n) = c_i^1 v_1 + ... + c_i^n v_n$. On the other hand By the construction $p(e_1, ..., e_d) = (e_1, ..., e_d) \in L$. So $q_i(e_1, ..., e_d) = e_i$ and [since $e_1, ..., e_d$ is a basis of V] we have $c_i^j = \delta_i^j$. Therefore $p = Id$. So $L = V^d \square$.

Now it's easy to finish the proof of the Proposition. Choose any $f \in End_K(V)$. We want to find $c_k \in K, g_k \in G, 1 \leq k \leq N$ such that $f = \sum_{k=1}^N c_k \rho(g)_k$. Consider $e_f := (f(e_1), ..., f(e_d)) \in V^d$. Since $L = V^d$ there exists $c_k \in K, g_k \in G, 1 \leq k \leq N$ such that

$$e_f = \sum_{k=1}^N c_k(\rho(g_k)e_1, ..., \rho(g_k)e_d)$$

So $f(e_i) = \sum_{k=1}^N c_k g_k e_i, 1 \leq i \leq d$ and therefore $f = \sum_{k=1}^N c_k \rho(g_k) \square$.

Definition 3.1. Let $\underline{G} = (G, A)$ be an affine algebraic group. We say that $\underline{G}$ is *unipotent* iff any element of G is unipotent.

Theorem 3.1 [Kolchin] Let $G \subset GL(V)$ be a unipotent subgroup. Then

a) there exists a line $L \subset V$ such that $gl = l$ for all $g \in G, l \in L$.

b) there exists an isomorphism $\alpha : V \to K^n$ such that $\alpha G \alpha^{-1} \subset U_n(K)$.

Proof of a). We can assume that the representation of G on V is irreducible [if not restrict your attention to any irreducible subspace of V]. It is clearly sufficient to prove that $G = Id$.

Since G is unipotent $tr_V(g) = dim(V)$ for all $g \in G$. Therefore $tr_V((g' - Id)g) = tr_V(g'g) - tr_V(g) = 0$ for all $g, g' \in G$. Since by Proposition 3.1 the space $End_k(V)$ is spanned by $g, g \in G$ we see that $tr_V((g' - Id)f) = 0$ for all $g' \in G, f \in End_K(V)$. But this implies (?) that $g' - Id = 0$ for all $g' \in G$. So $G = Id$.□.

Proof of b) We prove the result by induction in $n = dim(V)$. By a) there exist a 1-dimensional subspace $V_1 \subset V$ such that $gv = v$ for all $v \in V$. Let $\bar{V} := V/V_1$. By induction there exists an isomorphism $\bar{\alpha} : \bar{V} \to K^{n-1}$ such that $\bar{\alpha}\bar{G}\bar{\alpha}^{-1} \subset U_{n-1}(K)$ where $\bar{G}$ is the image of G in $Aut(\bar{V})$. Choose (?) now any isomorphism $\alpha : V \to K^{\alpha}$ such that $\alpha(V_1) \subset Ke_1$ and the induced map $V/V_1 \to K^{n-1}$ is equal to $\bar{\alpha}$. Then α satisfies the conditions of the Proposition. (?) □.

To move farther we need more results from commutative algebra.

Lemma 3.3 [Cayley-Hamilton theorem]. Let A be a commutative ring, $I \subset A$ an ideal, M a finitely generated A-module and $\phi : M \to M$ a morphism of A-modules such that $\phi(M) \subset IM$. Then there exists a monic polynomial $p(t) = t^n + \sum_{i=0}^{n-1} c_i t^i, c_i \in I^{n-i}$ such that $p(\phi) = 0$.

Proof. Choose generators $m_i, 1 \leq i \leq n$ of the A-module M. Since $\phi(M) \subset IM$ we can find $r_{i,j} \in I, 1 \leq i, j \leq n$ such that

$$\phi(m_i) = \sum_{i=1}^{n} r_{i,j} a_j$$

So the vector $\bar{x} = (x_1, ..., x_n) \in M^n$ is in the kernel of the endomorphism $\phi Id - R$ where $R := (r_{i,j})$. If we multiply both sides by the adjugate matrix to A and use that Cramer's rule we find that $det(\phi Id - R) = 0$.□

Definition 3.2. a) A commutative k- algebra A is *local* if it has unique maximal ideal.

b) If A is a commutative ring, $I \subset A$ an ideal we denote by and A_I the localization of A in respect to $S := A - I$.

