LOGIC

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1. Introduction

In this course we will discuss applications of the Model theory to Algebraic geometry and Analysis. There is long list of examples and I mention only some of applications:

1) Tarski (see [T]) proved the elimination of quantifiers in the theory of real closed fields. The following statement used by Hormander in his works on differential equations is a corollary of the Tarski's result

For any polynomial $P(x) \in \mathbb{R}[x_1, ..., x_n]$ there are positive constants c, r such that

$$|P(x)| \geq c|x - Z(P)|^r, \forall x \in \mathbb{R}^n, |x| \leq 1$$

where $Z(P) \subset \mathbb{R}^n$ is the set of zeros of P and $|x| = \sqrt{x_1^2 + ... + x_n^2}$.

2) Ax (see [A]) used the Model theory for the proof of the following result:

Let X be a complex algebraic variety, $f : X \to X$ a regular map which is an imbedding. Then f is onto .

3) Ax and Kochen (see [AK]) have shown that for any $n \in \mathbb{N}$ there exists $s(n) \in \mathbb{N}$ such that for any prime number $p > s(n)$ any homogeneous polynomial equation

$P(x_0, ..., x_{n^2}) = 0$, where $P \in \mathbb{Q}_p[x_0, ..., x_{n^2}]$ is a polynomial of degree n, has a non-zero solution.

d) In works of Denef ([D1]), Loeser and Cluckers [CL]) the Model theory is used to obtain new results about p-adic integrals and their "motivic" generalizations.

In spite of it successes, the Model theory did not enter into a "tool box" of mathematicians and even many of mathematicians working on "Motivic integrations" are content to use the results logicians without understanding the details of the proofs.

I don't know any mathematician who did not start as a logician and for whom it was "easy and natural " to learn the Model theory. Often the experience of learning of the Model theory is similar to the one of learning of Physics: for a [short] while everything is so simple and so easily reformulated in familiar terms that "there is nothing to learn" but suddenly one find himself in a place when Model theoreticians "jump from a tussock to a hummock" while we mathematicians don't see where to "put a foot" and are at a complete loss.

So we have two questions:

a) Why is the Model theory so useful in different areas of Mathematics?

b) Why is it so difficult for mathematicians to learn it ?

But really these two questions are almost the same- it is difficult to learn the Model theory since it appeals to different intuition. But exactly this new outlook leads to the successes of the Model theory.

One difficultly facing one who is trying to learn Model theory is disappearance of the "natural" distinction between the formalism and the substance. For example the *fundamental existence theorem* says that the syntactic analysis of a theory [the existence or non- existence of a contradiction] is equivalent to the semantic analysis of a theory [the existence or non- existence of a model].

The other novelty is related to a very general phenomena. A mathematical object never comes in a pure form but always on a definite background. Finding a new way of constructions usually lead to substantial achievements.

For example, a differential manifold is "something" which is locally like a ball. But we almost never construct a differential manifold X by gluing it from balls. For a long time the usual way to construct a differential manifold X was to realize it at a subvariety of a simple manifold M [a sphere, a projective space e.t.c].

A substantial progress in topology in the last 20 years comes from a "simple observation" due to physicists one can realize a differential manifold X as quotient of an "infinite-dimensional submanifold" Y of a "simple" infinite-dimensional manifold M. For example Donaldson's works on the invariants of differential 4-manifolds are based on the consideration of the moduli space of self-dual connections which is the quotient of the "infinite-dimensional submanifold" self-dual connections by the gauge group.

This tension between an abstract definition and a concrete construction is addressed in both the Category theory and the Model theory. The Category theory is directed to a removal of the importance of a concrete construction. It provides a language to compare different concrete construction and in addition provides a very new way to construct objects as "representable functors" which allows to construct objects internally. This construction is based on the Yoneda's lemma which I consider to be most important result of the Category theory.

On the other hand, the Model theory is concentrated on gap between an abstract definition and a concrete construction. Let $\mathcal{T}$ be a complete theory. On the first glance one should not distinguish between different models of $\mathcal{T}$, since all the results which are true in one model of $\mathcal{T}$ are true in any other model. One of main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty. Different models of complete theories are of different flavors and support different intuitions. So an attack on a problem often starts which a choice of an appropriate model. Such an approach lead to many non-trivial techniques for constructions of models which all are based on the *compactness theorem* which is almost the same as the fundamental existence theorem.

On the other hand the novelty creates difficulties for an outsider who is trying to reformulate the concepts in familiar terms and to ignore the differences between models.

In addition to these general consideration there are concrete reasons to use Model theory for the "Motivic integration". What is an integration? Let $\mathcal{C}$ be the category of pairs (X, μ) where X is an oriented n-manifold, μ is a smooth absolutely integrable $\mathbb{R}$-valued measure on X. If (X', μ') is another such pair we write

$$(X, \mu) \sim (X', \mu')$$

if there exists disjoint open subsets $U_i \subset X, U_i' \subset X', 1 \leq r$ such that

a) for any $i, 1 \leq r$ there exists a diffeomorphism $f_i : U_i \to U_i'$ such that $f_i^*(\mu') = \mu_i$ and

b) the complements $X - \cup_i U_i, X' - \cup_i U_i'$ are contained in subvarieties of dimension $n - 1$.

Let $K(\mathcal{C})$ the quotient of the free abelian group generated by equivalence classes $[(X, \mu)]$ of pairs (X, μ) by the relation

$$[(X, \mu)] + [(X, \mu')] = [(X, \mu + \mu')]$$

The theory of integration says that the natural map

$$(X, \mu) \to \int_X \mu$$

defines an isomorphism $K(\mathcal{C}) \to \mathbb{R}$. In other words one can say that a construction of the theory of integration is equivalent to the computation of the group $K(\mathcal{C})$.

Let F be a valued field with a valuation $v : F^* \to \Gamma$. [For example we can take $F = \mathbb{C}((t))$ be the field of formal Laurent series over $\mathbb{C}, \Gamma = \mathbb{Z}$ and $v(f), f \in F^*$ to be the order of f at 0.] One consider the category $\mathcal{C}_F(n)$ of *v-varieties* subsets X of F^n which are defined by a finite system of polynomial inequalities $v(P(x_1, ..., x_n)) > \gamma, \gamma \in \Gamma$. Let $K(\mathcal{C}_F)(n)$ be the quotient of the free abelian group generated by isomorphism classes $[X]$ of objects $\mathcal{C}_F(n)$ of by the relation $[X] + [Y] = [X \cup Y]$ where $[X \cup Y]$ is the isomorphism class of the disjoint union of X and Y. One of the questions in the theory of "Motivic integration" is the computation of the group $K(\mathcal{C}_F)_n$.

Why is Model theory useful for the study of the group $K(\mathcal{C}_F)_n$?

It is very convenient reduce the study of the group $K(\mathcal{C}_F)_n$ to the case of curves when $n = 1$. For such a reduction one has to consider fibers of the restriction to $X \subset F^n$ of the natural projection $p : F^n \to F^{n-1}$ over different "points" of F^{n-1}.

In the familiar case of Algebraic geometry when one studies n-dimensional algebraic varieties through the projections p onto $n-1$-dimensional varieties, it is important to consider fibers of p not only over geometric points of the base Y but also over *the generic point* of Y.

In the case when the valuation is not trivial we have to consider fibers of p over an wider set of "points". And one needs the Model theory to define such points and to be able to talk about fibers of these points.

Now I'll start the second part of my introduction to the course and present the basic concepts of the Model theory. One of many problems one faces while learning this theory is the necessity to remember a number of definitions. I wrote a relatively short list of them and also a couple of problems to play with these definitions. If these concepts are unfamiliar then it takes an effort to remember the definitions and it is almost impossible to grasp them on the first attempt. Please put efforts into playing with the definitions.

2. Basic definitions

Similarity types of Structure .

Definition 2.1. D:structure a) A similarity type τ is a 5-tuple

$$< I, J, K, n, m >$$

where I, J, K are sets and $n : I \to \mathbb{N}, m : J \to \mathbb{N}$ are maps.

b) A structure M of similarity type τ consists of
(i) A non-empty set U called the universe of M,
(ii) A family of $n(i)$-relations $\{R_i^M \subset U^{n(i)}, i \in I\}$,
(iii) A family of $m(j)$-functions $\{f_j^M : U^{m(j)} \to U\}, j \in J$ and
(iv) A subset of distinguished elements $\{c_k^M \in U, k \in K\}$.

We will write $R(a_1, ..., a_n)$ instead of $< a_1, ..., a_n > \in R$.

c) We denote by $U_0(M)$ for the minimal substructure of M. So U_0 consists of constants, evaluations of functions f_j^M on the constants, evaluations of these functions on the previously constructed subset of M and so on. In particular U_0 is empty if $K = \emptyset$.

d) We will always assume that our types contain a 2-placed relation $=$ and will not mention this symbol explicitely when defining a type.

e) For the simplicity of the exposition we will assume that our structures are infinite.

f) We denote by $U^\star(M)$ the union $\cup_{n=1}^\infty U^n(M)$

Example 2.2. E:f Consider a similarity type τ when

$$I = \emptyset, J = \{+, -, \times\}, K = \{0, 1\}, m(+) = m(\times) = 2, m(-) = 1$$

Then a τ-structure has form

$$M =< U, +, -, \times, 0, 1 >$$

where U is a set, $+, \times$ are 2-place functions on U, $-$ is a 1-place function on U, and $0, 1$ are distinguished elements of U. The concept of field can be formulated in terms of M but U need not be a field since the the functions and distinguished elements of M need not satisfy the axioms of fields.

