In this course we will discuss applications of the Model theory to Algebraic geometry and Analysis. There is long list of examples and I mention only some of applications:

1) Tarski proved the elimination of quantifiers in the theory of real closed fields. The following statement used by Hormander in his works on differential equations is a corollary of the Tarski's result

For any polynomial $P(x) \in \mathbb{R}[x_1, ..., x_n]$ there are positive constants c, r such that

$$|P(x)| \geq c|x - Z(P)|^r, \forall x \in \mathbb{R}^n, |x| \leq 1$$

where $Z(P) \subset \mathbb{R}^n$ is the set of zeros of P and $|x| = \sqrt{x_1^2 + ... + x_n^2}$.

2) Ax [Ax, James Injective endomorphisms of varieties and schemes. Pacific J. Math. 31 1969 1–7] used the Model theory for the proof of the following result:

Let X be a complex algebraic variety, $f : X \to X$ a regular map which is an imbedding. Then f is onto .

3) Ax and Kochen [Ax, James; Kochen, Simon Diophantine problems over local fields. I. Amer. J. Math. 87 1965 605–630] have shown that for any $n \in \mathbb{N}$ there exists $s(n) \in \mathbb{N}$ such that for any prime number $p > s(n)$ any homogeneous polynomial equation

$P(x_0, ..., x_{n^2}) = 0$, where $P \in \mathbb{Q}_p[x_0, ..., x_{n^2}]$ is a polynomial of degree n, has a non-zero solution.

d) In works of Denef [Denef, J. On the evaluation of certain p-adic integrals. Sminaire de thorie des nombres, Paris 1983–84, 25–47, Progr. Math., 59, Birkhuser Boston, Boston, MA, 1985.], Loeser and Cluckers [Fonctions constructibles exponentielles, transformation de Fourier motivique et principe de transfert. R. Cluckers, F. Loeser] the Model theory is used to obtain new results about p-adic integrals and their "motivic" generalizations.

In spite of it successes, the Model theory did not enter into a "tool box" of mathematicians and even many of mathematicians working on "Motivic integrations" are content to use the results logicians without understanding the details of the proofs.

I don't know any mathematician who did not start as a logician and for whom it was "easy and natural " to learn the Model theory. Often the experience of learning of the Model theory is similar to the one of learning of Physics: for a [short] while everything is so simple and so easily reformulated in familiar terms that "there is nothing to learn" but suddenly one find himself in a place when Model theoreticians "jump from a tussock to a hummock" while we mathematicians don't see where to "put a foot" and are at a complete loss.

So we have two questions:

a) Why is the Model theory so useful in different areas of Mathematics?

b) Why is it so difficult for mathematicians to learn it ?

But really these two questions are almost the same- it is difficult to learn the Model theory since it appeals to different intuition. But exactly this new outlook leads to the successes of the Model theory.

One difficultly facing one who is trying to learn Model theory is disappearance of the "natural" distinction between the formalism and the substance. For example the *fundamental existence theorem* says that the syntactic analysis of a theory [the existence or non- existence of a contradiction] is equivalent to the semantic analysis of a theory [the existence or non- existence of a model].

The other novelty is related to a very general phenomena. A mathematical object never comes in a pure form but always on a definite background. Finding a new way of constructions usually lead to substantial achievements.

For example, a differential manifold is "something" which is locally like a ball. But we almost never construct a differential manifold X by gluing it from balls. For a long time the usual way to construct a differential manifold X was to realize it at a subvariety of a simple manifold M [a sphere, a projective space e.t.c].

A substantial progress in topology in the last 20 years comes from a "simple observation" due to physicists on can realize a differential manifold X as quotient of an "infinite-dimensional submanifold" Y of a "simple" infinite-dimensional manifold M. For example Donaldson's works on the invariants of differential 4-manifolds are based on the consideration of the moduli space of self-dual connections which is the quotient of the "infinite-dimensional submanifold" self-dual connections by the gauge group.