Remark. There is striking inconsistence in my [and everybody] notations. I denote the localization of A in respect to $S := A - I$ by A_I while according to the notations of the first lecture I should write A_{A-I}. But the notation A_I is much shorter then A_{A-I}. Just remember that we always localize by subsets contains 1.

c) If M is an A-module we define an A_I-module M_I-module as the tensor product

$$M_I := M \otimes_A A_I$$

Problem 3.3. a) Show that $A_{\mathfrak{m}}$ is a local ring if $\mathfrak{m}$ is a maximal ideal of A.

b) If A is a local ring with a maximal ideal $\mathfrak{m}$ then any element $r \in A - \mathfrak{m}$ is invertible.

Lemma 3.4 [Nakayama's lemma]. Let A be a local ring with the maximal ideal $\mathfrak{m}$ and $M \neq \{0\}$ be a finitely generated A-module. Then $\mathfrak{m}M \neq M$.

Proof. Assume that $\mathfrak{m}M = M$. We want to show that any $m \in M$ is equal to 0. If we apply Lemma 3.3 to the case when $\phi = Id$ we see that there exists a monic polynomial $p(t) = t^n + \sum_{i=0}^{n-1} c_i t^i, c_i \in \mathfrak{m}$ such that $p(Id) = 0$. But $p(Id)(m) = rm$ for all $m \in M$ where $r := 1 + \sum_{i=0}^{n-1} c_i \in 1 + \mathfrak{m}$. Since the ring A is local and $1 \notin \mathfrak{m}$ we see [Problem 3.3] that $r \in A$ is invertible. So $m = 0$ for all $m \in M$.$\square$

Lemma 3.5 Let $\underline{X} = (X, A), \underline{Y} = (Y, B)$ be irreducible affine algebraic varieties and $(f, f^\star) : \underline{X} \to \underline{Y}$ a morphism such that $f^\star$ is an imbedding and B is finitely generated as an A-module. Then the map $f : X \to Y$ is surjective.

Proof. Let $\mathfrak{m} \in X$ be a maximal ideal of A. As follows from Problem 3.3 the map $A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is an imbedding. Since B is finitely generated A-module $B_{\mathfrak{m}}$ is finitely generated $A_{\mathfrak{m}}$-module. It follows now from the Nakayama's lemma that $B_{\mathfrak{m}} \neq \mathfrak{m}B_{\mathfrak{m}}$ and therefore $B/\mathfrak{m}B = B_{\mathfrak{m}}/\mathfrak{m}B_{\mathfrak{m}} \neq \{0\}$. Let $\bar{\mathbb{N}}$ be a maximal ideal of the ring $B/\mathfrak{m}B$ and $\mathbb{N} \subset B$ it preimage. Then $\mathbb{N}$ is a maximal ideal of B, So $\mathbb{N} \in Y$ and by the construction (?) $\mathfrak{m} = f(\mathbb{N})$.$\square$

Lemma 3.6. Let $\underline{X} = (X, A)$ be an irreducible affine algebraic variety, $\underline{p} : \underline{X} \times \mathbb{A}^1 \to \underline{X}$ the natural projection and $\underline{Y} = (Y, B) \subset$

$\underline{X} \times \mathbb{A}^1$ a proper closed subset. Then there exists a non-zero $a \in A$ such that the localization B_a is a finite A_a-module.

Proof. Since $\underline{Y} \neq \underline{X} \times \mathbb{A}^1$ there exists $f = \sum_{i=0}^{n} a_i t^i \in A[t], a_n \neq 0$ such that the restriction of f on Y is equal to zero. Define $a = a_n$. Then B_a is a generated by $t^i, 0 \leq i < n$ as an A_a-module(?).□

Corollary. Let K be a finitely generated k-algebra which is a field. Then $K = k$.