First order languages.

To any similarity type τ one associates a language L_τ.

Definition 2.3. D:language a) The *primitive symbols* of L_τ are
(i) variables $x_1, ..., x_n$
(ii) logical symbols $\wedge$ (and),$\neg$(negation), $\exists$
(iii) $n(i)$-place relation symbol $R_i, i \in I$
(iv) $m(j)$-place functional symbol $f_j, j \in J$
(v) constants $c_k, k \in K$.

b) The *terms* of L_τ are generated by two rules:
(i) all variables and constants are terms,
(ii) if f_j is an m-place functional symbol and $t_1, ..., t_m$ are terms then $f(t_1, ..., t_m)$ is a term.

c) The *atomic formulas* of L_τ are defined by the following inductive rules-
(i) if t_1, t_2 are terms then $t_1 = t_2$ is an atomic formula,
(ii) if R_i is an n-place relation symbol and $t_1, ..., t_m$ are terms then then $R(a_1, ..., a_n)$ is an atomic formula.

d) The *formulas* of L_τ are generated by three rules:
(i) every atomic formula is a formula,
(ii) if ψ, ϕ are formulas then $\neg\psi$ and $\psi \wedge \phi$ are formulas,
(iii) if ψ is a formula, x_s a variable then $\exists x_s \psi$ is a formula.

e) The notion of *free variables* in a formula is defined by the induction on the number of steps needed to generate the formula:
(i) if ψ is atomic and x_s occurs in ψ then x_i is a free variable,
(ii) if x_s is a free variable of ψ and $t \neq s$ then x_s is a free variable of $\exists x_t \psi$,
(iii) if x_s is a free variable of ψ then it is a free variable of $\neg\psi$ and $\psi \wedge \phi$.

In other words the only way to kill a free variable x_s of ψ is to prefix ψ by $\exists x_s$.

f) If t is a term such that all it's free variables are from $x_1, ..., x_r$ then for any $u_1, ..., u_r \in U$ the *value* $t^M[u_1, ..., u_r] \in U$ of a term t at $(u_1, ..., u_r)$ is defined as follows
(i) if $t^M = x_i$ then $t^M[u_1, ..., u_r] = u_i$,
(ii) if t is a constant symbol $c_k, k \in K$ then $t^M[u_1, ..., u_r] = c_k^M$,
(iii) if $t = f(t_1, ..., t_m)$ where f is an m-placed function symbol then $t^M[u_1, ..., u_r] = f^M(t_1^M[u_1, ..., u_r], ..., t_m^M[u_1, ..., u_r])$.

g) A *sentence* is a formula without free variables.

h) A formula is *open* if it contains no expression of the form $\exists x$.

i) The cardinality of L_τ is the cardinality of the set of all formulas of L_τ.

Remark 2.4. a) We will often use abbreviations
$\psi \vee \phi := \neg(\neg\psi \wedge \neg\phi)$ (or),$\psi \rightarrow \phi := \neg\psi \vee \phi$ (follows), and
$\forall x_i \psi := \neg\exists x_i \neg\psi$

b) if x is not a free variable in a formula ψ we will identify ψ with the formula obtained by the replacement of x by any other variable. In particular we can assume that names of non-free variables are different from names of free variables. $\square$.

Let M be a structure of the similarity type τ with the universe U and ϕ be a formula of L_τ with free variables $x_1, ..., x_r$. For a sequence $u_1, ..., u_r \in U$ we will now define the predicate

ϕ is satisfied by $(u_1, ..., u_r)$ in M

which we denote $M \models \phi(u_1, ..., u_r)$.

The definition this predicate is also by the induction on the length of a formula.

Definition 2.5. D:model (i) if ϕ is an atomic formula $t_1 = t_2$ where t_1, t_2 are terms then
$M \models \phi(u_1, ..., u_r)$ iff $t_1^M[u_1, ..., u_r] = t_2^M[u_1, ..., u_r]$,
(ii) if ϕ is an atomic formula $R(t_1, ..., t_n)$ where R is an n-place relation and $t_i(x_1, ..., x_r), 1 \leq i \leq n$ are terms then
$M \models \phi(u_1, ..., u_r)$ iff $R^M(t_1^M(u_1, ..., u_r), ..., t_n^M(u_1, ..., u_r))$.

(iii) if $\phi = \theta_1 \wedge \theta_2$ then
$M \models \phi(u_1, ..., u_r)$ iff both $M \models \theta_1(u_1, ..., u_r)$ and $M \models \theta_2(u_1, ..., u_r)$,
(iv) if $\phi = \neg\theta$ then
$M \models \phi(u_1, ..., u_r)$ iff θ is not satisfied by $u_1, ..., u_r$ in M,
(v) if ψ is a formula and x_i is a free variable of $\psi, \phi := \exists_{x_i}\psi$ then
$M \models \phi(u_1, ..., u_r)$ iff there exists $u \in U$ such that

$$M \models \psi(u_1, ..., u_{i-1}, u, u_{i+1}, ..., u_r)\square$$

Definition 2.6. D:model1 (a) If ψ is a formula with free variables $x_1, ..., x_r$ then for any τ- structure M we define

$$\psi(M) = \{(u_1, ..., u_r) \in U^r | M \models \psi(u_1, ..., u_r)\}$$

(b) if ψ is a formula, $x_1, ..., x_r$ be variables and $t_1, ..., t_r$ are terms of L we denote by $\psi_{x_1,...,x_r}[t_1, ...t_r]$ the formula obtained by the substitution of t_i for $x_i, 1 \leq i \leq r$.

(c) we say that two sentences ϕ, ϕ' of L_τ are *equivalent* if for any τstructure M we have $M \models \phi \leftrightarrow M \models \phi'$.

(d) we say that two structures M, M' of the same type τ are *elementary equivalent* iff for any sentence ψ of L_τ we have $M \models \psi$ iff $M' \models \psi$.$\square$

Remark 2.7. a) Since by our assumptions $=$ is a part of the relations of τ the atomic formula $\phi = [x_1 = x_2]$ is a part of L and $M \models \phi(u_1, u_2)$ iff $u_1 = u_2$.

b) We will not distinguish equivalent sentences.

c) all sentences $\psi \wedge \neg\psi$ where ψ is a sentence of L_τ are equivalent. We denote this equivalence class "false" of sentences by $\perp$.

Problem 2.8. *P:min The universe $U_0(M)$ of minimal substructure $U_0 \subset M$ consists of elements t^M where t runs through terms of L_τ without free variables.*

Theories and Models. If S is a set of sentences of a language L_τ we can deduce new sentences of L_τ using the *logical axioms*

Definition 2.9. D:la (i) the *propositional axioms*: $\psi \vee \neg\psi$ for all sentences ψ of L_τ,

(ii) the *substitutional axioms*: $\psi_x[t] \to \exists x\psi$ where t is any term and ψ is a formula of L_τ for which x is a free variable,

(iii) the *identity axiom* $[x = x]$

(iv) the *equality axioms*

$$x = y \to [t(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n) = t(x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n)]$$

and

$$x = y \to [\phi((x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)) \to \phi((x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n))]$$

Definition 2.10. D:theory Let S be a set of formulas of a language L.

a) We say that a formula ψ of L is a *logical consequence* of S and write $S \vdash \psi$ or $\vdash_S \psi$ if ψ is among the formulas generated from S by the following rules

(i) if $\psi \in S$ then $S \vdash \psi$,

(ii) if $S \vdash \psi$ and $S \vdash [\psi \to \phi]$ then $S \vdash \phi$

(iii) if x is not free in ψ and $S \vdash [\forall x\phi \to \psi]$ then $\vdash [\exists x\phi \to \psi]$.

(iv) if $S \vdash \psi_i, 1 \leq i \leq r$ and ψ is a result of applying some logical rules of inference to the sequence $\psi_1, ..., \psi_r$.

b) We say that S is *consistent* if $\perp$ is not a logical consequence of S.

c) A *theory* T is a consistent set of sentences of a language which contains all it's logical consequence.

d) If T consists of logical consequences of a set S of sentences we say that S is a *system of axioms* of T and write $T = T(S)$. If T is a theory in the language L and R a set of sentences of L we denote by $T(R)$ the theory which consists of logical consequences of $T \cup R$.

e) A *model* of a theory T is a structure M of the similarity type τ such that every sentence in T is true in M,

f) A theory T is *complete* if for every sentence ϕ of L, either ϕ or its negation $\neg\phi$ belongs to T.□

Definition 2.11. D:const Let T be a theory of a type $\tau =< I, J, K, n, m >$.

if $k_1, .., k_r$ are constants which did not appear in τ we denote by $T[k_1, ...k_r]$ the theory of the type $< I, J, K \cup k_1 \ldots \cup k_r, n, m >$ with T as a set of axioms.

