We start with the following general result.

Lemma 2.1 Let $\phi : X \to Y$ be a morphism of affine varieties such that $\phi^* : k[Y] \to k[X]$ is surjective. Then $Z := Im(\phi) \subset Y$ is a closed subset of Y and ϕ defines an isomorphism $\underline{X} \to \underline{Z}$ of affine algebraic varieties.

Proof. Let $I := Ker(\phi^*)$. Then I is a primitive ideal of $k[Y]$. It defines a closed subset Z of Y and moreover $\underline{Z} = (Z, k[Y]/I)$ is an affine variety. Since ϕ^* is surjective it defines an isomorphism $k[Y]/I \to k[X]\Box$.

Definition 2.1. a) An affine algebraic group is an affine variety $\underline{G} = (G, A)$ and a group structure $m : G \times G \to G, inv : G \to G, e \in G$ on G such that the maps m, inv are morphisms of affine varieties. All algebraic groups in this course are affine.

b) Let $\mathbb{X} = (X, B)$ be an affine variety. An *action* of $\underline{G}$ on $\underline{X}$ is an action $a : G \times X \to X$ of the group G on X such that $a : G \times X \to X$ is a morphism of affine varieties. For any $g \in G$ we define $a(g) : X \to X$ by $a(g)(x) := a(g, x)$.

c) Given an action $a : \underline{G} \times \underline{X} \to \underline{X}$ of an algebraic group $\underline{G}$ on an affine variety $\underline{X} = (X, B)$ we denote by $\rho_a : G \to Aut(B)$ the representation of G on B given by $\rho_a(g)(f)(x) := f(a(g^{-1})(x)), f \in B, x \in X, g \in G$.

d) In the case $\underline{X} = \underline{G}$ we obtain *the left regular representation* l of G on the space A of regular functions on G

$$l(g)(f)(x) := f(g^{-1}x), f \in A, x, g \in G$$

and *the right regular representation* r of G on the space A of regular functions on G

$$l(g)(f)(x) := f(xg), f \in A, x, g \in G\Box$$

Examples a) The additive group $\mathbb{G}_a$. We take $\underline{G} = \mathbb{A}^1$ with the addition as the group operation. Then $A = k[x], A \otimes A = k[x, y].inv^\star(f) = f(-x)$ and the map $m^\star : A \to, A \otimes A$ is given by $m^\star(f) = f(x + y)$.

b) The multiplicative group $\mathbb{G}_m$. We take $\underline{G} = \mathbb{A}^1 - \{0\}$ with the multiplication as the group operation. Then $A = k[x, x^{-1}], A \otimes A = k[x, x^{-1}, y, y^{-1}], inv^\star(f) = f(x^{-1})$ and the map $m^\star : A \to, A \otimes A$ is given by $m^\star(f) = f(xy)$

c) The matrix group $\underline{GL_n}$. Let $GL_n \subset M_n$ be the basic open set $X \in M_n | D(X) \neq 0$ where $D(X)$ is the determinant of X. The group operation is the matrix multiplication. Then $k[\underline{GL_n}] = k[T_{ij}, D^{-1}]$ and the map $m^\star$ is given by

$$m^\star(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj}$$

It is clear that $\underline{GL_1} = \mathbb{G}_m$.

d) Let $m^\star_{GL_n} : k[GL_n] \to k[GL_n] \otimes k[GL_n]$ be as in c). Since $D(xy) = D(x)D(y)$ we have $m^\star_{GL_n}(D) = D \otimes D$. So the closed subset

$$SL_n(k) = \{x \in GL_n(k) | D(x) = 1\}$$

inherits the structure of an algebraic group.□

Problems 2.1 a) Let L be a finite-dimensional k-vector space, Define the structure of algebraic group $\underline{Aut}(L)$ on the group $Aut(L)$ of invertible linear transformations of L.

b) Let (G, A) be an algebraic group, $\rho : G \to GL_n(k)$ a representation. Show that ρ defines a homomorphism of algebraic groups iff there exists $a_i^j \in A, 1 \leq i, j \leq n$ such that

$$\rho(g) = (ev_g(a_i^j))_{i,j=1}^n, g \in G$$

c) let $\underline{G} = (G, A), \underline{H} = (H, B)$ be algebraic groups. Define a notion of a homomorphism $\underline{\phi} : \underline{G} \to \underline{H}$ of algebraic groups without any appeal to groups G and H.

d) Let $\underline{G}$ be an affine algebraic variety, $m : \underline{G} \times \underline{G} \to \underline{G}, inv : \underline{G} \to \underline{G}$ algebraic morphisms and $e \in G$. We denote by and $\gamma : G \to G$ be the constant map onto $e \in G$. Write diagrams using the multiplication $\mu : A \otimes A \to A$ and the maps $m^\star, inv^\star$ and γ whose commutativity express the the group axioms for G□.

