Some notations.

Let \( G_a := (K, K[x], +) \) be the additive group,
\( G_m := (K^*, K[x, x^{-1}], \times) \) be the multiplicative group,
\( T_n(K) := \{(a_{ij}) \in GL_n(K) | a_{ij} = 0, \forall 1 \leq j < i \leq n\} \),
\( D_n(K) := \{(a_{ij}) \in GL_n(K) | a_{ij} = 0, \forall 1 \leq j \neq i \leq n\} \),
\( U_n(K) := \{(a_{ij}) \in T_n(K) | a_{ii} = 1, \forall 1 \leq i \leq n\} \),
\( SL_n(K) := \{g \in GL_n(K) | \det(g) = 1\} \).

**Problem 1** Assume that \( \text{char}(K) = 0 \). Show that for any unipotent element \( u \in GL_n(K) \) there exists unique homomorphism \( \phi : G_a \to GL(n) \) such that \( \phi(1) = u \).

**Problem 2**

a) Assume that \( \text{char}(K) = p \neq 0 \). Construct a surjective homomorphism \( f : G_a \to G_a \) such that \( f(1) = 0 \).

b) Describe the ring \( R \) of automorphisms of \( G_a \).

c) Describe the ring of automorphisms of \( G_m \).

Now we will do some more of linear algebra.

**Lemma 3.1.** [Schur] Let \( V \) be a finite-dimensional \( K \)-vector space, \( G \subset Aut(V) \) a group which is acting irreducibly on \( V \). If \( f : V \to V \) is a linear map such that \( f \circ g = g \circ f \) for all \( g \in G \) then \( f = cId \) for some \( c \in K \).

**Proof.** Let \( \mu \) be an eigenvalue of \( f \). The \( \text{Ker}(f - \mu Id) \) is a non-zero \( G \)-invariant subspace of \( V \). Since \( G \) is acting irreducibly on \( V \) we see that \( V = \text{Ker}(f - \mu Id) \). So \( f = \mu Id \). \( \Box \)

**Corollary.** Let \( V \) be finite-dimensional \( K \)-vector space, \( \rho \) an irreducible representation of a group \( G \) on \( V \) and \( f : V^n \to V \) is a linear map such that \( f \circ \rho(g) = \rho(g) \circ f \) for all \( g \in G \). Then there exist \( c_i \in K, 1 \geq i \geq n \) such that \( f(v_1, ..., v_n) = c_1 v_1 + ... + c_n v_n \).

**Lemma 3.2.** For any irreducible representation of \( G \) on \( V \) the representation of \( G \) on \( V^n \) is completely reducible. [that is for any \( G \)-invariant subspace \( W \subset V^n \) there exists a \( G \)-invariant subspace \( W' \subset V^n \) such that \( V^n = W \oplus W' \)].

**Proof.** The proof is by induction in \( n \). If \( n = 1 \) there is nothing to prove [since \( V \) is irreducible]. Assume that the result is known for \( V^{n-1} \).
For any \( i, 1 \leq i \leq n \) we denote by \( V_i \subset V^n \) the subspace of vectors \((v_1, ..., v_n)\) such that \( v_j = 0 \) for \( j \neq i \) and denote by \( V^{(i)} \subset V^n \) the subspace of vectors \((v_1, ..., v_n)\) such that \( v_j = 0 \) for \( j = i \). So \( V_i, V^{(i)} \) are \( G \)-invariant subspaces of \( V^n \), the representation of \( G \) on \( V_i \) is equivalent to it’s representation on \( V \), the representation on \( V^{(i)} \) is equivalent to representation on \( V^{n-1} \) and \( V^n = V_i \oplus V^{(i)} \). We denote by \( p_i : V^n \to V^{(i)} \) the projection along \( V_i \).

Let \( W \subset V^n \) be a \( G \)-invariant subspace. If \( W = V^n \) there is nothing to prove. So assume that \( W \not\subset V^n \). Since \( V_i, 1 \leq i \leq n \) span \( V \) there exists \( i, 1 \leq i \leq n \) such that \( V_i \not\subset W \). Then \( W \cap V_i \) is a proper \( G \)-invariant subspace of \( V_i \). Since the representation of \( G \) on \( V_i \) is irreducible we have \( W \cap V_i = \{0\} \) and the projection \( p_i \) defines an isomorphism \( W \to p_i(W) \subset V^{(i)} \).