Remark 2.12. Even if a theory T is complete the theory $T[k_1, ...k_r]$ does not have to be complete.

Problem 2.13. *P:const a) [The Theorem on Constants]. For any formula ψ of L_τ such that $x_1, \ldots, x_r$ are all the free variables of ψ.*

$$\vdash_T \psi \leftrightarrow \vdash_{T[k_1,..,k_r]} \psi_{x_1,\ldots,x_r}[k_1, ...k_r]$$

(see Definition 2.6 b)

b) If ψ, ϕ, ν are sentences such that $T[\psi] \vdash \phi$, then $T \vdash [\psi \to \phi]$.

c) If ψ, ϕ, ν are sentences such that $S \vdash [\psi \to \phi] \to \perp$ then $S \vdash \psi \to \neg\phi$.

d) A theory T is complete iff all the models of T are elementary equivalent.

Definition 2.14. D:sorts Sometimes we will consider similarity types which consists of a family of "sorts" parameterized by a set Λ. Let $\mathcal{S}(\Lambda)$ be the set of finite sequences of elements of Λ. For any $\bar\lambda = (\lambda_0, ..., \lambda_n) \in \mathcal{S}(\Lambda)$ we write $[\bar\lambda] = \lambda_0, \bar\lambda' = (\lambda_1, ..., \lambda_n)$

a) a Λ-similarity type is a 5-tuple

$$< I, J, K, n, m >$$

where $I, J, K = \cup K_\lambda, \lambda \in \Lambda$ are sets and
$n : I \to \mathcal{S}(\Lambda), m : J \to \mathcal{S}(\Lambda)$ are maps.

b) A structure M of similarity type τ consists of
(i) A non-empty sets $U^\lambda, \lambda \in \Lambda$ called the universes of M,
For any $\bar\lambda = (\lambda_1, ..., \lambda_n) \in \mathcal{S}(\Lambda)$ we write $U^{\bar\lambda} = U^{\lambda_1} \times ... \times U^{\lambda_n}$.

(ii) A family of relations $\{R_i^M \subset U^{n(i)}, i \in I\}$,
(iii) A family of functions $\{f_j^M : U^{m(j)'} \to U_{[m(j)]}\}, j \in J$ and
(iv) A subset of distinguished elements $\{c_k^M \in U_\lambda, k \in K_\lambda\}$.□

Examples 2.15. E:basic

a) **Integral domains (ID)** Let τ be the type as in 2.2. The theory ID consists of following sentences

(1) $\forall x, y, z \; [(x + y) + z = x + (y + z)]$
(2) $\forall x \; [x + 0 = x]$
(3) $\forall x \; [x + (-x) = 0]$
(4) $\forall x, y \; [x + y = y + x]$
(5) $\forall x, y, z \; [(x \times y) \times z = x \times (y \times z)$
(6) $\forall x \; [x \times 1 = x]$
(7) $\forall x, y[x, y \neq 0 \to x \times y \neq 0]$
(8) $\forall x, y \; [x \times y = y \times x]$
(9) $\forall x, y, z \; [(x \times (y + z) = x \times y + x \times z]$
(10) $0 \neq 1$

Models of ID are integral domains. For any such a model M the minimal submodel $U_0(M)$ is the image of the natural homomorphism $\mathbb{Z} \to M$

b) **F fields (TF)** The theory TF is the theory ID augmented by the sentence

$\forall x \neq 0 \exists y[xy = 1]$

Models of TF are fields and that substructures of such models are subrings containing the unit element.

c) **Algebraically closed fields (ACF)**.

The theory ACF is the theory TF augmented by following list of sentences:

$$\{\star_n\} \forall y_1, ..., y_n \exists x | x^n + y_1 x^{n-1} + ... + y_n = 0$$

Models of ACF are algebraically closed fields.

d) **Linear ordering (LO)**. Consider a similarity type $\tau =< I, J, K, n, m >$ where $I = K = \emptyset, J = \star$ and $m(\star) = 2$. Then a τ-structure has a form

$$M =< X, \leq>$$

where X is a set and $<$ is a 2-place relation on X. We define LO as the theory generated by the following three axioms

i) $\forall x, y[x \leq y \vee y \leq x]$,

ii) $\forall x, y, z[x \leq y \wedge y \leq z \to x \leq z]$,

iii) $\forall x, y[x \leq y \wedge y \leq x \to x = y]$.

e) **Abelian groups (AG)**. Consider a structure

$$M =< \Gamma, +, -, 0 >$$

where $+$ is a 2-place function on Γ, $-$ is a 1-place function on Γ, and 0 is a distinguished element of Γ.

Write a set of axioms such that models of the corresponding theory AG are Abelian groups.

f) The theory OAG of **Ordered Abelian groups** is is the theory AG augmented by a linear ordering $\leq$ and axiom

$\forall \alpha, \beta, \gamma \in \Gamma, [\alpha < \beta \to \alpha + \gamma < \beta + \gamma]$
Models of OAG are ordered Abelian groups.

We denote by OAG^+ the theory such that models of OAG^+ have a form $\Gamma \cup \infty$ where Γ is an ordered Abelian group, $x < \infty$ for any $x \in \Gamma$ and $x + \infty = \infty$ for any x

g) **ordered Abelian divisible groups (OADG)**. The theory $OADG$ is the theory OAG augmented by the axioms

$$\star_n = [\forall x \exists y : x = ny]$$

for all $n \in \mathbb{N}$. Models of $OADG$ are ordered abelian divisible groups.

h) **Valued integral domains (VID)**.

In this case the similarity type $\tau =< I, J, K, n, m >$ has two sorts F and Γ. More precisely

$\Lambda = \{F, \Gamma\}, I = \{<\}, J = \{+, -, \times, \bar{+}, \bar{-}, <, v\} K = K_F \cup K_\Gamma, K_F = \{0, 1\}, K_\Gamma = \{\bar{0}, \infty\}$

and

$n(<) = (\Gamma, \Gamma), m(+) = m(\times) = (F, F, F), m(-) = (F, F), m(\bar{+}) = (\Gamma, \Gamma, \Gamma), m(\bar{-}) = (\Gamma, \Gamma), m(v$

So a τ-structure M has a form

$$M = (F \cup \Gamma, +, -, \times, 0, 1, \bar{+}, \bar{-}, \bar{0}, <, v)$$

where $(F, +, -, \times, 0, 1)$ is a structure as in $TF, (\Gamma, \bar{+}, \bar{-}, \bar{0}, < \infty)$ is a structure of OAG^+ and v is a 1-place function from F to $\Gamma \cup \infty$.

We consider the theory which consists of sentences which say that $(F, +, -, \times, 0, 1)$ is the theory (ID), $(\Gamma, \bar{+}, \bar{-}, \bar{0}, >)$ is the theory (OAG) augmented by axioms

$\forall x, y \in F \ [v(x \times y) = v(x) \bar{+} v(y)]$

$v(x + y) \geq min(v(x), v(y)), \forall x, y \in F^*$

$v(x) = \infty \leftrightarrow x = 0$

We call Γ the valuation group of a valued fields F. When there is no danger of mixing elements of Γ with the ones of F we will write $+, -, 0$ instead of $\bar{+}, \bar{-}, \bar{0}$.

i) **Valued fields (VF)**

The theory (VF) is the theory (VID) augmented by axioms of TF for V

j) **Algebraically closed valued fields (AVF)**

The theory (AVF) is the theory (VID) augmented by axioms of ACF for V

and the axioms

$\forall \gamma \in \Gamma \exists x \in F[v(x) = \gamma]$ and

$$\exists \gamma \neq 0$$

Models of AVF are algebraically closed valued fields.

k) Show that for any algebraically closed valued field the that the group Γ is uniquely divisible.

l) The theory OF of **ordered fields (OF)** is the theory (TF) augmented by a 2-place symbol $<$ and the following axioms

(1) $\forall x\ [\neg(x < x)]$

(2) $\forall x, y, z\ [x < y \wedge y < z \rightarrow x < z]$

(3) $\forall x, y\ [x < y \vee x = y \vee y < x]$

(4) $\forall x, y\ [0 < x \wedge 0 < y \rightarrow 0 < x \times y]$

(5) $\forall x, y, z\ [x < y \rightarrow x + z < x + y]$

m) The theory of **real closed fields (RCF)** is the theory OF augmented by axioms
(6) $\forall x\ [0 < x, \exists y | x = y^2]$
$(\star_n)$ for all odd $n \in \mathbb{N}$.□

Problem 2.16. *Find a model of (RCF) different from $\mathbb{R}$*

Definition 2.17. D:mono Let τ be a similarity type and M a structure of type τ.

a) A *monomorphism* ρ of M into another structure M' of type τ is a map $\rho : U(M) \rightarrow U(M')$ such that

(i) $R_i^M(u_1, ..., u_n) \leftrightarrow R_i^{M'}\rho((u_1), ..., \rho(u_n)), \forall i \in I$,

(ii) $\rho(f_j^M(u_1, ..., u_m)) = f_j^{M'}(\rho(u_1), ..., \rho(u_m)), \forall j \in J$,

(iii) $\rho(c_k^M) = c_k^{M'}, \forall k \in K$.

b) A monomorphism $\rho : M \to M'$ is *elementary* if for any formula $\psi(x_1, ..., x_r)$ and any $(a_1, ..., a_r) \in M$ we have

$$M \models \psi_{x_1,...,x_r}[a_1, ..., a_r] \leftrightarrow M' \models \psi_{x_1,...,x_r}[a_1, ..., a_r]$$

In this case we say that M' is an *elementary extension* of M and write $M \prec M'$.

