We start with the following general result.

**Lemma 2.1** Let \( \phi : X \to Y \) be a morphism of affine varieties such that \( \phi^* : k[Y] \to k[X] \) is surjective. Then \( Z := \text{Im}(\phi) \subseteq Y \) is a closed subset of \( Y \) and \( \phi \) defines an isomorphism \( X \to Z \) of affine algebraic varieties.

**Proof.** Let \( I := \text{Ker}(\phi^*) \). Then \( I \) is a primitive ideal of \( k[Y] \). It defines a closed subset \( Z \) of \( Y \) and moreover \( Z = (Z, k[Y]/I) \) is an affine variety. Since \( \phi^* \) is surjective it defines an isomorphism \( k[Y]/I \to k[X] \).

**Definition 2.1.**

a) An affine algebraic group is an affine variety \( G = (G, A) \) and a group structure \( m : G \times G \to G, \text{inv} : G \to G, e \in G \) on \( G \) such that the maps \( m, \text{inv} \) are morphisms of affine varieties. All algebraic groups in this course are affine.

b) Let \( X = (X, B) \) be an affine variety. An *action* of \( G \) on \( X \) is an action \( a : G \times X \to X \) of the group \( G \) on \( X \) such that \( a : G \times X \to X \) is a morphism of affine varieties. For any \( g \in G \) we define \( a(g) : X \to X \) by \( a(g)(x) := a(g, x) \).

c) Given an action \( a : G \times X \to X \) of an algebraic group \( G \) on an affine variety \( X = (X, B) \) we denote by \( \rho_a : G \to \text{Aut}(B) \) the representation of \( G \) on \( B \) given by \( \rho_a(g)(f)(x) := f(a(g^{-1})(x)), f \in B, x \in X, g \in G \).

d) In the case \( X = G \) we obtain the *left regular representation* \( l \) of \( G \) on the space \( A \) of regular functions on \( G \)

\[
l(g)(f)(x) := f(g^{-1}x), f \in A, x, g \in G
\]

and the *right regular representation* \( r \) of \( G \) on the space \( A \) of regular functions on \( G \)

\[
l(g)(f)(x) := f(xg), f \in A, x, g \in G
\]

**Examples**

a) The additive group \( \mathbb{G}_a \). We take \( G = \mathbb{A}^1 \) with the addition as the group operation. Then \( A = k[x], A \otimes A = k[x, y], \text{inv}^*(f) = f(-x) \) and the map \( m^* : A \to A \otimes A \) is given by \( m^*(f) = f(x + y) \).

b) The multiplicative group \( \mathbb{G}_m \). We take \( G = \mathbb{A}^1 - \{0\} \) with the multiplication as the group operation. Then \( A = k[x, x^{-1}], A \otimes A = k[x, x^{-1}, y, y^{-1}], \text{inv}^*(f) = f(x^{-1}) \) and the map \( m^* : A \to A \otimes A \) is given by \( m^*(f) = f(xy) \).

c) The matrix group \( GL_n \). Let \( GL_n \subseteq M_n \) be the basic open set \( X \in M_n | D(X) \neq 0 \) where \( D(X) \) is the determinant of \( X \). The group operation is the matrix multiplication. Then \( k[GL_n] = k[T_{ij}, D^{-1}] \) and
the map \( m^* \) is given by

\[
m^*(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj}
\]

It is clear that \( GL_1 = \mathbb{G}_m \).

d) Let \( m^*_{GL_n} : k[GL_n] \rightarrow k[GL_n] \otimes k[GL_n] \) be as in c). Since \( D(xy) = D(x)D(y) \) we have \( m^*_{\mathbb{GL}_n}(D) = D \otimes D \). So the closed subset

\[
SL_n(k) = \{x \in GL_n(k)|D(x) = 1\}
\]

inherits the structure of an algebraic group. □

Problems 2.1

a) Let \( L \) be a finite-dimensional \( k \)-vector space, Define the structure of algebraic group \( Aut(L) \) on the group \( Aut(L) \) of invertible linear transformations of \( L \).