This tension between an abstract definition and a concrete construction is addressed in both the Category theory and the Model theory. The Category theory is directed to a removal of the importance of a concrete construction. It provides a language to compare different concrete construction and in addition provides a very new way to construct objects as "representable functors" which allows to construct objects internally. This construction is based on the Yoneda's lemma which I consider to be most important result of the Category theory.

On the other hand, the Model theory is concentrated on gap between an abstract definition and a concrete construction. Let $\mathcal{T}$ be a complete model. On the first glance one should not distinguish between different models of $\mathcal{T}$, since all the results which are true in one model of $\mathcal{T}$ are true in any other model. One of main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty. Different models of complete theories are of different flavors and support different intuitions. So an attack on a problem often starts which a choice of an appropriate model. Such an approach lead to many non-trivial techniques for constructions of models which all are based on the *compactness theorem* which is almost the same as the fundamental existence theorem.

On the other hand the novelty creates difficulties for an outsider who is trying to reformulate the concepts in familiar terms and to ignore the differences between models.

In addition to these general consideration there are are concrete reasons to use Model theory for the "Motivic integration". What is an integration? Let $\mathcal{C}$ be the category of pairs (X, μ) where X is an oriented n-manifold, μ is a smooth absolutely integrable $\mathbb{R}$-valued measure on X. If (X', μ') is another such pair we write

$$(X, \mu) \sim (X', \mu')$$

if there exists disjoint open subsets $U_i \subset X, U'_i \subset X', 1 \leq r$ such that

a) for any $i, 1 \leq r$ there exists a diffeomorphism $f_i : U_i \to U'_i$ such that $f_i^*(\mu') = \mu_i$ and

b) the complements $X - \cup_i U_i, X' - \cup_i U'_i$ are contained in subvarieties of dimension $n - 1$.

Let $K(\mathcal{C})$ the quotient of the free abelian group generated by equivalence classes $[(X, \mu)]$ of pairs (X, μ) by the relation

$$[(X, \mu)] + [(X, \mu')] = [(X, \mu + \mu')]$$

The theory of integration says that the natural map

$$(X, \mu) \to \int_X \mu$$

defines an isomorphism $K(\mathcal{C}) \to \mathbb{R}$. In other words one can say that a construction of the theory of integration is equivalent to the computation of the group $K(\mathcal{C})$.

Let F be a valued field with a valuation $v : F^* \to \Gamma$. [For example we can take $F = \mathbb{C}((t))$ be the field of formal Laurent series over $\mathbb{C}, \Gamma = \mathbb{Z}$ and $v(f), f \in F^*$ to be the order of f at 0.] One consider the category $\mathcal{C}_F$ of *v-varieties* subsets X of F^n which are defined by a finite system of polynomial inequalities $v(P(x_1, ..., x_n)) > \gamma, \gamma \in \Gamma$. Let $K(\mathcal{C}_F)_n$ be the quotient of the free abelian group generated by isomorphism classes $[X]$ by the relation $[X] + [Y] = [X \cup Y]$ where $[X \cup Y]$ is the isomorphism class of the disjoint union of X and Y. One of the questions in the theory of "Motivic integration" is the computation of the group $K(\mathcal{C}_F)_n$.

Why is Model theory useful for the study of the group $K(\mathcal{C}_F)_n$?

It is very convenient reduce the study of the group $K(\mathcal{C}_F)_n$ to the case of curves when $n = 1$. For such a reduction one has to consider fibers of the restriction to $X \subset F^n$ of the natural projection $p : F^n \to F^{n-1}$ over different "points" of F^{n-1}.

In the familiar case of Algebraic geometry when the [when say $F = \mathbb{C}$ and the valuation is trivial] for an analysis of an n-dimensional algebraic variety X through it's projection $p : X \to Y$ to an $n-1$-dimensional Y it is important to consider fibers of p not only over $\mathbb{C}$-points of Y but also over points with values in extensions of $\mathbb{C}$ such as *the generic point* of Y.