Remark 2.18. R:subst Since = [and therefore $\neq$] are part of the language any monomorphism $M \to M'$ comes from an imbedding $\rho : U(M) \to U(M')$

Definition 2.19. D:direct a) A *directed set* is a partially ordered set $(I, \leq)$ such that for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

b) A *directed system of sets* parametrized by a directed set $(I, \leq)$ is a family $U_i, i \in I$ of sets and maps $m_{ij} : U_i \to U_j, i, j \in I$ such that

i) $m_{ii} = Id$ for all $i \in I$ and

ii) $m_{ik} = m_{jk} \circ m_{ij}$ for all $i \leq j \leq k \in I$.

c) If $\{U_i, m_{ij} : U_i \to U_j\}, i, j \in I$ is directed system of sets we define the *direct limit* $U = \varinjlim U_i$ as the quotient of $\cup_{i \in I} U_i \times \{i\}$ be the equivalence relation

$$(u', i') \sim (u'', i'') \leftrightarrow \exists i \in I, i \geq i', i \geq i'', m_{i'i}(u') = m_{i''i}(u'')$$

and denote by $\rho_i : U_i \to U$ the natural imbeddings.

d) A *directed system of structures* parametrized by a directed set $(I, \leq)$ is a family $M_i, i \in I$ of τ-structures and monomorphisms $\rho_{ij} : M_i \to M_j, i, j \in I$ such that the imbeddings $\rho_{ij} : M_i \to M_j, i, j \in I$ constitute a directed system of sets.

e) If $M_i, \rho_{ij} : M_i \to M_j, i, j \in I$ is directed system of structures we define a structure $M = \varinjlim M_i$ as a structure with the universe $U = \varinjlim U(M_i)$ and such that

i) a relation

$$R^M(m_{i_1}(u_1), \ldots, \rho_{i_n}(u_n), u_{i_r}) \in U_{i_r}, 1 \leq r \leq n$$

holds iff for some $k, k \geq i_r$ the relation

$$R^M(\rho_{i_1,k}(u_1, k), \ldots, \rho_{i_n,k}(u_n))$$

holds .

ii) an equality $f(\rho_{i_1}(u_1), \ldots, \rho_{i_n}(u_n)) = \rho_i(u)$ holds iff for some $k, k \geq i_r, i$ the relation $f(\rho_{i_1,k}(u_1), \ldots, \rho_{i_n,k}(u_n)) = \rho_{i,k}(u)$ holds .

iii) $c^M = \rho_i(c^{M_i})$ for any $i \in I$.

f) A directed system of structures is *elementary* if all monomorphisms m_{ij} are elementary.

Problem 2.20. *P:direct Show that for any elementary directed system M_i, ρ_{ij} of structures then the monomorphsims $\rho_i : M_i \to M$ are elementary.*

Definition 2.21. D:expand Let A be a subset the universe U of a structure M of a similarity type $\tau =< I, J, K, n, m >$. We define

a) τ_A to be the similarity type $< I, J, K \cup A, n, m >$.

b) M_A to be the τ_A-structure obtained from M in such a way that $c_{k_a}^{M_A} = a, a \in A$.

c) the *diagram* $D(A)$ as the set of all atomic sentences and negations of such sentences in the language $L_A := L_{\tau_A}$ which are true in M_A.

d) the theory T_A the [in the language L_A] as the extension of T by $D(A)$ (see Definition 2.10 d).

e) we denote by $T(M, A)$ the theory in the language L_A consisting of the set of all the formulas in the language L_A which are true in M_A and will write $T(M)$ instead of $T(M, U(M))$.

Example 2.22. Let k be an algebraically closed field. Then k is a structure of ACF and models of ACF_k are algebraically closed extensions $K \supset k$ where K.

Remark 2.23. a) Even if a theory T is complete the theory T_A does not have to be complete.

b) From now on we will always assume that our theories have a form $T = T'(M, A)$ where the language of T' is countable and A is a subset of a universe of some model M of T'.

Problem 2.24. *P:complete Show that*

a) For any subset A of U the languages L_A and $L_{<A>}$ are equivalent. where $< A > \subset U$ is the minimal substructure containing A.

b) if A is a substructure of M then models of the theory T_A consist of triples (M', A', r) where M' is a model of T and $A' \leq M'$ is a substructure and $r : A \to A'$ is an isomorphism of structures.

c) for any structure M the theory $T(M)$ is complete.

d) Let ψ be a formula in L_M such that $x_1, ..., x_r$ are all the free variables of ψ. Then $u_1, ..., u_r \in \psi(M)$ iff the sentence $\psi[u_1, ..., u_r]$ in L_M is true.

e) let M be a structure, t a variable-free term of L_M and ψ a formula of L_M in which no variables except x occurs. Then
$\psi_x[t](M) = \psi_x[t(M)](M)$.

f) models of $T(M)$ are elementary extensions of M.

3. The basics of the set theory

Definition 3.1. D:well a) A *well-ordered* set is a linearly ordered set $S, <$ such that any non-empty subset $S' \subset S$ has a minimal element.

b) If $S, <$ is a well-ordered set an *initial segment* of S is a subset $S' \subset S$ such that for any any $x \in S'$ and any $y < x$ we have $y \in S'$.

c) An *ordinal* is an isomorphism class of a well-ordered set.

d) Given tow ordinals α, β we say that $\alpha < \beta$ if α is isomorphic to an initial segment of β.

Remark 3.2. a) It is clear that any initial segment of a well-ordered set is a well-ordered set.

b) We denote ordinals by Greek letters from the beginning of the alphabet.

The following result is well known.

Lemma 3.3. *L:well a) A well-ordered set does not have non-trivial automorphisms.*

b) If R, S are well-ordered sets such that R is isomorphic to an initial segment S' of S then $S' \subset S$ is uniquely defined.

c) If R, S are well-ordered sets then either there are isomorphic, or R is isomorphic to an initial segment of S or S is isomorphic to an initial segment of R.

d) If $S' \subset S$ is a proper initial segment of a well-ordered set S then there exists $x \in S$ such that $S' = \{y \in S | y < x\}$.

e) If α is an ordinal with the minimal element 0 and $F_\beta, \beta \leq \alpha$ is a statement such that ,

i) F_0 is true,

ii) if F_β is true then $F_{\beta+1}$ is true,

iii) for any limit ordinal $\gamma \leq \alpha$ such that F_β is true for all $\beta < \gamma$ the statement F_β is true,

then F_α is also true.

Remark 3.4. a) As follows from Lemma 3.3 the *class* of ordinals is well-ordered. If we don't want to talk about classes we can assume that all the ordinals are initial segments of a fixed very big ordinal Λ.

b) The last claim of Lemma 3.3 is the principle of the *transfinite induction.*

Definition 3.5. D:well1 a) For any ordinal α we define the *successor* $\alpha+1$ of α as the ordinal obtained from α by an addition of an element bigger then any element of α.

b) An ordinal is a *limit ordinal* if it is not a successor of another ordinal,

c) An ordinal α is *even* if it has a form $\alpha = \beta + n$ where β is a limit ordinal and n is even. An ordinal α is *odd* if it has a form $\alpha = \beta + n$ where β is a limit ordinal and n is odd.

d) A *cardinal* is an ordinal which can not be put into one-to-one correspondence with a lesser ordinal.

h) If κ is a cardinal we denote by κ^+ the least cardinal greater then κ and by 2^κ the cardinality of the set of all subsets of κ.

Remark 3.6. a) The class of infinite cardinals is well-ordered. So we can number all the infinite cardinals by ordinals $\alpha \to \omega_\alpha$ in such a way that ω_0 is the countable cardinal.

b) We will use the generalized continuum hypothesis [GCH] which says that for any cardinal κ we have $\kappa^+ = 2^\kappa$.

c) For a wide class of statements we know that their truth is independent of the validity of GCH. To be precise consider two kinds of mathematical objects: finite objects x such as integers and objects X such as sets of integers, real numbers, continuous function on $\mathbb{R}$ which can be specified using countably many bits.

Next, classify mathematical statements regarding such objects according to their logical complexity. A Δ_0-sentence is one that can be verified "by a computer". Π^0_1 sentence has the form "for all $n, \dots$" where $\dots$ is Δ_0¿ It is clear that the usual statement of Fermat's last theorem is Π^0_1. The usual statement of Euclid's theorem on the infinitude of primes is Π^0_2, i.e. requires two such quantifiers: $(\forall m)(\exists p)(p > m\&pisprime)$. More generally, an *arithmetic* sentence allows any finite number of quantifiers on integers.

A Π^1_1 sentence has the form "for all X, ...", where X is essentially countable, and ... is arithmetic. More generally, Π^1_n sentence allows n such quantifiers.