b) Let \( (G, A) \) be an algebraic group, \( \rho : G \rightarrow GL_n(k) \) a representation. Show that \( \rho \) defines a homomorphism of algebraic groups iff there exists \( a^i_j \in A, 1 \leq i, j \leq n \) such that

\[
\rho(g) = (ev_g(a^i_j))_{i,j=1}^{n}, g \in G
\]

c) Let \( G = (G, A), H = (H, B) \) be algebraic groups. Define a notion of a homomorphism \( \phi : G \rightarrow H \) of algebraic groups without any appeal to groups \( G \) and \( H \).

d) Let \( G \) be an affine algebraic variety, \( m : G \times G \rightarrow G, inv : G \rightarrow G \) algebraic morphisms and \( e \in G \). We denote by and \( \gamma : G \rightarrow G \) be the constant map onto \( e \in G \). Write diagrams using the multiplication \( \mu : A \otimes A \rightarrow A \) and the maps \( m^*, inv^* \) and \( \gamma \) whose commutativity express the the group axioms for \( G \). □

Lemma 2.2 Let \( g : G \times X \rightarrow X \) be an action of an algebraic group on an affine variety \( (X, B) \) and \( \rho_a : G \rightarrow Aut(B) \) the corresponding representation. Show that

a) A subspace \( L \subset B \) is \( \rho \)-invariant iff \( a^*(L) \subset A \otimes L \).

b) For any finite-dimensional subspace \( L' \subset B \) there exists a finite-dimensional \( \rho_a \)-invariant subspace \( L \subset B \) containing \( L \).

c) Let \( (G, A) \) be an algebraic group, \( c : A \rightarrow A \) an algebra automorphism commuting with \( r(g) \) for all \( g \in G \). Then there exists unique \( y \in G \) such that \( c = l(y) \).

Proof. a') Assume that \( L \subset B \) is \( \rho(g) \)-invariant for all \( g \in G \). We want to show that \( a^*(L) \subset A \otimes L \). Fix \( l \in L \). Since the action
a : G × X → X is algebraic we have

\[ a^*(l) = \sum_{i=1}^{n} f_i \otimes b_i, f_i \in k[G], b_i \in B = k[X] \]

Moreover we can assume that the functions \( f_i \in k[G], 1 \leq i \leq n \) are linearly independent. Since \( \rho(g)(l) \in L \) for all \( g \in G \) we see that \( \sum_{i=1}^{n} f_i(g)b_i(= \rho(g)(l)) \) lies in \( L \) for all \( g \in G \). It is now easy to see (?) that \( b_i \in L \) for all \( 1 \leq i \leq m \).

a') Conversely, assume that \( a^*(L) \subset A \otimes L \). Then for any \( l \in L \) we have

\[ a^*(l) = \sum_{i=1}^{n} f_i \otimes b_i, f_i \in k[G], b_i \in L \]

But then then \( \rho(g)(l) = \sum_{i=1}^{n} f_i(g)b_i = \rho(g)(l) \in L \).

b) Let \( l_j, 1 \leq j \leq m \) be a basis of \( L' \). Since the action \( a : G \times X \to X \) is algebraic we have

\[ a^*(l_j) = \sum_{i=1}^{n_j} f_j^i(g) \otimes b_j^i, f_j^i \in k[G], b_j^i \in B = k[X], 1 \leq i \leq n_j \]

Then (?) the subspace \( L \) of \( B \) spanned by \( b_j^i, 1 \leq j \leq m, 1 \leq i \leq n_j \) contains \( L' \) and is \( \rho \)-invariant.

c) Consider the homomorphism \( \alpha : A \to k, f \to c(f)(e) \). Then \( \ker(\alpha) \) is a maximal ideal of \( A \). Therefore [ by the Nullstellensatz] there exists \( g \in G \) such that \( \alpha(f) = f(y), f \in A \). So \( c(f)(e) \equiv f(y) \). Since \( c \) commutes with with \( r(g) \) for all \( g \in G \) we see that

\[ c(f)(g) = r(g)(c(f))(e) = c(r(g)(f))(e) = r(g)(f)(y) = f(yg) = l(y)(f)(g) \]