By the inductive assumption there exists \( W'' \subset V^{(i)} \) such that \( V^{(i)} = p_i(W) \oplus W'' \). But this implies that \( V^n = W \oplus W'' \) where \( W' := V_i \oplus W'' \).

**Proposition 3.1.** [Burnside] Let \( \rho : G \subset Aut(V) \) an irreducible representation of a group \( GV \) be finite-dimensional vector space \( V \). Then the space \( End_k(V) \) is spanned by \( (\rho(g)e_1, ..., \rho(g)e_d), g \in G \). We start with the following result.

**Claim.** \( L = V^d \).

**Proof of the Claim.** As follows from Lemma 3.2 there exists a \( G \)-invariant subspace \( W \subset V^d \) such that \( V^d = L \oplus W \). Let \( p : V^d \to L \) be the projection along \( W \) and \( q_i := p_i \circ p \). As follows from Corollary 1 for any \( i, 1 \leq i \leq d \) there exists \( c^i_j \in K, 1 \leq j \leq d \) such that \( q_i(v_1, ..., v_n) = c^i_1 v_1 + ... + c^i_n v_n \). On the other hand By the construction \( p(e_1, ..., e_d) = (e_1, ..., e_d) \in L \). So \( q_i(e_1, ..., e_d) = e_i \) and [since \( e_1, ..., e_d \) is a basis of \( V \)] we have \( c^i_j = \delta^i_j \). Therefore \( p = Id \). So \( L = V^d \).

Now it’s easy to finish the proof of the Proposition. Choose any \( f \in End_k(V) \). We want to find \( c_k \in K, g_k \in G, 1 \leq k \leq N \) such that \( f = \sum_{k=1}^N c_k \rho(g_k) \). Consider \( e_f := (f(e_1), ..., f(e_d)) \in V^d \). Since \( L = V^d \) there exists \( c_k \in K, g_k \in G, 1 \leq k \leq N \) such that

\[
e_f = \sum_{k=1}^N c_k (\rho(g_k)e_1, ..., \rho(g_k)e_d)\]

So \( f(e_i) = \sum_{k=1}^N c_k g_k e_i, 1 \leq i \leq d \) and therefore \( f = \sum_{k=1}^N c_k \rho(g_k) \).

□
**Definition 3.1.** Let $G = (G, A)$ be an affine algebraic group. We say that $G$ is unipotent iff any element of $G$ is unipotent.

**Theorem 3.1 [Kolchin]** Let $G \subseteq GL(V)$ be a unipotent subgroup. Then

a) there exists a line $L \subset V$ such that $gl = l$ for all $g \in G, l \in L$.

b) there exists an isomorphism $\alpha : V \to K^n$ such that $\alpha G \alpha^{-1} \subset U_n(K)$.

**Proof of a).** We can assume that the representation of $G$ on $V$ is irreducible [if not restrict your attention to any irreducible subspace of $V$]. It is clearly sufficient to prove that $G = Id$.

Since $G$ is unipotent $tr_V(g) = dim(V)$ for all $g \in G$. Therefore $tr_V((g' - Id)g) = tr_V(g'g) - tr_V(g) = 0$ for all $g, g' \in G$. Since by Proposition 3.1 the space $End_K(V)$ is spanned by $g, g \in G$ we see that $tr_V((g' - Id)f) = 0$ for all $g', f \in End_K(V)$. But this implies (?) that $g' - Id = 0$ for all $g' \in G$. So $G = Id$. □

Proof of b) We prove the result by induction in $n = dim(V)$. By a) there exist a 1-dimensional subspace $V_1 \subset V$ such that $gv = v$ for all $v \in V$. Let $\bar{V} := V/V_1$. By induction there exists an isomorphism $\bar{\alpha} : \bar{V} \to K^{n-1}$ such that $\bar{\alpha} G \bar{\alpha}^{-1} \subset U_{n-1}(K)$ where $\bar{G}$ is the image of $G$ in $Aut(\bar{V})$. Choose (?) now any isomorphism $\alpha : V \to K^n$ such that $\alpha(V_1) \subset Ke_1$ and the induced map $V/V_1 \to K^{n-1}$ is equal to $\bar{\alpha}$. Then $\alpha$ satisfies the conditions of the Proposition. (?) □

To move farther we need more results from commutative algebra.