Most statements of number theory and geometry can be stated as Π^1_1 sentences, and often with a little effort, as arithmetic sentences. For example the usual formulation of Riemann's hypothesis is of Π^1_1 form but one can find a Π^0_1 formulation of Riemann's hypothesis. formulation. See, for example,[M].

Shoenfield and Levy have shown that that the truth of a Π^1_2 sentence cannot depend on questions such as the axiom of choice, or the continuum hypothesis. Moreover given a proof of a Π^1_2 sentence which is based on the axiom of choice or on the continuum hypothesis there is an algorphm how to write a proof which does not relay on the axiom of choice or on the continuum hypothesis. But, of course, the new proof could be much longer.

Definition 3.7. D:regular An infinite cardinal κ is *regular* for any subset X of κ such that $\kappa(X) < \kappa$ there exists $y \in \kappa$ such that $x < y$ for all $x \in X$.

Lemma 3.8. *L:regular For any infinite cardinal κ the cardinal κ^+ is regular.*

Proof. We can write $X = \{x_\delta\}, \delta \in \Delta, \kappa(\Delta) \leq \kappa$. For any $\delta \in \Delta$ we define $Z_\delta = \{\gamma | \gamma \leq x_\delta\}$. Since $x_\delta < \kappa^+$ and κ^+ is a cardinal we see that $\kappa(Z_\delta) < \kappa^+$. So $\kappa(Z_\delta) \leq \kappa$ and therefore $\kappa(\cup_{\delta \in \Delta}) \leq \kappa^2$. Since κ is infinite $\kappa^2 = \kappa$. On the other hand, the union Z of intervals $Z_\delta, \delta \in \Delta$ is an interval $\{\gamma | \gamma \leq x$ for some $z \leq \kappa^+$. Since $\kappa(Z) \leq \kappa < \kappa^+$ we see that $z < \kappa^+$.□

4. The fundamental existence theorem

The following result which is called the fundamental existence theorem is a backbone of the Model theory.

Theorem 4.1. *T:fund Let τ be a similarity type, and S be a consistent set of sentences in the language L_τ. Then there exists a model M of the theory $T(S)$ with the universe U of cardinality $\kappa(U) \leq max(\aleph_0, \kappa(S))$.*

Before proving this result we show how to use it.

Corollary 4.2. *C:comp [The compactness theorem]*

If S is a set of sentences such that every finite subset S' of S has a model then S has a model.

Proof. I claim that S is consistent. Really, if S were inconsistent we could derive $\perp$ from. But any deduction uses only a finite number $S' \subset S$ of axioms. Since, by the assumption, S' has a model such a deduction is not possible.

So S is consistent and the existence of a model of S follows from the fundamental existence theorem.

Corollary 4.3. *C:upw [The upward Skolem-Lowenheim theorem]. Let M be an infinite structure of type τ. Then for any $\kappa \geq max(\kappa(M), \kappa(L_\tau))$ there exists an elementary extension M' of M of cardinality κ.*

Proof. Let R be a set of cardinality $\kappa, \nu :=< I, J, K \cup R, n, m >$ and $S = T(M) \bigcup \cup_{r \neq r' \in R}[r \neq r']$ where $T(M)$ is the complete theory of M. I claim that any finite subset S' of S has a model. Indead for any finite subset S' of S only a finite set $r_1, ..., r_N$ of distinguished elements from R appear in S'. So

$$S' \subset T(M) \cup \cup_{1 \leq i < j \leq N}[r_i \neq r_j]$$

Since the model M is infinite we can find N distinct elements $u_1, ..., u_N \in U(M)$. But then an extension of M to an $L[r_1, ..., r_N]$-structure by $r_l \to u_i, 1 \leq l \leq N$ is a model of S'.

Now it follows from the compactness theorem that there exists a model N of S with the universe U' of cardinality $\leq \kappa$. On the other hand axioms $r \neq r' \in R[r \neq r']$ imply that $\kappa(U') \geq \kappa$. So $\kappa(U') = \kappa$.

Since $S \supset T(M)$ the model M' is an elementary extension of M.$\square$ (see Problem 2.24 f)

Our proof of the fundamental existence theorem is based of the notion of the *Henkin* set of sentences.

Definition 4.4. D:Henkin Let S be a consistent set of sentences in the language L_τ. We say that the set S is *Henkin* if for any formula $\psi \in L_\tau$ containing a free variable x there exists a constant $k_\psi \in K$ such that

$$\vdash_S \exists x \psi \to \psi_x[k_\psi]$$

Lemma 4.5. *L:fuH If S is a Henkin set of sentences in the language L_τ then there exists a model M of $T(S)$ of cardinality not bigger then $max(\aleph_0, \kappa(S))$.*

Proof of Lemma 4.5. Let $\mathcal{T}$ be the set of consistent theories in the language L_τ containing S. As follows from the Zorn's lemma the set $\mathcal{T}$ contains a maximal element T. We fix T and define an equivalence relation $\sim$ on K by

$$k \sim k' \leftrightarrow \vdash_T k = k'$$

and denote by U the set of equivalence classes of $\sim$. For any $k \in K$ we denote by $[k] \in U$ the equivalence class of $\sim$ containing k.

The following statement is not difficult. I'll leave it as an exercise.

Sublemma 1.

a) T is a complete theory.

b) For any $i \in I$ and any $(k_1, ...k_{n(i)}), (k'_1, ...k'_{n(i)}) \in K$ such that $k_l \sim k'_l, 1 \leq l \leq n(i)$ we have

$$\vdash_T R_i(k_1, ...k_{n(i)}) \leftrightarrow R_i(k'_1, ...k'_{n(i)})$$

c) For any $j \in J$ and any $(k_1, ...k_{m(j)}) \in K$ there exists $k \in K$ such that $f_j(k_1, ...k_{m(j)}) = k$.

d) for any $j \in J$ and any $(k_1, ...k_{m(j)}), (k'_1, ...k'_{m(j)}) \in K$ such that $k_l \sim k'_l, 1 \leq l \leq m(j)$ we have

$$\vdash_T f_j(k_1, ...k_{m(j)}) \leftrightarrow f_j(k'_1, ...k'_{m(j)})\square$$

e) For any $i \in I, ([k_1], ...[k_{n(i)}]) \in U^{n(i)}$ the validity of $\vdash_T R_i(k_1, ...k_{n(i)})$ does not depend on a choice of $k_r \in [k_r], 1 \leq r \leq n(i)$,

f) $j \in J, ([k_1], ...[k_{m(j)}], [k]) \in U^{m(j)}, [f_j(k_1, ...k_{m(j)})] \in U$ does not depend on a choice of $k_r \in [k_r], 1 \leq r \leq n(i)$.$\square$

Remark 4.6. Only the proof of c) using the assumption that S is Henkin.

Consider now the τ-structure $M = (U, R_i^M, f_j^M, c_k^M)$ where
(i) U is the set of equivalence classes of elements in K,
(ii) $R_i^M([k_1], ..., [k_{n(i)}])$ iff $\vdash_T R_i(k_1, ...k_{n(i)}), i \in I$
(iii) $f_j^M(([k_1], ..., [k_{m(j)}]) = [k]$ iff $\vdash_T f_j(k_1, ...k_(m(j)) = k, j \in J$
(iv) $c_k^M = [k]$.

Sublemma 2. M is a model of T and for any sentence $\psi \in L_\tau$ we have $M \models \psi$ iff $\psi \in T$.

The proof of Sublemma 2 is by the induction in the number of steps needed to generate a sentence $\psi \in L_\tau$.

Lemma 4.5 follows immediately from Sublemma 2.$\square$

Lemma 4.7. *L:exH If S is a consistent set of sentences in the language L_τ then there exists an extension τ' of τ and a Henkin set S' of sentences in $L_{\tau'}$ containing S such that $\kappa(S') \leq max(\aleph_0, \kappa(S))$.*

Proof of Lemma 4.7 . Let $\kappa = max(\aleph_0, \kappa(S))$. We define

$$K' := K \bigcup \cup_{\delta<\kappa} k_\delta$$

where k_δ is a set of distinct elements, non of which occurs in S and define

$$\tau' =< I, J, K', n, m >$$

Let $\psi_\delta(x), \delta < \kappa$ be the list of all formulas in $L_{\tau'}$ whose only free variable is x. By the transfinite induction we can choose a function $h : \kappa \to \kappa$ such that

(i) $\gamma < \delta$ implies $h(\gamma) < h(\delta)$,

(ii) $\gamma \leq \delta$ implies $k_{h(\delta)}$ does not occur in $\psi_\gamma(x)$.

Using the function h we define for any $\delta < \kappa$ a set S_δ of sentences in $L_{\tau'}$ by

$$S_\delta = S \bigcup \cup_{\gamma<\delta}\{\exists x\psi_\gamma(x) \to \psi_\gamma(k_{h(\gamma)})\}$$

Remark 4.8. The constant $k_{h(\delta)}$ does not occur in S_δ.

Claim. The theory S_δ is consistent for all $\delta \leq \kappa$.

We prove the Claim by the transfinite induction in δ.