Proposition 3.2. Let $\underline{X} = (X, A)$ be an irreducible affine algebraic variety, $\underline{p} : \underline{X} \times \mathbb{A}^n \to \underline{X}$ the natural projection and $\underline{Y} = (Y, B) \subset \underline{X} \times \mathbb{A}^n$ a such that $p(Y)$ is dense in X. Then $p(Y)$ contains a non-empty open subset U of X.

Proof. It is easy to reduce the proof to the case when $n = 1$ and $\underline{Y} \neq \underline{X}$. As follows from Lemma 3.6 there exists a non-zero $a \in A$ such that the localization B_a is a finite A_a-module and $B_a \neq \{0\}$ since $p(Y)$ is dense in X But then it follows from Lemma 3.5 that $p(Y)$ contains the basic open set $U_a \subset X$.□

Problem 3.4. Show that

a) Let $\underline{X} = (X, A)$ be an irreducible affine algebraic variety, $\underline{p} : \underline{X} \times \mathbb{A}^n \to \underline{X}$ the natural projection and $\underline{Y} = (Y, B) \subset \underline{X} \times \mathbb{A}^n$. Then the subset $p(Y) \subset X$ is constructible.

b) For any morphism $(f, f^\star : \underline{X} \to \underline{Y}$ the image $f(Y) \subset X$ is constructible.

c) Prove the Nullstellensatz. [use the Corollary to Lemma 3.6].

d) Show that under the conditions of Lemma 3.4 the either the image $p(Y) \subset X$ is dense or there exists non-zero $a \in A$ such that $B_a = \{0\}$.

Now we will apply these general results to algebraic groups.

Lemma 3.7. a) Let G be an algebraic group, $H \subset G$ a dense constructible subset. Then $HH = G$.

Proof. Let $H^{-1} := \{h^{-1}\}, h \in H$. To show that an element $g \in G$ belongs to HH it is sufficient to prove that $gH^{-1} \cap H \neq \emptyset$. But both H and gH^{-1} are dense constructible subsets of G. Therefore (?) $gH^{-1} \cap H \neq \emptyset$.□

Problem 3.5. a) let $\phi : H \to G$ be a morphism of algebraic groups. The $\phi(H) \subset G$ is closed.

b) Any connected algebraic group is irreducible.

c) Let $G^0 \subset G$ be the connected component of G containing e. Show that $G^0 \subset G$ is a normal subgroup and define a bijection between the set of connected components of G and the quotient G/G^0.

Proposition 3.3 Let G be an algebraic group, $H \subset G$ be an irreducible subset containing e and such that $H^{-1} = H$. Then the subgroup $G' \subset G$ generated by H is closed and connected.

Proof. By the definition $G' = H \cup H^2 \cup ... \cup H^j \cup ...$ where $H^j = HH...H$ (j-times). Consider the increasing sequence

$$H \subset H^2 \subset ... \subset H^j \subset ...$$

Since

$$dim(H) \leq dim(H^2)... \leq dim(H^j)... \leq dim(G)$$

there exists m such that $dim(H^m) = dim(H^j)$ for $j \geq m$. As follows from Problem 2.4 all subsets H^j are irreducible. Let $\overline{H^m}$ be closure of H^m. It follows from Lemma 2.8 that $\overline{H^m} = \overline{H^j}$ for $j \geq m$. So by Lemma 3.7 $H^{2m} = \overline{H}^{2m}$. So $G' = \overline{H}^{2m}$ is closed and irreducible. $\square$

Definition 3.2 a) Given two subgroups $G_1, G_2 \subset G$ we define

$$(G_1, G_2) := \{g_1 g_2 g_1^{-1} g_2^{-1} | g_1 \in G_1, g_2 \in G_2\}$$

b) Let G be a connected algebraic group. We define $D^0(G) = C^0(G) := G, D^1(G) = C^1(G) := (G, G), D^{i+1}(G) := (D^i(G), D^i(G)), C^{i+1}(G) := (G, C^i(G))$.