**Lemma 3.3 [Cayley-Hamilton theorem].** Let $A$ be a commutative ring, $I \subset A$ an ideal, $M$ a finitely generated $A$-module and $\phi : M \to M$ a morphism of $A$-modules such that $\phi(M) \subset IM$. Then there exists a monic polynomial $p(t) = t^n + \sum_{i=0}^{n-1} c_i t^i$ such that $p(\phi) = 0$.

**Proof.** Choose generators $m_i, 1 \leq i \leq n$ of the $A$-module $M$. Since $\phi(M) \subset IM$ we can find $r_{i,j} \in I, 1 \leq i, j \leq n$ such that

$$\phi(m_i) = \sum_{j=1}^n r_{i,j} a_j$$

So the vector $\bar{x} = (x_1, ..., x_n) \in M^n$ is in the kernel of the endomorphism $\phi Id - R$ where $R := (r_{i,j})$. If we multiply both sides by the adjugate matrix to $A$ and use that Cramer’s rule we find that $det(\phi Id - R) = 0$. □
Definition 3.2. a) A commutative k-algebra \( A \) is local if it has unique maximal ideal.

b) If \( A \) is a commutative ring, \( I \subset A \) an ideal we denote by \( A_I \) the localization of \( A \) in respect to \( S := A - I \).

Remark. There is striking inconsistence in my [and everybody] notations. I denote the localization of \( A \) in respect to \( S := A - I \) by \( A_I \) while according to the notations of the first lecture I should write \( A_{A-I} \). But the notation \( A_I \) is much shorter then \( A_{A-I} \). Just remember that we always localize by subsets contains 1.

c) If \( M \) is an \( A \)-module we define an \( A_I \)-module \( M_I \) as the tensor product

\[ M_I := M \otimes_A A_I \]

Problem 3.3. a) Show that \( A_m \) is a local ring if \( m \) is a maximal ideal of \( A \).

b) If \( A \) is a local ring with a maximal ideal \( m \) then any element \( r \in A - m \) is invertible.

Lemma 3.4 [Nakayama’s lemma]. Let \( A \) be a local ring with the maximal ideal \( m \) and \( M \neq \{0\} \) be a finitely generated \( A \)-module. Then \( mM \neq 0 \).

Proof. Assume that \( mM = M \). We want to show that any \( m \in M \) is equal to 0. If we apply Lemma 3.3 to the case when \( \phi = Id \) we see that there exists a monic polynomial \( p(t) = t^n + \sum_{i=0}^{n-1} c_i t^i, c_i \in m \) such that \( p(Id) = 0 \). But \( p(Id)(m) = rm \) for all \( m \in M \) where \( r := 1 + \sum_{i=0}^{n-1} c_i \in 1 + m \). Since the ring \( A \) is local and \( 1 \notin m \) we see [Problem 3.3] that \( r \in A \) is invertible. So \( m = 0 \) for all \( m \in M \). ☐

Lemma 3.5 Let \( X = (X, A), Y = (Y, B) \) be irreducible affine algebraic varieties and \( (f, f^*) : X \to Y \) a morphism such that \( f^* \) is an imbedding and \( B \) is finitely generated as an \( A \)-module. Then the map \( f : X \to Y \) is surjective.

Proof. Let \( m \in X \) be a maximal ideal of \( A \). As follows from Problem 3.3 the map \( A_m \to B_m \) is an imbedding. Since \( B \) is finitely generated \( A \)-module \( B_m \) is finitely generated \( A_m \)-module. It follows now from the Nakayama’s lemma that \( B_m \neq mB_m \) and therefore \( B/mB = B_m/mB_m \neq \{0\} \). Let \( N \) be a maximal ideal of the ring \( B/mB \) and \( N \subset B \) it preimage. Then \( N \) is a maximal ideal of \( B \), So \( N \in Y \) and by the construction \( m = f(N) \). ☐

Lemma 3.6. Let \( X = (X, A) \) be an irreducible affine algebraic variety, \( p : X \times \mathbb{A}^1 \to X \) the natural projection and \( Y = (Y, B) \subset \)
$X \times A^1$ a proper closed subset. Then there exists a non-zero $a \in A$ such that the localization $B_a$ is a finite $A_a$-module.