By the assumption $S_0 = S$ is consistent. If λ is a limit ordinal and S_δ is consistent for all $\delta < \lambda$, then the consistency

$$S_\lambda = \cup_{\delta<\lambda} S_\delta$$

is clear since any sentence of S_λ belongs to S_δ for some $\delta < \lambda$.

To show that the consistency of S_δ implies the consistency of $S_{\delta+1}$ is equivalent showing that the inconsistency of $S_{\delta+1}$ implies the inconsistency of S_δ.

Assume that $S_{\delta+1}$ is inconsistent. Then there exists a sentence ϕ in $L_{\tau'}$ such that $S_{\delta+1} \vdash \phi \wedge \neg\phi$. So

$$S_\delta \vdash [\exists x\psi_\delta(x) \to \psi_\delta(k_{h(\delta)})] \to [\phi \wedge \neg\phi]$$

and therefore [see Problem 2.13]

$$S_\delta \vdash \exists x\psi_\delta(x) \wedge \neg\psi_\delta(k_{h(\gamma)})$$

Since $k_{h(\gamma)}$ does not occur in S_δ we have [see Problem 2.13]

$$S_\delta \vdash \exists x\psi_\delta(x) \wedge \forall y \neg\psi_\delta(y)$$

So an inconsistency of $S_{\delta+1}$ implies an inconsistency of S_δ.

We see that S_δ is consistent for all $\delta < \kappa$ and therefore $S_\kappa := \cup_{\delta<\kappa} S_\delta$ is also consistent. By the construction we see that S_κ is a Henkin set.□

It is clear that Lemma 4.5 and Lemma 4.7 imply the validity of the fundamental existence theorem.□

5. Applications of the fundamental existence theorem to the algebraic geometry

We start with the following well known result.

Lemma 5.1. *Let k be a field and K an extension of k such that K is algebraically closed and the cardinality $\kappa(K)$ of K is strictly bigger then $\kappa(k)$. Then there exist a transcendence basis of K over k. That is there exist elements $u_i i \in \kappa(K)$ of K which are algebraically independent over k K is an algebraic closure of the field $k(u_i), i \in I$.*

Lemma 5.2. *L:acf a) Let $K' \supset k$ be another algebraically closed field such that $\kappa(K') \geq \kappa(K)$. Then there exists a field homomorphism $\phi : K \to K'$ such that $\phi(a) = a, \forall a \in k$.*

b) If $\kappa(K') = \kappa(K)$ then there exists a field isomorphism $\phi : K \to K'$ such that $\phi(a) = a, \forall a \in k$.

Proof. a) Choose a transcendence bases $u_i \in K, i \in \kappa(K)$ and $u'_i \in K', i \in \kappa(K')$ of K, K' over k. Since $\kappa(K') \geq \kappa(K)$ we can find an imbedding f of $\kappa(K)$ into $\kappa(K')$. Such an imbedding defines a field isomorphism $\phi_0 : k(u_i) \to k(u'_i)$ such that $\phi(u_i) = u'_{f(i)}, i \in \kappa K$. As follows from the essential uniqueness an algebraic closure we can extend ϕ_0 to a k-homomorphism $\phi : K \to K'$. The proof of b) is completely analogous.□

Definition 5.3. D:categorical Let T be a theory and κ an ordinal. We say that T is *κ-categorical* if any two models of T of cardinality κ are isomorphic.

Example 5.4. As follows from Lemma 5.2, for any algebraically closed field k the theory ACF_k is κ-categorical for any $\kappa > \kappa(k)$.

Lemma 5.5. *L:categorical If T is a theory which is κ-categorical for all sufficiently big κ then any two models M, M' of T are elementary equivalent.*

Proof. Choose a cardinality κ such that T is κ-categorical and κ is bigger then $max(\kappa(M), (\kappa(M'))$. As follows from the upward Skolem-Lowenheim theorem, there exists elementary extensions

$E \supset M, E' \supset M'$ of T such that $\kappa(E) = \kappa(E') = \kappa$. Since the theory T is κ-categorical the models E, E' of T are elementary equivalent. Therefore the models M, M' of k are also elementary equivalent. □

Corollary 5.6. *C:elacf For any algebraically closed field k any two models of the theory ACF_k are elementary equivalent.*

Corollary 5.7. *C:acfcom The ACF_p of algebraically closed fields of characteristic p is complete for any characteristic p.*

Proof. As follows from the Problem 2.24 it is sufficient to show that any two models F, F' of ACF_p are elementary equivalent. But the elementary equivalence of F, F' follows from Lemma 5.5 applied to the case when k is the prime field $[k = \bar{\mathbb{F}}_p$ or $k = \mathbb{Q}]$.□

Corollary 5.8. *C:nu [Nullstellenzatz]. Let $k \subset K$ be algebraically closed fields and Y a constructible "subset" of the affine space $\mathbb{A}^n$ such that $Y(K) \neq \emptyset$. Then $Y(k) \neq \emptyset$.*

Proof. By the definition of a constructible "subset" there exists an open formula ψ in L_k such that for any algebraically closed extension K of k we have

$$Y(K) = \psi(K)$$

(on the left K is a field and on the right it is a model of ACF_k).

But by Lemma 5.5, the models $k \subset k$ and $k \subset K$ of ACF_k are elementary equivalent. So $\psi(K) \neq \emptyset$ implies that $\psi(k) \neq \emptyset$.□

Lemma 5.9. *L:fchar Let D be a definable "set" in TF such that such that $D(K) \neq \emptyset$ for some field K of characteristic zero. Then there exists $r \in \mathbb{N}$ such that $D(k) \neq \emptyset$ for any algebraically closed field k of characteristic $p > r$.*

Proof. We can assume that the field K is algebraically closed. By Corollary 5.7 we know that the theory ACF_0 is complete. Since $D(K) \neq \emptyset$ we have $ACF_0 \vdash D \neq \emptyset$.

On the other hand $ACF_0 = ACF \bigcup \cup_{n \in \mathbb{N}} \psi_n$ where $\psi_n = [n \neq 0]$. Since the derivation $ACF_0 \vdash D \neq \emptyset$ is finite there exists $r \in \mathbb{N}$ such that one can derive $ACF_0 \vdash D \neq \emptyset$ in $ACF \bigcup \cup_{1 \leq n < r} \psi_n$. So $D(K) \neq \emptyset$ for any algebraically closed field of characteristic $p > r$.□

Definition 5.10. a) An *affine algebraic variety* Y over a field k is a constructible subset of $\mathbb{A}^n$ given by a system of polynomial equations $P_i(x_1, ..., x_n) = 0, 1 \leq i \leq l$.

b) a *regular automorphism* $f : Y \to Y$ of an affine algebraic variety $Y$is one given by

$$y \to f(y) = (Q_1(x_1, ..., x_n), ..., Q_n(x_1, ..., x_n))$$

where $Q_j(x_1, ..., x_n) \in k[(x_1, ..., x_n], 1 \leq j \leq n$ are such that $f(y) \in Y(k)$ for all $y \in Y(k)$.

Lemma 5.11. *L:ax Let $Y \subset \mathbb{C}^n$ be an affine algebraic subvariety over $\mathbb{C}$ and $f : Y \to Y$ a regular map such that the map $f_{\mathbb{C}} : Y(\mathbb{C}) \to Y(\mathbb{C})$ is an imbedding. Then $f_{\mathbb{C}}$ is onto.*

Proof. Assume that there exists a regular map $f : Y \to Y$ which is not surjective and $f(y') \neq f(y''), \forall y' \neq y'' \in Y(\mathbb{C})$. We want to show that that such as assumption leads to a contradiction.

The subset $Y \subset \mathbb{C}^n$ is defined by a system of equations

$$\{P_i(x_1, ..., x_n) = 0\}, P_i(x_1, ..., x_n) \in k[(x_1, ..., x_n]1 \leq i \leq l$$

and the map $f : Y \to Y$ is given by polynomials $Q_j(x_1, ..., x_n), 1 \leq j \leq n$ such that $f(y) = (Q_1(y), ..., Q_n(y)$.

Let r be the maximal degree of polynomials $P_i, Q_j, 1 \leq i \leq l, 1 \leq j \leq n$ and $\mathbb{A}^N$ the affine space of polynomials $P_i, Q_j, 1 \leq i \leq l, 1 \leq j \leq n$ of degree $\leq r$. We denote by $Z \subset \mathbb{A}^N$ the subset consisting of polynomials $P_i, Q_j, 1 \leq i \leq l, 1 \leq j \leq n$ of degree $\leq r$ such that

(i) if $(x_1, ..., x_n) \in \mathbb{C}^n$ are such that $P_i(x_1, ..., x_n) = 0, 1 \leq i \leq l$ then $P_i(Q_1(x_1, ..., x_n), ..., Q_n(x_1, ..., x_n)) = 0, 1 \leq i \leq l$

(ii) if $(x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{C}^n$ are such that $P_i(x_1, ..., x_n) = P_i(y_1, ..., y_n) = 0, 1 \leq i \leq l$ and $Q_j(x_1, ..., x_n) = Q_j(y_1, ..., y_n), 1 \leq j \leq n$ then

$(x_1, ..., x_n) = (y_1, ..., y_n)$

(iii) there exists $(y_1, ..., y_n) \in \mathbb{C}^n$ is such that $P_i(y_1, ..., y_n) = 0, 1 \leq i \leq l$ and

$$(Q_1(x_1, ..., x_n), ..., Q_n(x_1, ..., x_n)) \neq (y_1, ..., y_n)$$

for any $(x_1, ..., x_n) \in \mathbb{C}^n$ such that $P_i(x_1, ..., x_n) = 0, 1 \leq i \leq l$.