In the case when the valuation is not trivial we have to consider fibers of p over an wider set of "points". And one needs the Model theory to define such points and to be able to talk about fibers of these points.

Now I'll start the second part of my introduction to the course and present the basic concepts of the Model theory. One of many problems one faces while learning this theory is the necessity to remember a number of definitions. I wrote a relatively short list of them and also a couple of problems to play with these definitions. If these concepts are unfamiliar then it takes an effort to remember the definitions and it is almost impossible to grasp them on the first attempt. Please put efforts into playing with the definitions.

But I'll start with an informal presentation of a model theoretic proof of a corollary of the Chevalley's theorem. The presentation is informal, since we did not yet discuss any concepts of the Model theory, and almost does not use the language of Logic. An attempt to avoid the language of Model theory makes the proof much less clear and less general. But on the other hand it is, I hope, easier to grasp and will provide a possibility to meet some Model theoretic concept on the "familiar territory".

Problem 1. Let k be a field and K an extension of k such that K is algebraically closed and the cardinality $\kappa(K)$ of K is strictly bigger then $\kappa(k)$. Show that there exist $u_i \in K, i \in \kappa(K)$ which are algebraically independent over k and such that that K is an algebraic closure of the field $k(u_i), i \in I$.

Such a set is called a *transcendence basis* of K over k.

Lemma 1. a) Let $K' \supset k$ be another algebraically closed field such that $\kappa(K') \geq \kappa(K)$. Then there exists a field homomorphism $\phi : K \to K'$ such that $\phi(a) = a, \forall a \in k$.

b) If $\kappa(K') = \kappa(K)$ then there exists a field isomorphism $\phi : K \to K'$ such that $\phi(a) = a, \forall a \in k$.

Proof. a) Choose a transcendence bases $u_i \in K, i \in \kappa(K)$ and $u'_i \in K', i \in \kappa(K')$ of K, K' over k. Since $\kappa(K') \geq \kappa(K)$ we can find an imbedding f of $\kappa(K)$ into $\kappa(K')$. Such an imbedding defines a field isomorphism $\phi_0 : k(u_i) \to k(u'_i)$ such that $\phi(u_i) = u'_{f(i)}, i \in \kappa K$. As follows from the essential uniqueness an algebraic closure we can extend ϕ_0 to a k-homomorphism $\phi : K \to K'$. The proof of b) is completely analogous.$\Box$

Definition 1. A subset $X \subset K^n$ is *k-constructible* if it is a finite Boolean combination of sets defined by a finite number of polynomial equalities $P_i(t_1, ..., t_n) = 0, 1 \leq i \leq M$ where $P_i, Q_j \in k[t_1, ..., t_n]$,

b) A subset $X \subset K^n$ is *almost k-constructible* if it can be obtained as a finite Boolean combination of images of constructible subsets Y of K^{m+n} under the natural projection $p : K^{m+n} \to K^n$.

c) For a pair $\bar{a}, \bar{b} \in K^n$ we say that $\bar{a} \sim^c \bar{b}$ iff for any k-constructible subset $X \in K^n, \bar{a} \in X$ iff $\bar{b} \in X$ and we say that $\bar{a} \sim \bar{b}$ iff for any almost k-constructible subset $X \in K^n, \bar{a} \in X$ iff $\bar{b} \in X$.

d) We denote by $S^c_n(k, K)$ the set of equivalence classes under $\sim^c$, by $S_n(k, K)$ the set of equivalence classes under $\sim$ and by $\pi^c : K^n \to S^c_n, \pi^d : K^n \to S_n$ and $\pi : S_n \to S^c_n$ the natural projections.

Actually the sets $S^c_n(k, K)$ and $S_n(k, K)$ do not depend on a choice of a field K.

