Definition 1. a) An operator $\mathcal{L}$ on a Hilbert space $\mathcal{H}$ is *compact* if it can be written in the form

$$\mathcal{L}(h) = \sum_{n=1}^{\infty} \rho_n \langle f_n, h \rangle g_n$$

where $f_1, \ldots, f_n, ..., g_1, \ldots, g_n, ...$ are orthonormal sets, $\rho_1, \ldots, \rho_n, ...$ are real numbers such that $\rho_n \to 0$ for $n \to \infty$ where the sum converges in norm.

b) An operator $\mathcal{L}$ is *Nuclear* if it is compact and $\sum_{n=1}^{\infty} |\rho_n| < \infty$.

A nuclear operator on a Hilbert space has the important property that its trace may be defined so that it is finite and is independent of the basis. Given any orthonormal basis $\{\psi_n\}$ for the Hilbert space, one may define the trace as $\mathrm{Tr}\mathcal{L}$ as the sum $\sum_n \langle \psi_n, \mathcal{L}\psi_n \rangle$. Furthermore, this trace is identical to the sum over the eigenvalues of $\mathcal{L}$ (counted with multiplicity).

The definition of trace-class operator was extended to Banach spaces by Alexander Grothendieck in 1955.

Definition 2. a) If A is a Banach space we denote by A' be the dual Banach space which is the set of all continuous or (equivalently) bounded linear functionals on A with the usual norm.

b) Let A and B be Banach spaces. An operator $\mathcal{L} : A \to B$ is *nuclear* if there exist sequences of vectors $\{g_n\} \in B$ with $\|g_n\| \leq 1$, functionals $\{f_n^*\} \in A'$ with $\|f_n^*\| \leq 1$ and complex numbers $\{\rho_n\}$ with $\sum_n |\rho_n| < \infty$, such that $\mathcal{L}(a) = \sum_n \rho_n f_n^*(a) g_n, a \in A$ with the sum converging in the operator norm.

Definition 3. a) A seminorm on a vector space V is a function p on V, such that

$$p(v) \geq 0, p(cv) = |c|p(v), p(v + w) \leq p(v) + p(w), v, w \in V, c \in \mathbb{R}$$

It is clear that $Ker(p) := \{v \in V | p(v) = 0\}$ is a subspace of V. A seminorm is a norm if $Ker(p) = \{0\}$.

b) A *locally convex topological vector space* is a vector space V whose topology is defined by some family $p_i, i \in I$ of seminorms. For any seminorm p, the unit ball $B_p = \{v \in V | p(v) \leq 1\}$ is a closed convex symmetric neighborhood of 0. Conversely any closed convex symmetric neighborhood of 0 is the unit ball of some continuous seminorm. If p is a seminorm on V, we write V_p for the Banach space given by completing V using the seminorm p. There is a natural map from V to V_p whose kernel is $Ker(p)$.

c) A nuclear space is a locally convex topological vector space such that for any seminorm p we can find a larger seminorm q so that the natural map from V_q to V_p is nuclear.

d) A locally convex topological vector space V is Frechet if V is complete and the topology on V is given by a countable family of seminorms.

Examples.

Any finite-dimensional vector space is nuclear.

There are no infinite-dimensional Banach spaces that are nuclear. In practice a sort of converse to this is often true: if a "naturally occurring" topological vector space is not a Banach space, then there is a good chance that it is nuclear.

The simplest infinite example of a nuclear space is the space C of rapidly decreasing sequences $c = (c_1, c_2, ...)$ ("Rapidly decreasing" means that sequences $(c_n P(n))$ are bounded for any polynomial P). For each real number s, we denote by C_s the completion of C in the norm $p_s(C) := \sup |c_n| n^s$.

Whenever $s \geq t$ there is a natural map from C_s to C_t which is nuclear if $s > t+1$. So the space C is nuclear.

The space of smooth functions on any compact manifold is nuclear.

The Schwartz space of smooth functions on $\mathbf{R}^n$ for which the derivatives of all orders are rapidly decreasing is a nuclear space.

The space of entire holomorphic functions on the complex plane is nuclear.

The inductive limit of a sequence of nuclear spaces is nuclear.

The strong dual of a nuclear Frechet space is nuclear.

The product of a family of nuclear spaces is nuclear.

The completion of a nuclear space is nuclear (and in fact a space is nuclear if and only if its completion is nuclear).

Properties.

Nuclear spaces are in many ways similar to finite-dimensional spaces and have many of their good properties.

A closed bounded subset of a nuclear Frechet space is compact. (A subset B of a topological vector space V is *bounded* if for any neighborhood U of 0 in V there exists a positive real scalar c such that $B \subset cU$.)