Problem 3.6. a) Let G be a connected algebraic group, $G_1, G_2 \subset G$ closed subgroups such that G_1 is irreducible, Then (G_1, G_2) is closed and connected subgroup of G.

b) $C^i(G), D^i(G)$ are closed, connected normal subgroups of G.

Definition 3.3 We say that an algebraic group G

a) is *solvable* if $D^i(G) = \{e\}$ for some $j > 0$,

b) is *nilpotent* if $C^i(G) = \{e\}$ for some $j > 0$.

Example. The subgroup $T_n(k) \subset GL_n(k)$ of upper-triangular matrices is a closed connected solvable subgroup of $GL_n(k)$.

Remark As follows from the theorem 3.1 any unipotent group is nilpotent.

Theorem 3.2 (Lie-Morozov) Let G be a connected solvable algebraic group, $\rho : G \to Aut(V)$ a morphism of algebraic groups where V is a k-vector space, dim $(V) = n \geq 1$. Then

a) there exists a G-invariant line $V_1 \subset V$,

b) there exists a G-invariant flag $V_1 \subset V_2 \subset ... \subset V_n = V$ where $V_i \subset V$ are subspaces of dimension $i, 1 \leq i \leq n$,

c) there exists an isomorphism $\alpha : V \to k^n$ such that $\alpha G \alpha^{-1} \subset T_n(k)$.

Proof of a). We can assume that ρ is irreducible (?). The proof goes by induction in a number i such that $D^i(G) = \{e\}$. If $i = 1$ then G is commutative and the result follows from the Schur's lemma. So assume that $i > 1$.

Since $D^{i+1}(G) = D^i(D^1(G))$ we know that the result is true for the restriction of ρ to $D^1(G)$. So there exist a $D^1(G)$-invariant line $L_1 \subset V$. Let $W \subset V$ be the subspace generated by all $D^1(G)$-invariant lines. We have seen that $W \neq \{0\}$ and it is easy to check (?) that W is G-invariant. Since V is irreducible we have $W = V = L_1 \oplus L_2 \oplus ... \oplus L_n$ where L_i are $D^1(G)$-invariant lines. In other words there exists a basis in V such that $\rho(g_1)$ is diagonal for all $g_1 \in G_1$. In particular we see that all $\rho(g_1), g_1 \in G_1$ commute.

Fix any $g_1 \in G_1$. For any $g \in G$ we have $gg_1g^{-1} \in D^1(G)$. Therefore all matrices $R(g) := \rho(gg_1g^{-1})$ belong to set S of diagonal matrices conjugate to $R(e) = \rho(g_1)$. So the set S is finite. Consider the map $R : G \to S, g \to \rho(gg_1g^{-1})$. Since G is connected the image $Im(R) \subset S$ is connected. Since S is a finite we see that $Im(R)$ is a point. So $\rho(g_1)\rho(g) = \rho(g)\rho(g_1)$ for all $g \in G, g_1 \in G_1$.

Since V is irreducible it follows from the Schur's lemma that there exists a algebraic morphism $\lambda : D^1(G) \to k^*$ such that $\rho(g_1) = \lambda(g_1)Id$ for all $g_1 \in D^1(G)$. On the other hand it is clear that $det(\rho(g_1)) = 1$ for all $g_1 \in D^1(G)$. So $\lambda(D^1(G)) \subset \mu_n$ where $\mu_n = \{x \in k^* | x^n = 1\}$. Since μ_n is finite and $D^1(G)$ is connected we see that $\lambda \equiv 1$. So $gg'g^{-1}g'^{-1} \in D^1(G)$ for all $g, g' \in G$ and all $\rho(g), g \in G$ commute. Therefore there exists a G-invariant line $V_1 \subset V$.

b) and c) follow from a) as in the proof of Theorem 3.1)□

Corollary Let G be a connected solvable algebraic group and $G_u \subset G$ is the subset of unipotent elements. Then G_u is a closed normal subgroup of G which is nilpotent.