By the construction Z is a definable "set" in TF. The existence of a regular imbedding $f : Y \to Y$ such that $f_{\mathbb{C}}$ is not surjective shows that $Z(\mathbb{C}) \neq \emptyset$. As follows from Lemma 5.9 there exists a prime p such that $Z(\bar{\mathbb{F}}_p) \neq \emptyset$ where $\bar{\mathbb{F}}_p$ is the algebraic closure of the field $\mathbb{F}_p$. In other words there exist polynomials

$P_i(x_1, ..., x_n), Q_j(x_1, ..., x_n) \in \bar{\mathbb{F}}_p, 1 \leq i \leq l, 1 \leq j \leq n$ which satisfy (i),(ii),(iii) when we replace $\mathbb{C}$ by $\bar{\mathbb{F}}_p$.

Since we consider only a finite set of polynomial there exists a finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$ such that

$P_i(x_1, ..., x_n), Q_j(x_1, ..., x_n) \in \mathbb{F}_q[x_1, \ldots, x_n], 1 \leq i \leq l, 1 \leq j \leq n$.

By enlarging q we can assume that there exists $(y_1, ..., y_n) \in Y(\mathbb{F}_q^n)$ such that for any $(x_1, ..., x_n) \in Y(\bar{\mathbb{F}}_p)$ we have
$(Q_1(x_1, ..., x_n), ..., Q_n(x_1, ..., x_n)) \neq (y_1, ..., y_n)$.
But as follows from (i),(ii) the map

$$(x_1, ..., x_n) \to (Q_1(x_1, ..., x_n), ..., Q_n(x_1, ..., x_n))$$

defines an imbedding of a finite set $g : Y(\mathbb{F}_q) \to Y(\mathbb{F}_q)$ into itself. So $g : Y(\mathbb{F}_q) \to Y(\mathbb{F}_q)$ onto. This contradiction proves the non existence of a regular imbedding $f : Y \to Y$ such that $f_{\mathbb{C}}$ is not surjective shows that $Z(\mathbb{C}) \neq \emptyset$.□

6. Types

For the simplicity of exposition I give a definition of *types* only for theories with only one sort (see Definition 2.14). If a theory T has a number of sorts parameterized by a set Λ then one has to understand n as a map from Λ to $\mathbb{N}$.

Definition 6.1. D:"set" a) Given a type τ we denote by $\mathcal{M}(\tau)$ the category such that objects of $\mathcal{M}(\tau)$ are τ-structures and morphisms are elementary monomorphisms. We fix a τ-theory T and denote by $\mathcal{M}(T)$ the subcategory of $\mathcal{M}(\tau)$ consisiting of models of T.

b) A *definable set* D is a functor from the category $\mathcal{M}(T)$ to the category of sets of the form $M \mapsto \phi(M)$, where ϕ is a formula of $L = L_\tau$.

c) A definable set is *constructible* if can be defined by an open formula.

d) An ∞- *definable set* is an intersection of definable sets.

e) we say that two formulas ψ, ψ' of L are *equivalent* if $\phi(M) = \phi'(M)$ for any model M of T.

f) We denote by $B_n(\tau)$ the set of definable sets which come from formulas ψ of L whith free variables $x_1, \dots, x_n$. For any such formula ψ we denote by $[\psi] \in B_n(T)$ the corresponding equivalence class. An element of $B_n(T)$ is an equivalence class of formulas but often we call an element of $B_n(T)$ " a formula".

g) If $R \subseteq D \times D'$ and for any model $M \models T$, $R(M)$ is the graph of a function $D(M) \to D(M')$, we say R is a *definable function of* T.

h) a definable "set" D is *finite* if $D(M)$ is finite for any $M \models T$.

i) for any model M of T and a subset A of $U(M)$ we denote by $dcl(A, T)$ [or simply $dcl(A)$] the *definable closure of A in M* as the subset of all points $u \in U(M)$ for which there exists a definable set D in the language L_A such that for any extension M' of M we have $D(M') = \{u\}$.

j) for any model M of T and a substructure $A \subset U(M)$ we denote by $acl(A)$ the *algebraic closure* of A in M as the subset of all points $u \in U(M)$ for which there exists a finite definable set D such that $u \in D(M)$.

k) For any theory T and any $n \in \mathbb{N}$ we denote by $\mathbb{A}^n$ the "set" $M \to U(M)^n$.

Problem 6.2. *P:types a) Let T be a complete theory, M a model of T and D, D' definable sets such that $D(M) = D'(M)$. Show that $D = D'$.*

b) Give an example of two ∞-definable sets D, D' such that $D(M) = D'(M)$ but $D \neq D'$.

c) If $M' \subset M$ is an elementary submodel then $U(M')$ is algebraic closed in $U(M)$. [That is $U(M')$ is equal to its algebraic closure in $U(M)$.]

Remark 6.3. a) It is often the case that one formula is equivalent to a different one. For a example, in the theory TF the formula $\{\exists y | xy = 1\}$ is equivalent to an open formula $\{x \neq 0\}$.

b) The set of formulas depends only on the language τ, but the notion of the eqivalence of formulas and therefore the set $B_n(T)$ depends on a choice of a theory T.

c) In the case of complete theories we will often identify a definable set $D \subset \mathbb{A}^n$ with a subset $D(M)$ of $U(M)^n$ for a particular model M of T.

d) $dcl(\emptyset)$ contains $U_0(M)$ but in general $dcl(\emptyset)$ is strictly bigger then $U_0(M)$. For example for the model $\mathbb{Q}$ of TF the set $U_0(M)$ is equal to $\mathbb{Z}$ but $dcl(\emptyset) = \mathbb{Q}$.

Examples 6.4. If K is a field then models of TF_K are fields extensions $F \supset K$, substructures are subrings of F containing K, and monomorphisms between models are K-homomorphisms of fields.

Constructible "subsets" of the "set" $\mathbb{A}^n$ are functors from the category of the field extensions of K to *Sets* which can be represented as finite Boolean combinations of "sets" defined by equations

$$P_i(x_1, ..., x_n) = 0, 1 \leq i \leq s, P_i \in K[x_1, ..., x_n] \square$$

Lemma 6.5. *L:finite If D is finite, then for some integer m we have $|D(M)| \leq m$ for any $M \models T$.*

Proof. Let $D = D_\psi \subset \mathbb{A}^n$ be a definable "set"such that for any integer m there exists a model M_m of T such that $|D(M_m)| > m$. We have to show that there exists a model M of T such that the set $D(M)$ is infinite. Consider a new theory T' which is obtained from T by additions of sentences

$$\star_m = \exists x_l^i \psi_m$$

where ψ_m is the formula with free variables $x_l^i, 1 \leq l \leq n, 1 \leq i \leq m$ given by

$$\psi_m = \wedge_{i=1}^m \psi(x_1^i, \ldots, x_n^i) \wedge_{1 \leq j < k \leq m} (\vee_{l=1}^n x_l^j \neq x_l^k)$$

for all $m \in \mathbb{N}$. It follows from the existence of models M_m for all $m \in \mathbb{N}$ that any finite subset of T' has a model. So by the compactness theorem the theory T' has a model M. Then $D(M)$ is an infinite model of T.□

Problem 6.6. *P:open a) Describe all open sentences in TF,*

b) show that any term t with one free variable x in the language of TF is of the form $t = p(x), p \in \mathbb{Z}[x]$,

c) show that any open formula with one free variable x in the language of TF_K, where K is a field, is equivalent to a Boolean combinations of sentences of the form $p(x) = 0, p(x) \in K[x]$.

d) show that any open formula in the language of OF_K, where K is an ordered field, is equivalent to a Boolean combinations of sentences of either of the form $p(x) = 0$ or of the form $a < p(x)$ where $p(x) \in K[x], a \in K$.

e) Let F be an algebraically closed valued field [that is a model M of AVF]. Show that any open formula of AVF_M with one variable x of the sort F is equivalent to a Boolean combination of the of formulas of types
(i) $v(p(x)) < v(q(x))$,
(ii) $v(p(x)) = v(q(x))$
(iii) $p(x) = 0$ where $p(x), q(x) \in F[x]$.

f) describe open sentences and formulas with only free variable in the theory OAG.