Any subspace of a nuclear space and any quotient of a nuclear space by a closed subspace is nuclear.

Definition 4. Let A and B be Banach spaces.

a) The algebraic tensor product $A \otimes B$ is the tensor product of A and B as vector spaces without topology. Elements of $A \otimes B$ are finite sums $x = \Sigma_{i=1}^n a_i \otimes b_i, a_i \in A$ and $b_i \in B$ for $i = 1, \ldots, n$.

When A and B are Banach spaces a *cross norm* p on the algebraic tensor product $A\otimes$ B is any norm satisfying the conditions

$$p(a \otimes b) = \|a\| \|b\|, p'(a' \otimes b') = \|a'\| \|b'\|, a \in A, a' \in A', b \in B, b' \in B'.$$

where A', B' are the Banach spaces dual to A and B , and p' is the dual norm of p.

There is the largest cross norm π called the *projective cross norm*, given by

$$\pi(x) = \inf\{\Sigma_{i=1}^n \|a_i\| \|b_i\| : x = \Sigma a_i \otimes b_i\}, x \in A \otimes B$$

and also the smallest cross norm ε called the *injective cross norm*, given by

$$\varepsilon(x) = \sup\{|(a' \otimes b')(x)| : a' \in X', b' \in Y', \|a'\| = \|b'\| = 1\} x \in A \otimes B$$

The completions of the algebraic tensor product in these two norms are called the projective and injective tensor products, and are denoted by $A\hat{\otimes}_\pi B$ and $A\hat{\otimes}_\varepsilon B$. It is easy to see that $\pi(x) \geq \varepsilon(x)$ for all $x \in A \otimes B$ and therefore there is a canonical map from $A\hat{\otimes}_\pi B$ to $A\hat{\otimes}_\varepsilon B$.

Remark. The norm used for the Hilbert space tensor product is not equal to either of these norms.

Tensor products of locally convex topological vector spaces.

The topologies of locally convex topological vector spaces A and B are given by families of seminorms. For each choice of seminorms on A and on B we define the projective and injective seminorms on the algebraic tensor product $A \otimes B$. These families of norms define the projective and injective tensor products denoted by $A\hat{\otimes}_\pi B$ and $A\hat{\otimes}_\varepsilon B$. As before there is a canonical map from $A\hat{\otimes}_\pi B$ to $A\hat{\otimes}_\varepsilon B$.

Theorem 1. If A is a nuclear space then for any locally convex topological vector space B the canonical map from $A\hat{\otimes}_\pi B$ to $A\hat{\otimes}_\varepsilon B$ is an isomorphism.

Definition 5. Let V, W be locally convex topological vector spaces. We say that a continuous bilinear form $\beta : V \times W \to \mathbb{R}$ is *nuclear* if there exist linear forms $\lambda_n \in V', \nu_n \in W', 1 \leq n < \infty$ and continuous seminorms $p_{V'}, p_{W'}$ on V', W' such that
$\sum_n p_{V'}(\lambda_n) p_{W'}(\nu_n) < \infty$ and $\beta(v, w) = \sum_n \lambda_n(v)\nu_n(w)$.

Theorem 2 (The kernel theorem). Any bilinear form on $V \times W$ where the space V is nuclear is a nuclear bilinear form.

Measures. **Definition 6.** a) For any normalized (that is of the total volume one) measure μ on a locally convex topological vector space V we denote by $\tilde{\mu}$ the *Fourier transform* of μ which is a complex-valued function on V given by

$$\tilde{\mu}(y) := \int_{v \in V} e^{i\langle v, y\rangle} d\mu$$

b) A continuous function ϕ on a locally convex topological vector space A is called a *characteristic functional* if $\phi(0) = 1$, and for any complex numbers z_j and vectors $x_j \in A, j, k = 1, ..., n$ we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} z_j \bar{z}_k \phi(x_j - x_k) \geq 0.$$

It is easy to see that for any normalized measure μ on V the function $\tilde{\mu}$ on V' is a characteristic functional.

Theorem 3 (The $BochnerMinlos$ **theorem).** Any characteristic functional on a nuclear space V is the Fourier transform of a measure on the dual space V'.

In particular, if A is the nuclear space of the form $A = \bigcap_{k=0}^{\infty} H_k$, where H_k are Hilbert spaces, the *Bochner Minlos* theorem guarantees the existence of a probability measure on the dual space A' with the characteristic function $e^{-\frac{1}{2}\|y\|^2_{H_0}}$, that is, the existence of the Gaussian measure on the dual space. Such measure is called white noise measure. When A is the Schwartz space, the corresponding random element is a random distribution.