**Proof.** Since $Y \neq X \times A^1$ there exists $f = \sum_{i=0}^{n} a_i t^i \in A[t], a_n \neq 0$ such that the restriction of $f$ on $Y$ is equal to zero. Define $a = a_n$. Then $B_a$ is a generated by $t^i, 0 \leq i < n$ as an $A_a$-module. □

**Corollary.** Let $K$ be a finitely generated $k$-algebra which is a field. Then $K = k$.

**Proposition 3.2.** Let $X = (X, A)$ be an irreducible affine algebraic variety, $p : X \times A^n \to X$ the natural projection and $Y = (Y, B) \subset X \times A^n$ a such that $p(Y)$ is dense in $X$. Then $p(Y)$ contains a non-empty open subset $U$ of $X$.

**Proof.** It is easy to reduce the proof to the case when $n = 1$ and $Y \neq X$. As follows from Lemma 3.6 there exists a non-zero $a \in A$ such that the localization $B_a$ is a finite $A_a$-module and $B_a \neq \{0\}$ since $p(Y)$ is dense in $X$. But then it follows from Lemma 3.5 that $p(Y)$ contains the basic open set $U_a \subset X$. □

**Problem 3.4.** Show that

a) Let $X = (X, A)$ be an irreducible affine algebraic variety, $p : X \times A^n \to X$ the natural projection and $Y = (Y, B) \subset X \times A^n$. Then the subset $p(Y) \subset X$ is constructible.

b) For any morphism $(f, f^* : X \to Y$ the image $f(Y) \subset X$ is constructible.

c) Prove the Nullstellensatz. [use the Corollary to Lemma 3.6].

d) Show that under the conditions of Lemma 3.4 the either the image $p(Y) \subset X$ is dense or there exists non-zero $a \in A$ such that $B_a = \{0\}$.

Now we will apply these general results to algebraic groups.

**Lemma 3.7.** a) Let $G$ be an algebraic group, $H \subset G$ a dense constructible subset. Then $HH = G$.

**Proof.** Let $H^{-1} := \{h^{-1}\}, h \in H$. To show that an element $g \in G$ belongs to $HH$ it is sufficient to prove that $gH^{-1} \cap H \neq \emptyset$. But both $H$ and $gH^{-1}$ are dense constructible subsets of $G$. Therefore (?) $gH^{-1} \cap H \neq \emptyset$. □

**Problem 3.5.** a) let $\phi : H \to G$ be a morphism of algebraic groups. The $\phi(H) \subset G$ is closed.

b) Any connected algebraic group is irreducible.
c) Let $G^0 \subset G$ be the connected component of $G$ containing $e$. Show that $G^0 \subset G$ is a normal subgroup and define a bijection between the set of connected components of $G$ and the quotient $G/G^0$.

**Proposition 3.3** Let $G$ be an algebraic group, $H \subset G$ be an irreducible subset containing $e$ and such that $H^{-1} = H$. Then the subgroup $G' \subset G$ generated by $H$ is closed and connected.

**Proof.** By the definition $G' = H \cup H^2 \cup \ldots \cup H^j \cup \ldots$ where $H^i = HH\ldots H$ (j-times). Consider the increasing sequence

$$H \subset H^2 \subset \ldots \subset H^j \subset \ldots$$

Since

$$\dim(H) \leq \dim(H^2) \ldots \leq \dim(H^j) \ldots \leq \dim(G)$$

there exists $m$ such that $\dim(H^m) = \dim(H^j)$ for $j \geq m$. As follows from Problem 2.4 all subsets $H^j$ are irreducible. Let $\overline{H^m}$ be closure of $H^m$. It follows from Lemma 2.8 that $\overline{H^m} = \overline{H^j}$ for $j \geq m$. So by Lemma 3.7 $H^{2m} = \overline{H^{2m}}$. So $G' = \overline{H^{2m}}$ is closed and irreducible. □

**Definition 3.2** a) Given two subgroups $G_1, G_2 \subset G$ we define

$$(G_1, G_2) := \{g_1g_2g_1^{-1}g_2^{-1} | g_1 \in G_1, g_2 \in G_2\}$$

b) Let $G$ be a connected algebraic group. We define $D^0(G) = C^0(G) := G, D^1(G) = C^1(G) := (G, G), D^{i+1}(G) := (D^i(G), D^i(G)), C^{i+1}(G) := (G, C^i(G))$.