Lemma 2.2 Let $\underline{a} : \underline{G} \times \underline{X} \to \underline{X}$ be an action of an algebraic group on an affine variety (X, B) and $\rho_a : G \to Aut(B)$ the corresponding representation. Show that

a) A subspace $L \subset B$ is ρ-invariant iff $a^*(L) \subset A \otimes L$.

b) For any finite-dimensional subspace $L' \subset B$ there exists a finite-dimensional ρ_a-invariant subspace $L \subset B$ containing L.

c) Let (G, A) be an algebraic group, $c : A \to A$ an algebra automorphism commuting with $r(g)$ for all $g \in G$. Then there exists unique $y \in G$ such that $c = l(y)$.

Proof. a') Assume that $L \subset B$ is $\rho(g)$-invariant for all $g \in G$. We want to show that $a^*(L) \subset A \otimes L$. Fix $l \in L$. Since the action

$a : G \times X \to X$ is algebraic we have

$$a^\star(l) = \sum_{i=1}^{n} f_i \otimes b_i, f_i \in k[G], b_i \in B = k[X]$$

Moreover we can assume that the functions $f_i \in k[G], 1 \leq i \leq n$ are linearly independent. Since $\rho(g)(l) \in L$ for all $g \in G$ we see that $\sum_{i=1}^{n} f_i(g)b_i (= \rho(g)(l))$ lies in L for all $g \in G$. It is now easy to see (?) that $b_i \in L$ for all $1 \leq i \leq m$.

a”) Conversely, assume that $a^*(L) \subset A \otimes L$. Then for any $l \in L$ we have

$$a^*(l) == \sum_{i=1}^{n} f_i \otimes b_i, f_i \in k[G], b_i \in L$$

But then then $\rho(g)(l) = \sum_{i=1}^{n} f_i(g)b_i = \rho(g)(l) \in L$.

b) Let $l_j, 1 \leq j \leq m$ be a basis of L'. Since the action $a : G \times X \to X$ is algebraic we have

$$a^\star(l_j) = \sum_{i=1}^{n_j} f_j^i(g) \otimes b_j^i, f_j^i \in k[G], b_j^i \in B = k[X], 1 \leq i \leq n_j$$

Then (?) the subspace L of B spanned by $b_j^i, 1 \leq j \leq m, 1 \leq i \leq n_j$ contains L' and is ρ-invariant.

c) Consider the homomorphism $\alpha : A \to k, f \to c(f)(e)$. Then $ker(\alpha)$ is a maximal ideal of A. Therefore [by the Nullstellensatz] there exists $y \in G$ such that $\alpha(f) = f(y), f \in A$. So $c(f)(e) \equiv f(y)$. Since c commutes with with $r(g)$ for all $g \in G$ we see that

$$c(f)(g) = r(g)(c(f))(e) = c(r(g)(f))(e) = r(g)(f)(y) = f(yg) = l(y)(f)(g)\square$$

Theorem 2.1. If $\underline{G} = (G, A)$ is an [affine] algebraic group then there exists a morphism $\phi : G \to GL_n(k)$ of algebraic groups such that $Im(\phi)$ is a closed subgroup of $GL_n(k)$ and ϕ defines an isomorphism $\phi : G \to Im(\phi)$ of algebraic groups.$\square$

Proof. Let $l : G \times G \to G$ be the left regular representation and $a_1, ..., a_r$ be generators of the algebra A. As follows from Lemma 2.2 b) there exists a finite-dimensional l-invariant subspace $L \subset A$ containing $a_1, ..., a_r$. Choose a basis $l_i, 1 \leq i \leq n$ of L. Since the left regular representation is algebraic there exist $a_i^j \in A, 1 \leq i, j \leq n$ such that

$$l_i(g'g'') = \sum_{j=1}^{n} (a_i^j)(g')l_j(g''), g', g'' \in G$$

Consider now the map

$$\psi : G \to GL_n(k), g \to (a_i^j(g))$$

It is clear (?) that ψ is a morphism of algebraic groups and that the map $\psi^* : k[GL_n] \to A$ is surjective.

Theorem 2.1 follows now from Lemma 2.1.□

We discuss now some results familiar from Linear algebra.