**Theorem 2.1.** If \( G = (G, A) \) is an [affine] algebraic group then there exists a morphism \( \phi : G \to GL_n(k) \) of algebraic groups such that \( \text{Im}(\phi) \) is a closed subgroup of \( GL_n(k) \) and \( \phi \) defines an isomorphism \( \phi : G \to \text{Im}(\phi) \) of algebraic groups.□

Proof. Let \( l : G \times X \to G \) be the left regular representation and \( a_1, ..., a_r \) be generators of the algebra \( A \). As follows from Lemma 2.2 b) there exists a finite-dimensional \( l \)-invariant subspace \( L \subset A \) containing \( a_1, ..., a_r \). Choose a basis \( l_i, 1 \leq i \leq n \) of \( L \). Since the left regular representation is algebraic there exist \( a_i^j \in A, 1 \leq i, j \leq n \) such that

\[ l_i(g'g'') = \sum_{j=1}^{n} (a_i^j)(g')l_j(g''), g', g'' \in G \]
Consider now the map 
\[ \psi : G \to GL_n(k), g \to (a^1_j(g)) \]
It is clear (?) that \( \psi \) is a morphism of algebraic groups and that the map \( \psi^* : k[GL_n] \to A \) is surjective.

Theorem 2.1 follows now from Lemma 2.1. □

We discuss now some results familiar from Linear algebra.

**Definition 2.2.** Let \( V \) be a finite-dimensional \( k \)-vector space, 
\[ d := \text{dimension } (V), x \in \text{End}(V). \]

a) \( x \) is semisimple if \( x \) is diagonalizable,
b) \( x \) is unipotent if all eigenvalues of \( x \) are equal to 1,
c) For any \( \mu \in k \) we define 
\[ V_\mu := \{ v \in V | (x - \mu Id)^d v = 0 \} \]
d) We denote by \( P_x(t) \) the characteristic polynomial \( P_x(t) := \det(t - x) \) and by \( \text{Spect}(x) \subset k \) the set of roots of \( P_x(t) \). Then we have \( P_x(t) = \prod_{\mu \in \text{Spect}(x)} (t - \mu)^{m_\mu} \) where \( m_\mu \) is the order of zero of \( P_x(t) \) at \( \mu \). □

**Lemma 2.3** a) Let \( V \) be a finite-dimensional \( k \)-vector space, \( x \in \text{End}(V) \). Then \( P_x(x) = 0. \)

**Proof.** For any \( x \in \text{End}(V) \) we denote by \( \text{ad}(x) : \text{End}(V) \to \text{End}(V) \) the linear map \( y \to [x,y] := xy - yx \) and by \( \text{disc}(x) \) the characteristic polynomial of \( \text{ad}(x) \). Let 
\[ U := \{ x \in \text{End}(V) | \text{disc}(x) \neq 0 \} \]
It is clear that any \( x \in U \) the subspaces \( V_\mu, \mu \in \text{Spect}(x) \) are one-dimensional and the matrix \( x \) is conjugate to a diagonal matrix with the set of diagonal entries equal to \( \text{Spect}(x) \). So the restriction of \( P_x(x) : \text{End}(V) \to \text{End}(V) \) to \( U \) is equal to zero. Choose a basis \( v_1, ..., v_d \) in \( V \) and consider the matrix coefficients of \( P_x(x) \) as polynomial function \( f_{i,j}, 1 \leq i, j \leq d \) on the algebraic variety \( \text{End}(V) \). We want to prove that \( f_{i,j} \equiv 0 \) for all \( 1 \leq i, j \leq d \).

Since the restriction of \( P_x(x) : \text{End}(V) \to \text{End}(V) \) is equal to zero we see that the restriction of on the basic set \( U = U_{\text{disc}} \) is equal to zero. In other words the image of \( f_{i,j} \) in the ring \( k[\text{End}(V)]_{\text{disc}} \) is equal to zero. But is follows from Problem 1.1 b) that the kernel of the localization map \( k[\text{End}(V)] \to k[\text{End}(V)]_{\text{disc}} \) is equal to \( \{0\} \). So \( f_{i,j} \equiv 0 \) for all \( 1 \leq i, j \leq d \). □
Definition 2.3. a) Let $V$ be a [not necessary finite-dimensional] $k$-vector space. We say that an automorphism $g : V \to V$ is locally finite if for any vector $v \in V$ there exists a finite-dimensional $g$-invariant subspace $W \subset V$ containing $v$.