**Problem 3.6.** a) Let $G$ be a connected algebraic group, $G_1, G_2 \subset G$ closed subgroups such that $G_1$ is irreducible, Then $(G_1, G_2)$ is closed and connected subgroup of $G$.

b) $C^i(G), D^i(G)$ are closed, connected normal subgroups of $G$.

**Definition 3.3** We say that an algebraic group $G$

a) is solvable if $D^i(G) = \{e\}$ for some $j > 0$,

b) is nilpotent if $C^i(G) = \{e\}$ for some $j > 0$.

**Example.** The subgroup $T_n(k) \subset GL_n(k)$ of upper-triangular matrices is a closed connected solvable subgroup of $GL_n(k)$.

**Remark** As follows from the theorem 3.1 any unipotent group is nilpotent.
Theorem 3.2 (Lie-Morozov) Let $G$ be a connected solvable algebraic group, $\rho : G \to Aut(V)$ a morphism of algebraic groups where $V$ is a $k$-vector space, $\dim(V) = n \geq 1$. Then

a) there exists a $G$-invariant line $V_1 \subset V$,

b) there exists a $G$-invariant flag $V_1 \subset V_2 \subset \ldots \subset V_n = V$ where $V_i \subset V$ are subspaces of dimension $i, 1 \leq i \leq n$,

c) there exists an isomorphism $\alpha : V \rightarrow k^n$ such that $\alpha G \alpha^{-1} \subset T_n(k)$.

Proof of a). We can assume that $\rho$ is irreducible (?). The proof goes by induction in a number $i$ such that $D^i(G) = \{e\}$. If $i = 1$ then $G$ is commutative and the result follows from the Schur’s lemma. So assume that $i > 1$.

Since $D^{i+1}(G) = D^i(D^1(G))$ we know that the result is true for the restriction of $\rho$ to $D^1(G)$. So there exist a $D^1(G)$-invariant line $L_1 \subset V$. Let $W \subset V$ be the subspace generated by all $D^1(G)$-invariant lines. We have seen that $W \neq \{0\}$ and it is easy to check (?) that $W$ is $G$-invariant. Since $V$ is irreducible we have $W = V = L_1 \oplus L_2 \oplus \ldots \oplus L_n$ where $L_i$ are $D^1(G)$-invariant lines. In other words there exists a basis in $V$ such that $\rho(g_1)$ is diagonal for all $g_1 \in G_1$. In particular we see that all $\rho(g_1), g_1 \in G_1$ commute.

Fix any $g_1 \in G_1$. For any $g \in G$ we have $gg_1g^{-1} \in D^1(G)$. Therefore all matrices $R(g) := \rho(gg_1g^{-1})$ belong to set $S$ of diagonal matrices conjugate to $R(e) = \rho(g_1)$. So the set $S$ is finite. Consider the map $R : G \rightarrow S, g \rightarrow \rho(gg_1g^{-1})$. Since $G$ is connected the image $Im(R) \subset S$ is connected. Since $S$ is a finite we see that $Im(R)$ is a point. So $\rho(g_1)\rho(g) = \rho(g)\rho(g_1)$ for all $g \in G, g_1 \in G_1$.

Since $V$ is irreducible it follows from the Schur’s lemma that there exists a group morphism $\lambda : D^1(G) \rightarrow k^*$ such that $\rho(g_1) = \lambda(g_1)Id$ for all $g_1 \in D^1(G)$. On the other hand it is clear that $\det(\rho(g_1)) = 1$ for all $g_1 \in D^1(G)$. So $\lambda(D^1(G)) \subset \mu_n$ where $\mu_n = \{x \in k^*| x^n = 1\}$.

Since $\mu_n$ is finite and $D^1(G)$ is connected we see that $\lambda \equiv 1$. So $gg'g^{-1}g'^{-1} \in D^1(G)$ for all $g, g' \in G$ and all $\rho(g), g \in G$ commute. Therefore there exists a $G$-invariant line $V_1 \subset V$.

b) and c) follow from a) as in the proof of Theorem 3.1) □

Corollary Let $G$ be a connected solvable algebraic group and $G_u \subset G$ is the subset of unipotent elements. Then $G_u$ is a closed normal subgroup of $G$ which is nilpotent.