Definition 2.2. Let V be a finite-dimensional k-vector space, $d :=$ dimension $(V), x \in End(V)$.

a) x is *semisimple* if x is diagonalizable,

b) x is *unipotent* if all eigenvalues of x are equal to 1,

c) For any $\mu \in k$ we define

$$V_\mu := \{v \in V | (x - \mu Id)^d v = 0\}$$

d) We denote by $P_x(t)$ the characteristic polynomial $P_x(t) := det(t - x)$ and by $Spect(x) \subset k$ the set of roots of $P_x(t)$, Then we have $P_x(t) = \prod_{\mu \in Spect(x)} (t - \mu)^{m_\mu}$ where m_μ is the order of zero of $P_x(t)$ at μ.□

Lemma 2.3 a) Let Let V be a finite-dimensional k-vector space, $x \in End(V)$. Then $P_x(x) = 0$.□

Proof. For any $x \in End(V)$ we denote by $ad(x) : End(V) \to End(V)$ the linear map $y \to [x, y] := xy - yx$ and by $disc(x)$ the characteristic polynomial of $ad(x)$. Let

$$U := \{x \in End(V) | disc(x) \neq 0\}$$

It is clear that any $x \in U$ the subspaces $V_\mu, \mu \in Spect(x)$ are one-dimensional and the matrix x is conjugate to a diagonal matrix with the set of diagonal entries equal to $Spect(x)$. So the restriction of $P_x(x) : End(V) \to End(V)$ to U is equal to zero. Choose a basis $v_1, ..., v_d$ in V and consider the matrix coefficients of $P_x(x)$ as polynomial function $f_{i,j}, 1 \leq i, j \leq d$ on the algebraic variety $End(V)$. We want to prove that $f_{i,j} \equiv 0$ for all $1 \leq i, j \leq d$.

Since the restriction of $P_x(x) : End(V) \to End(V)$ is equal to zero we see that the restriction of on the basic set $U = U_{disc}$ is equal to zero. In other words the image of $f_{i,j}$ in the ring $k[End(V)]_{disc}$ is equal to zero. But is follows from Problem 1.1 b) that the kernel of the localization map $k[End(V)] \to k[End(V)]_{disc}$ is equal to $\{0\}$. So $f_{i,j} \equiv 0$ for all $1 \leq i, j \leq d$.□

Definition 2.3. a) Let V be a [not necessary finite-dimensional] k-vector space. We say that an automorphism $g : V \to V$ is *locally finite* if for any vector $v \in V$ there exists a finite-dimensional g-invariant subspace $W \subset V$ containing v.

b) we say that a locally finite automorphism $s : V \to V$ is *locally semisimple* if for any s-invariant finite-dimensional subspace $W \subset V$ the automorphism s_W is semisimple.

c) we say that a locally finite automorphism $s : V \to V$ is *locally unipotent* if for any s-invariant finite-dimensional subspace $W \subset V$ the automorphism s_W is unipotent.$\square$

In the next problem all vector spaces in the part a) -m) are finite-dimensional.

Problem 2.2 a) Let $g \in Aut(V)$ be an element of finite order m. Then g is semisimple iff m is prime to the characteristic p of k. [We assume that every number is prime to 0].

b) $V_\mu \neq \{0\}$ iff $\mu \in Spect(g)$.

c) $V = \oplus_{\mu \in Spect(g)} V_\mu$.

d) If $s' : V \to V$ are semisimple and $u, u' : V \to V$ a unipotent automorphisms such that $g = u's' = s'u'$ and $g = us = su$ then $s' = s, u' = u$.

e) Let $s_g \in Aut(V)$ be such that $s_g(v) = \mu v$ for $v \in V_\mu$. Then s_g is semisimple, $u_g := s_g^{-1} g$ is unipotent and $g = u_g s_g = s_g u_g$.

Remark This decomposition $g = su = us$ is called *the Jordan decomposition* of g.

f) Let $R_g[t] \in k[t]$ be a polynomial such that $R_g(\mu) = \mu$ for all $\mu \in Spect(g)$ and the order of zero of $R_g[t] - \mu$ at μ is not less then m_μ for all $\mu \in Spect(g)$. Then $s_g = R_g[g]$.

g) Show the existence of a polynomial $Q_g[t]$ such that $u_g = Q_g[g]$

h) For any $x \in Aut(V)$ such that $gx = xg$ we have $s_g x = x s_g, u_g x = x u_g$

l) If $W \subset V$ is g-invariant then it is s_g-invariant and u_g-invariant.

m) Let V, V' be k-vectors spaces, $g \in Aut(V)), g' \in Aut(V'))$ and $r : V \to V'$ a k-linear map such that $g' \circ r = r \circ g$. Then

$$s' \circ r = r \circ s, u' \circ r = r \circ u$$

where $g = su, g' = s'u'$ are the Jordan decompositions of g, g'.