b) we say that a locally finite automorphism $s : V \to V$ is locally semisimple if for any $s$-invariant finite-dimensional subspace $W \subset V$ the automorphism $s_W$ is semisimple.

c) we say that a locally finite automorphism $s : V \to V$ is locally unipotent if for any $s$-invariant finite-dimensional subspace $W \subset V$ the automorphism $s_W$ is unipotent.□

In the next problem all vector spaces in the part a) -m) are finite-dimensional.

Problem 2.2 a) Let $g \in \text{Aut}(V)$ be an element of finite order $m$. Then $g$ is semisimple iff $m$ is prime to the characteristic $p$ of $k$. [We assume that every number is prime to 0].

b) $V_\mu \neq \{0\}$ iff $\mu \in \text{Spect}(g)$.

c) $V = \bigoplus_{\mu \in \text{Spect}(g)} V_\mu$.

d) If $s' : V \to V$ are semisimple and $u, u' : V \to V$ a unipotent automorphisms such that $g = u's' = s'u'$ and $g = us = su$ then $s' = s, u' = u$.

e) Let $s_g \in \text{Aut}(V)$ be such that $s_g(v) = \mu v$ for $v \in V_\mu$. Then $s_g$ is semisimple, $u_g := s_g^{-1}g$ is unipotent and $g = u_gs_g = s_gu_g$.

Remark This decomposition $g = su = us$ is called the Jordan decomposition of $g$.

f) Let $R_g[t] \in k[t]$ be a polynomial such that $R_g(\mu) = \mu$ for all $\mu \in \text{Spect}(g)$ and the order of zero of $R_g[t] - \mu$ at $\mu$ is not less then $m_\mu$ for all $\mu \in \text{Spect}(g)$. Then $s_g = R_g[g]$.

g) Show the existence of a polynomial $Q_g[t]$ such that $u_g = Q_g[g]$.

h) For any $x \in \text{Aut}(V)$ such that $gx = xg$ we have $s_gx = xs_g, u_gx = xu_g$.

i) If $W \subset V$ is $g$-invariant then it is $s_g$-invariant and $u_g$-invariant.

m) Let $V, V'$ be $k$-vectors spaces, $g \in \text{Aut}(V)), g' \in \text{Aut}(V'))$ and $r : V \to V'$ a $k$-linear map such that $g' \circ r = r \circ g$. Then 

$$s' \circ r = r \circ s, u' \circ r = r \circ u$$

where $g = su, g' = s'u'$ are the Jordan decompositions of $g, g'$.
n) Let \( A \) be a \( k \)-vector space, \( r \in Aut(A) \) a locally finite automorphism of \( A \). Then there exists unique pair \( S, U \) of locally finite automorphism of \( A \) such that \( r = SU = US \), \( S \) is locally semisimple and \( U \) is locally unipotent. Moreover in the case when \( A \) is a \( k \)-algebra and \( r \) is a locally finite automorphism of \( A \) the linear automorphisms \( S \) and \( U \) are algebra automorphisms of \( A \).

o)* Show the existence of the Jordan decomposition for any perfect field \( k \).

p) Let \( L = \mathbb{F}_2(t) \) [ so \( L \) is not perfect],

\[
g = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}
\]

Show that one can’t find \( s, u \in GL_2(L) \), \( s \) semisimple, \( u \) unipotent, such that \( g = su = us \).

q) Formulate and prove the additive version of the Jordan decomposition for for \( M_n(k) \) where \( k \) is an algebraically closed field. □

**Definition 2.4.** Let \((G,A)\) be an [affine] algebraic group. As follows from Lemma 2.2 operators \( r(g) : A \to A, g \in G \) are locally finite. We say that \( g \in G \) is **semisimple** if \( r(g) : A \to A \) is semisimple and that \( g \in G \) is **unipotent** if \( r(g) : A \to A \) is unipotent. □

**Theorem 2.2.** Let \((G,A)\) be an [affine] algebraic group, \( g \in G \). Then there exists unique elements \( s, u \in G \) such that \( g = su = us \) where \( s \) is semisimple and \( u \) is unipotent.