Really, let $K' \supset k$ be another algebraically closed field such that $\kappa(K) \leq \kappa(K')$. As follows from the proof of Lemma 1 there exists a k- homomorphism $\phi : K \to K'$. It is clear that ϕ induces a bijection

$\tilde{\psi}$ between k–constructible subsets of K^n and K'^n and an imbedding $S_n^c(k, K) \to S_n^c(k, K')$.

Lemma 2.The map ϕ induces also an imbedding of $S_n(k, K)$ into $S_n(k, K')$.

bf Proof. It is sufficient to show that for any $\bar{a} \in K^n$ and a k-constructible set $Y \subset K^{m+n}$ we have $\bar{a} \in p(Y)$ iff $\phi(\bar{a}) \in p(\tilde{\psi}(Y))$. It is obvious that $\phi(\bar{a}) \in p(\tilde{\psi}(Y))(K')$ if $\bar{a} \in p(Y)$. Suppose now that $\phi(\bar{a}) \in p(\tilde{\psi}(Y))$. We want to show that $\bar{a} \in p(Y)(K)$.

Since $\phi(\bar{a}) \in p(\tilde{\psi}(Y))(K')$ there exist $b_{n+1}, ..., b_{n+m} \in K'$ such that $(\phi(\bar{a}), b_{n+1}, ..., b_{n+m}) \in \psi(Y)$. Let $K'' \subset K'$ be the algebraic closure of the field $k(\bar{a}, b_{n+1}, ..., b_{n+m})$. It is clear that $\kappa(K'') < \kappa(K)$. By lemma 1 there exists a $k(\bar{a})$-isomorphism $g : K'' \to K'$. Then $(\bar{a}, g(b_{n+1}), ..., g(b_{n+m})) \in Y\square$

Problem 2. Show that these imbeddings are bijections which don't depend on a choice of $\phi : K \to K'$.$\square$

Remark a) By the construction $\pi^c = \pi \circ \pi^d$ and for any $\bar{a} \in K^n, g \in Gal(K/k)$ we have $\bar{a} \sim g(\bar{a})$ [and therefore $\bar{a} \sim^c g(\bar{a})$] .

b) If $n = 1$ and $a, b \in K$ are such that $a \sim^c b$ then either both a, b are transcendent over k or there there exists an irreducible polynomial $p(t) \in k[t]$ such that both a, b are roots of $p(t)$. So the Galois theory implies that $a \sim^c b$ iff there exists $g \in Gal(K/k)$ such that $g(\bar{a}) = \bar{b}$.

c) I'll write S_n^c or S_n instead of $S_n^c(k, K)$ and $S_n(k, K)$.

Theorem 1 [a corollary to the Chevalley's theorem]. Any almost k-constructible subset of K^n is k-constructible.

We start with the following result.

Lemma 2_n. Let $K \supset k$ be a field extension such that K is an 'algebraically closed field. Assume that $\bar{a} = (a_1, ..., a_n), \bar{b} = (b_1, ..., b_n) \in K^n$ are such that $\bar{a} \sim^c \bar{b}$. Then there exists $g \in Gal(K/k)$ such that $g(\bar{a}) = \bar{b}$.

Proof. We prove the result by induction in n.

The case when $n = 1$ follows from the previous Remark. Assume now that the Lemma 2_{n-1} is known.

Let $\bar{a} = (a_1, ..., a_n), \bar{b} = (b_1, ..., b_n) \in K^n$ be such that $\bar{a} \sim^c \bar{b}$. It is clear that in such a case $(a_1, ..., a_{n-1}) \sim^c (b_1, ..., b_{n-1})$. Therefore there exists $g' \in Gal(K/k)$ such that $g'(a_1, ..., a_{n-1}) = (b_1, ..., b_{n-1})$. So by replacing $\bar{a} = (a_1, ..., a_n)$ by $\bar{a}' = (g'(a_1), ..., g'(a_n))$ we may assume that $a_i = b_i$ for $1 \leq i \leq n-1$.