n) Let A be a k-vector space, $r \in Aut(A)$ a locally finite automorphism of A. Then there exists unique pair S, U of locally finite automorphism of A such that $r = SU = US, S$ is locally semisimple and U is locally unipotent. Moreover in the case when A is a k-algebra and r is a locally finite automorphism of A the linear automorphisms S and U are algebra automorphisms of A.

o)* Show the existence of the Jordan decomposition for any perfect field k.

p) Let $L = \mathbb{F}_2(t)$ [so L is not perfect],

$$g = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

Show that one can't find $s, u \in GL_2(L)$, s semisimple, u unipotent, such that $g = su = us$.

q) Formulate and prove the additive version of the Jordan decomposition for for $M_n(k)$ where k is an algebraically closed field.□

Definition 2.4. Let (G, A) be an [affine] algebraic group. As follows from Lemma 2.2 operators $r(g) : A \to A, g \in G$ are locally finite. We say that $g \in G$ is *semisimple* if $r(g) : A \to A$ is semisimple and that $g \in G$ is *unipotent* if $r(g) : A \to A$ is unipotent.□

Theorem 2.2. Let (G, A) be an [affine] algebraic group, $g \in G$. Then there exists unique elements $s, u \in G$ such that $g = su = us$ where s is semisimple and u is unipotent.

Proof. I leave for you to prove the uniqueness.

Let $r(g) = SU$ be the decomposition as in Lemma Problem 2.2 n). As follows from Problem Lemma 2.2 h) the automorphisms S and U commute with operators $l(g), g \in G$. Therefore it follows from Lemma 2.2 c) that there exist $s, u \in G$ such that $S = r(s), U = r(u)$. By the definition S is semisimple, u is unipotent and it is easy to see that $g = su = us$.□

Problem 2.3 a) Show that in the case $G = GL_n(k)$ the definition 2.4 coincides with the definition 2.2

Remark This decomposition $g = su$ is called *the Jordan decomposition* of g

b) Let $(G, A), (H, B)$ be algebraic groups and $\phi : G \to H$ be a morphism of algebraic groups. Let $g = su$ be the Jordan decomposition of an element $g \in G$. Then $\phi(g) = \phi(s)\phi(u)$ is the Jordan decomposition of $\phi(g) \in H$□.

The construction of the Jordan decomposition of elements of G is a little ad hoc. So I'll give a more "categorical" explanation.

Definition* 2.5 a) Let $\underline{G} = (G, A)$ be an [affine] algebraic group. A representation of $\underline{G}$ is a representation $\rho_V : G \to Aut(V)$ where V is a finite-dimensional k-vector space and $\rho_V : G \to Aut(V)$ is a morphism of algebraic groups.

b) A $\underline{G}$-data is rule which α associates with any representation $\rho_V : G \to Aut(V)$ of $\underline{G}$ an automorphism $\alpha_V \in Aut(V)$ such that

a) $\alpha_{V' \oplus V''} = \alpha_{V'} \oplus \alpha_{V''}$,

b) $\alpha_{V' \otimes V''} = \alpha_{V'} \otimes \alpha_{V''}$ and

c) for any G-morphism $f : V' \to V''$ we have $\alpha_{V''} \circ f = f \circ \alpha_{V'}$.

It is clear that any $g \in G$ defines a $\underline{G}$-data $\alpha(g)$ such that $\alpha_V(g) := \rho_V(g)$.$\square$

Lemma 2.7 For any $\underline{G}$-data α there exists unique $g \in G$ such that $\alpha = \alpha(g)$.$\square$

Proof. Let α be a $\underline{G}$-data. As follows form Lemma 2.2 we can write A as a union $A = \cup_n V_n$ where $V_n \subset A, V_{n-1} \subset V_n, n > 0$ are finite dimensional subspaces invariant under the left regular representation. So we can define $\alpha_{V_n} \subset Aut(V_n), n > 0$. As follows from the condition c) the automorphisms $\alpha_{V_n}, n > 0$ come from an automorphisms α_A of the vector space A. Using the conditions b) and c) we see that

$$\alpha_A(f'f'') = \alpha_A(f')\alpha_A(f'')$$

for all $f', f'' \in A$. In other words we see that α_A an automorphism of the algebra A. As follows from the condition c) α_A commutes with the right shifts. As follows from Lemma 2.2 c) there exists $g \in G$ such that $\alpha_A(f) = r(g)(f)$ for all $f \in A$$\square$

Using Lemma 2.7 we can give a very short [an conceptual] definition of the Jordan decomposition of $g \in G$. For any representation $\rho_V : G \to Aut(V)$ of $\underline{G}$ we consider the Jordan decomposition $\rho_V(g) = s_V u_V$ of $\rho_V(g)$ and define $\alpha(s) := s_V, \alpha(u) := u_V$. As follows from Lemma 2.5 g) $\alpha(s)$ and $\alpha(u)$ are $\underline{G}$-datas and therefore there exists $s, u \in G$ such that $\alpha(s)_V \equiv \rho_V(s), \alpha(u)_V \equiv \rho_V(u)$. Then $g = su$ is the Jordan decomposition of an element $g \in G$$\square$.