Proof. I leave for you to prove the uniqueness.

Let \( r(g) = SU \) be the decomposition as in Lemma Problem 2.2 n). As follows from Problem Lemma 2.2 h) the automorphisms \( S \) and \( U \) commute with operators \( l(g), g \in G \). Therefore it follows from Lemma 2.2 c) that there exist \( s, u \in G \) such that \( S = r(s), U = r(u) \). By the definition \( S \) is semisimple, \( u \) is unipotent and it is easy to see that \( g = su = us \). □

**Problem 2.3** a) Show that in the case \( G = GL_n(k) \) the definition 2.4 coincides with the definition 2.2

Remark This decomposition \( g = su \) is called the **Jordan decomposition** of \( g \).

b) Let \((G,A), (H,B)\) be algebraic groups and \( \phi : G \to H \) be a morphism of algebraic groups. Let \( g = su \) be the Jordan decomposition of an element \( g \in G \). Then \( \phi(g) = \phi(s)\phi(u) \) is the Jordan decomposition of \( \phi(g) \in H \). □
The construction of the Jordan decomposition of elements of $G$ is a little ad hoc. So I’ll give a more “categorical” explanation.

**Definition** 2.5 a) Let $G = (G, A)$ be an [affine] algebraic group. A representation of $G$ is a representation $\rho_V : G \to Aut(V)$ where $V$ is a finite-dimensional $k$-vector space and $\rho_V : G \to Aut(V)$ is a morphism of algebraic groups.

b) A $G$-data is rule which associates with any representation $\rho_V : G \to Aut(V)$ of $G$ an automorphism $\alpha \in Aut(V)$ such that

\[
\begin{align*}
\alpha_V \otimes V'' &= \alpha_V \otimes \alpha_{V''} \\
\text{for any } G\text{-morphism } f : V' \to V'' \text{ we have } \alpha_{V''} \circ f = f \circ \alpha_{V'}.
\end{align*}
\]

It is clear that any $g \in G$ defines a $G$-data $\alpha(g)$ such that $\alpha_V(g) := \rho_V(g)$.

**Lemma** 2.7 For any $G$-data $\alpha$ there exists unique $g \in G$ such that $\alpha = \alpha(g)$.

**Proof.** Let $\alpha$ be a $G$-data. As follows form Lemma 2.2 we can write

$A$ as a union $A = \bigcup_n V_n$ where $V_n \subset G, V_{n-1} \subset V_n, n > 0$ are finite dimensional subspaces invariant under the left regular representation. So we can define $\alpha_{V_n} \subset Aut(V_n), n > 0$. As follows from the condition c) the automorphisms $\alpha_{V_n}, n > 0$ come from an automorphism $\alpha_A$ of the vector space $A$. Using the conditions b) and c) we see that

$\alpha_A(f'f'') = \alpha_A(f')\alpha_A(f'')$

for all $f', f'' \in A$. In other words we see that $\alpha_A$ an automorphism of the algebra $A$. As follows from the condition c) $\alpha_A$ commutes with the right shifts. As follows from Lemma 2.2 c) there exists $g \in G$ such that $\alpha_A(f) = r(g)(f)$ for all $f \in A$.

Using Lemma 2.7 we can give a very short [an conceptual] definition of the Jordan decomposition of $g \in G$. For any representation $\rho_V : G \to Aut(V)$ of $G$ we consider the Jordan decomposition $\rho_V(g) = s_V u_V$ of $\rho_V(g)$ and define $\alpha(s) := s_V, \alpha(u) := u_V$. As follows from Lemma 2.5 g) $\alpha(s)$ and $\alpha(u)$ are $G$-datas and therefore there exists $s, u \in G$ such that $\alpha(s) = \rho_V(s), \alpha(u) = \rho_V(u)$. Then $g = su$ is the Jordan decomposition of an element $g \in G$.