But it is easy to see that for any $\bar{a} = (a_1, ..., a_n), \bar{b} = (a_1, ..., a_{n-1}, b_n) \in K^n$ such that $\bar{a} \sim_k^c \bar{b}$ we have $a_n \sim_l^c b_n$ where $l := k(a_1, ..., a_{n-1})$.

Therefore there exists $g'' \in Gal(K/l)$ such that $a_n = g''(b_n)$.□

Corollary. The map $\pi : S_n \to S_n^c$ is a bijection.

Proof. By the construction $\pi : S_n \to S_n^c$ is a surjection. So we have to prove only that it is also an injection.

In other words we have to show that for any $\bar{a} = (a_1, ..., a_n), \bar{b} = (b_1, ..., b_n) \in K$ such that

$\bar{a} \sim^c \bar{b}$ we have $\bar{a} \sim \bar{b}$. As follows from Corollary to Lemma 1, there exists $g \in Gal(K/k)$ such that $g(\bar{a}) = \bar{b}$. But it is clear that $g(\bar{a}) \sim \bar{a}$.□

Definition 2. a) To any almost k-constructible subset $Y \subset K^n$ we associate a subset $U_Y \subset S_n$ by $U_Y := \{t \in S_n | t \in Y\}$.

b) to any k-constructible subset $X \subset K^n$ we associate a subset $U_X := \{t \in S_n^c | t \in X(K)\}$.

c) We denote by $\mathcal{T}$ the topology on S_n such that the sets $\{U_Y\}, Y$ is an almost k-constructible subset of K^n , is a basis of open sets in $\mathcal{T}$ and by $\mathcal{T}^c$ the topology on S_n^c such that the sets $\{U_X\}, X$ is k-constructible subset of K^n ,is a basis of open sets in $\mathcal{T}^c$.

One of basic results in Model theory is the *Compactness theorem*.

When applied to the theory ACF of algebraically closed fields it guarantees [in view of the validity of Lemma 1 or Problem 1] the compactness of the topological space S_n.

Lemma 3. a) For any almost k-constructible subset of K^n the subset U_Y of S_n is both open and closed,

b) conversely if $U \subset S_n$ is both open and closed then there exists an almost k-constructible subset Y of K^n such that $U = U_Y$.

Proof. The part a) is obvious. To prove b) consider a subset $U \subset S_n$ which is both open and closed. Since U is open there exists a family $Y_i, i \in I$ of almost k-constructible subsets of K^n such that

$$U = \cup_{i \in I} U_{Y_i}$$

On the other hand since U is a closed subset of a compact it is compact. Therefore there exists finite subsets $R_I \subset I$ such that

$$U = \cup_{i \in R_I} U_{Y_i}$$

But this implies that $U = U_Y$ where $Y := \cup_{i \in R_I} Y_i$.□

Lemma 4. The map $\pi : S_n \to S_n^c$ is a homeomorphism of topological spaces.

Proof. By the definition the map $\pi : S_n \to S_n^c$ is separated and continuous. Since the topological space S_n is compact π maps closed sets to closed sets. Since [by Corollary to Lemma 1] $\pi : S_n \to S_n^c$ is a bijection we see that it is a homeomorphism of topological spaces. .$\square$

Now Theorem 1 follows immediately from Lemma 3.

The Chevalley's theorem tells us that in the case when the field k is algebraically closed and almost constructible subset of k^n is constructible. It is easy to deduce the Chevalley's theorem from Theorem 1 if one knows the Hilbert's Nullstellenzatz theorem

For any k algebraically closed field and any constructible subset $X \subset k^n$ such that $X(k) = \emptyset$ we have $X(K) = \emptyset$ for any extension $K \supset k$.

We come back to these questions after learning some basics of the Model theory.