Definition 2.6.a) We say that a topological space X is *irreducible* if $X \neq \emptyset$ and X is not a union of two proper closed subsets. A subset $Y \subset X$ is irreducible if it is irreducible in the induced topology.

b) Let X be a topological space. A subset of X is *locally closed* if it is an intersection of an open and a closed subsets of X. A subset of

X is *constructible* if it is a union of a finite number of locally closed subsets.$\square$

Problem 2.4 a) If (X, A) is an affine variety then X is irreducible iff A is integral [= does not have zero divisor].

b) A constructible subset $Y \subset X$ is irreducible iff it's closure $\bar{Y}$ is irreducible.

c) Let X, X' be topological spaces such that X is irreducible and $f : X \to X'$ is a continuous map. Then $f(X)$ is irreducible.

d) If X is irreducible, $U \subset X$ is open non-empty then U is irreducible and dense in $X\square$.

e) Let X be a Noetherian topological space. Then there exists only a finite number of maximal irreducible subsets of X, X is a union of it's maximal irreducible subsets, and every irreducible subset of X is contained in a maximal irreducible subset.$\square$

f) If $Y \subset X$ is a constructible subset then there exists a set $U \subset Y$ which is open and dense in the closure $\bar{Y}$ of Y.$\square$.

Definition 2.8. a) Let (X, A) be an irreducible affine variety. Then $\dim(X) := \mathit{deg.tr}_k(Q_A)$ where Q_A is the field of fractions of A.

b) For any affine variety X we define $\dim(X)$ as the maximal dimension of it irreducible components. $\square$.

Lemma 2.8. Let (X, A) be an affine variety, $Z \subset X$ a closed irreducible subspace. Then $\dim(Z) \leq \dim(X)$ and if $\dim(Z) = \dim(X)$ then Z is an irreducible component of (X).$\square$

Proof. It is easy (?) to reduce the proof to the case when (X, A) is irreducible. We have to show that for any proper closed irreducible subspace $Z \subset X$ we have $\dim(Z) < \dim(X)$. Since $Z \subset X$ is closed it has the form $Z = \mathcal{V}(I)$ where $I \subset A$ is a non-zero ideal and we have to show that

$$\mathit{deg.tr}_k(Q_B) < \mathit{deg.tr}_k(Q_A), B := A/I$$

Let $d = \mathit{deg.tr}_k(Q_A)$. Then we can find algebraically independent elements $a_1, ..., a_d \in A$ such that for any non-zero $x \in A$ there exists a monic polynomial $P_a[t] = t^n + \sum_{i=0}^{n-1} c_i[a_1, ..., a_d]t^i$ with coefficients in the subring $k[a_1, ..., a_d] \subset A$ such that $c_0[a_1, ..., a_d] \neq 0$ and $P_x[x] = 0$. We denote by $b_1, ..., b_d \in B$ the images of a_i under the *reduction* $A \to B$.

Let x be a non-zero element of $I, P := P_x = \sum_{i=0}^{n} c_i[a_1, ..., a_d]t^i$. So

$$x^n + \sum_{i=0}^{n-1} c_i[a_1, ..., a_d]x^i = 0$$

Since $x \in I$ we see that $c_0[b_1, ..., b_d] = 0$. Therefore we can find $j_0, 1 \leq j_0 \leq d$ such that $b_j \in B$ is algebraic over $k[a_j, j \neq j_0$. Therefore $deg.tr_k(Q_B) < d.\Box$,

Problem 2.4 a) Let $\phi : X \to Y$ be a morphism of affine varieties and $\phi^* : k[Y] \to k[X]$ be the associated algebra homomorphism. Show that

a) If X is irreducible then the closure $\overline{\phi(X)}$ is also irreducible and $\dim\overline{\phi(X)} \leq \dim(X)$,

b) ϕ^* is injective iff $\phi(X)$ is dense in Y,
c) if X, Y are irreducible affine varieties then
$\dim(X \times Y) = \dim(X)+\dim(Y)$.