**Definition** 2.6 a) We say that a topological space $X$ is irreducible if $X \neq \emptyset$ and $X$ is not a union of two proper closed subsets. A subset $Y \subset X$ is irreducible if it is irreducible in the induced topology.

b) Let $X$ be a topological space. A subset of $X$ is locally closed if it is an intersection of an open and a closed subsets of $X$. A subset of
$X$ is constructible if it is a union of a finite number of locally closed subsets. □

**Problem 2.4**

a) If $(X, A)$ is an affine variety then $X$ is irreducible iff $A$ is integral $[=\text{does not have zero divisor}].$

b) A constructible subset $Y \subset X$ is irreducible iff it’s closure $\overline{Y}$ is irreducible.

c) Let $X, X'$ be topological spaces such that $X$ is irreducible and $f : X \rightarrow X'$ is a continuous map. Then $f(X)$ is irreducible.

d) If $X$ is irreducible, $U \subset X$ is open non-empty then $U$ is irreducible and dense in $X$. □

e) Let $X$ be a Noetherian topological space. Then there exists only a finite number of maximal irreducible subsets of $X$, $X$ is a union of it’s maximal irreducible subsets, and every irreducible subset of $X$ is contained in a maximal irreducible subset. □

f) If $Y \subset X$ is a constructible subset then there exists a set $U \subset Y$ which is open and dense in the closure $\overline{Y}$ of $Y$. □.

**Definition 2.8.**

a) Let $(X, A)$ be an irreducible affine variety. Then $\dim(X) := \deg.tr_k(Q_A)$ where $Q_A$ is the field of fractions of $A$.

b) For any affine variety $X$ we define $\dim(X)$ as the maximal dimension of its irreducible components. □.

**Lemma 2.8.** Let $(X, A)$ be an affine variety, $Z \subset X$ a closed irreducible subspace. Then $\dim(Z) \leq \dim(X)$ and if $\dim(Z) = \dim(X)$ then $Z$ is an irreducible component of $(X)$. □

**Proof.** It is easy (?) to reduce the proof to the case when $(X, A)$ is irreducible. We have to show that for any proper closed irreducible subspace $Z \subset X$ we have $\dim(Z) < \dim(X)$. Since $Z \subset X$ is closed it has the form $Z = \mathcal{V}(I)$ where $I \subset A$ is a non-zero ideal and we have to show that

$$\deg.tr_k(Q_B) < \deg.tr_k(Q_A), B := A/I$$

Let $d = \deg.tr_k(Q_A)$. Then we can find algebraically independent elements $a_1, ..., a_d \in A$ such that for any non-zero $x \in A$ there exists a monic polynomial $P_x[t] = t^n + \sum_{i=0}^{n-1} c_i[a_1, ..., a_d]t^i$ with coefficients in the subring $k[a_1, ..., a_d] \subset A$ such that $c_0[a_1, ..., a_d] \neq 0$ and $P_x[x] = 0$. We denote by $b_1, ..., b_d \in B$ the images of $a_i$ under the reduction $A \rightarrow B$. 

Let $x$ be a non-zero element of $I, P := P_x = \sum_{i=0}^{n} c_i[a_1, ..., a_d] t^i$. So

$$x^n + \sum_{i=0}^{n-1} c_i[a_1, ..., a_d] x^i = 0$$

Since $x \in I$ we see that $c_0[b_1, ..., b_d] = 0$. Therefore we can find $j_0, 1 \leq j_0 \leq d$ such that $b_j \in B$ is algebraic over $k[a_j, j \neq j_0]$. Therefore $\text{deg} tr_k(Q_B) < d$. □

**Problem 2.4**

a) Let $\phi : X \to Y$ be a morphism of affine varieties and $\phi^* : k[Y] \to k[X]$ be the associated algebra homomorphism. Show that

a) If $X$ is irreducible then the closure $\overline{\phi(X)}$ is also irreducible and $\dim \overline{\phi(X)} \leq \dim(X)$,

b) $\phi^*$ is injective iff $\phi(X)$ is dense in $Y$,

c) if $X, Y$ are irreducible affine varieties then $\dim(X \times Y) = \dim(X) + \dim(Y)$. 