Definition 6.7. D:type a) We define Boolean operations on $B_n(T)$ by:

$$[\psi'] \cup [\psi''] = [\psi' \vee \psi'']$$

$$[\psi'] \cap [\psi''] = [\psi' \wedge \psi'']$$

$$c[\psi] = [\neg\psi]$$

b) A formula $[\psi] \in B_n(T)$ is *consistent* with T if there exists a model M of T such that

$$M \models \exists x_1, ..., x_n \psi(x_1, ..., x_n)$$

c) a set $S \subset B_n(T)$ is *consistent* if a conjunction of any finite number of members of S is consistent with T.

d) an $n - type$ of T is a maximal consistent subset $p \subset B_n(T)$.

e) if M is a model of T and $\bar{u} = (u_1, \ldots, u_n) \in U^n$. We denote by $p_{\bar{u}}(T)$ the $n - type$ of $\bar{u}$ in T given by

$p_{\bar{u}} = \{[\psi] \in B_n(T) : \psi_{x_1,...,x_n}(u_1, ..., u_n)$ is true $\}$.

f) If p is an n-type of T and M is a model of T we say that p is *realizable* in M if there exists $\bar{u} \in U^n$ such that $p = p_{\bar{u}}$. In such a case we say that $\bar{u}$ *realizes* the type p.

h) We denote by $S_n(T)$ the set of $n - type$ of T.

i) We denote by $\mathcal{T}$ the topology on $S_n(T)$ such that the sets $V_\psi := \{p \ni [\psi]\}, [\psi] \in B_n(T)$ is a basis of open sets in $\mathcal{T}$.

j) We denote by $S_n^c(T)$ the quotient of $S_n(T)$ by an equivalence relation

$p \sim^c p'$ iff for any open formula ϕ we have $p \in [\phi] \leftrightarrow p' \in [\phi]$.

k) We denote by $\mathcal{T}^c$ the topology on $S_n^c(T)$ with a basis of open sets given by V_ϕ where ϕ is an open formula.

l) If Λ is a set of formulas with free variables $x_1, \ldots, x_n$ we define $V_\Lambda \subset S_n(T)$ as the intersection $V_\Lambda = \cap_{\psi \in \Lambda} V_\psi$.

m) If M is a model of $T, U = U(M)$ we say that a subset X of U^n is $\emptyset$ *definable* if there exists a formula $\psi(x_1, \ldots, x_n)$ in L such that $X = V_\psi(M)$ and we say that t X is *definable* if there exists a formula $\psi(x_1, \ldots, x_n)$ in L_M such that $X = V_\psi(M)$.

n) We denote by $\mathcal{T}(M)$ the topology on $U^n(M)$ such that $\emptyset$ definable sets are a basis of open sets in $\mathcal{T}(M)$.

Remark 6.8. a) As follows from the Zorn's lemma any consistent subset of $B_n(T)$ can be extended to an $n - type$.

b) For any $p \in S_n(T)$ there exists a model M of T such that p is realizable in M. On the other hand in general there exist types p, q which are not be in the same model.

c) The set $S_1(T)$ is not a model of T since function symbols $f_j, j \in J$ do not define maps $S_1^{m(j)}(T) \to S_1(T)$. For example if $T = ACF_k$ then the sum of two transcendental element is not defined.

d) A set S of formulas is consistent iff $V_s \neq \emptyset$.

e) the topological spaces $(S_n^c(T), \mathcal{T}^c)$ and $(S_n(T), \mathcal{T})$ are Hausdorff.

Problem 6.9. *P:consistent a) If $T = ACF_k$ then $S_1^c(T) = k \cup t$ where t is "an element transcendental over k".*

b) If $T = ACF_k$ then $S_n^c(T)$ is the set of prime ideals of $k[x_1, \ldots, x_n]$ and open sets of $\mathcal{T}^c$ are unions of constructible sets in the Zariski topology on the k-scheme $\mathbb{A}^n$.

c) If $T = OADG$ is the theory of ordered abelian divisible groups then $S_1(T)$ cosisits of three elements $p_<, p_0, p_>$ which are realized in the model $\mathbb{Q}$ of T by negative elements, 0 and positive elements.

d) For any $t \in \mathbb{R}$ we denote by $p_{(1,t)}$ the 2-type of $OADG$ realized by a pair $(1, t) \in \mathbb{R}^2$ where we consider $\mathbb{R}$ as a model of $OADG$. Then the 2-types $p_{(1,t)}, p_{(1,t')}$ are distinct if $t \neq t'$.

e) Describe $S_2(OADG)$.

f) A complete theory T a formula $[\psi] \in B_n(T)$ is consistent iff

$$T \vdash \exists x_1, ..., x_n \psi(x_1, ..., x_n)$$

g) Closed subsets of $B_n(T)$ are in one-to-one correspodence with ∞-definable subsets of $\mathbb{A}^n$.

Remark. We will see that for any countable field k the set $S_n(ACF_k)$ is countable for all $n \in \mathbb{N}$. On the other hand it follows from Problem 6.9 c) that the set $S_2(OADG)$ is uncountable.

Lemma 6.10. *L:compact a) For any type $p \in S_n(T)$ there exists an elementary extension M' of M which realizes the type p and $\kappa(M') = \kappa(M)$.*

b) The topological space $(S_n(T), \mathcal{T})$ is compact.

Proof . a) Let $k_1, ..., k_n$ new constants and $S = T(M) \cup \{F(k_1, ..., k_n)\}$ where $F(x_1, ..., x_n)$ runs through all the formulas in p. The set S is consistent since for any $F(x_1, ..., x_n) \in p$ we have $T(M) \vdash \exists x_1, ..., x_n F(x_1, ..., x_n)$. Using the same arguments as in the proof of the upward Skolem-Lowenheim theorem (Corollary 4.3) one shows the existence of a model M' of S of cardinality $\kappa(M)$ which is an elementary extension of M such that $\kappa(M') = \kappa(M)$. Then $u = (k_1^{M'}, ..., k_n^{M'})$ realizes the type p.

b) Let $\psi^i, i \in I$ be a set of formulas whose free variables are among $x_1, \dots, x_n$ such that $S_n(T) = \cup_{i\in I} V_{\psi^i}$. We want to show the existence of a finite subset $I' \subset I$ such that $S_n(T) = \cup_{i\in I'} V_{\psi^i}$.

If such a subset does not exist then for any finite subset $I' \subset I$ we have $S_n(T) \neq \cup_{i\in I'} V_{\psi^i}$ and the theory $T(\{\exists x_1, ..., x_n \neg\psi^i\}), i \in I'$ is consistent. Therefore the theory $T(\{\exists x_1, ..., x_n \neg\psi^i\}), i \in I$ is also consistent. Therefore there exists a model M of T and $u_1, ..., u_n \in U$ such that $\psi^i_{x_1,...,x_n}(u_1, ..., u_n)$ is false for all $i \in I$. But then $p_{u_1,...,u_n}$ does not belong to $\cup_{i\in I} V_{\psi^i}$. This condrudiction shows the existence of a finite subset $I' \subset I$ such that $S_n(T) = \cup_{i\in I'} V_{\psi^i}\Box$

Problem 6.11. *P:compact a) For any set B of types of T there exists an elementary extension M' of M which realizes all the types in B and such that $\kappa(M') = max(\kappa(M), (\kappa(B)$.*

b) Find a model M of ACF such that the topological space $(U(M), \mathcal{T}(M))$ is not quasi-compcat.

Lemma 6.12. *L:openclosed A subset $X \subset S_n(T)$ has a form V_ψ for some formula ψ iff X is open and closed.*

Proof. Since $S_n(T) - V_\psi = V_{\neg\psi}$ the set V_ψ is open and closed for any formula ψ.

Conversely if a subset X of S_n is both open and closed then there exists a family $\psi_i, i \in I$ of formulas such that $X = \cup_{i\in I} V_{\psi_i}$. Since X is closed it is compact and therefore there exists a finite subset $I' \subset I$ such that $X = \cup_{i\in I'} V_{\psi_i}$. But then $X = V_\psi$ for $\psi = \psi_{i\in I'}$.$\Box$

Corollary 6.13. *C:ccomp The topological space $(S^c_n(T), \mathcal{T}^c)$ is compact.*

Proof. $(S^c_n(T), \mathcal{T}^c)$ is an image of the compact $(S_n(T), \mathcal{T})$ under the continuous map $\pi : S_n(T) \to S^c_n(T)$. So it is compact.

Corollary 6.14. *C:bij If the theory T is complete and the natural continuous map $\pi : S_n(T) \to S^c_n(T)$ is a bijection then any formula of T is equivalent to an open formula.*

Proof. Let ψ be a formula with free variables $x_1, ..., x_n$. Since the topological space $S_n(T)$ is compact, any closed subset $X \subset S_n(T)$ is also compact and therefore $\pi(X) \subset S^c_n(T)$ is closed. So $\pi : S_n(T) \to S^c_n(T)$ is a homeomorphism. Since the set $V_\psi \subset S_n(T)$ is both open and closed in $\mathcal{T}$ it is both open and closed in $\mathcal{T}^c$.

The same arguments as in the proof of Lemma 6.12 show then the existence of an open formula ϕ such that $V_\psi = V_\phi$.$\Box$

Definition 6.15. If D an ∞-definable set in a complete theory T and M a model of T we say that D is L_M-definable if there exists a formula ψ in L_M such that $D(M') = V_\psi(M')$ for any extension M' of M.

Lemma 6.16. *L:inf Let T be a complete theory, M a model of T and D is an L_M-definable set such that there exist a set $\psi_i, i \in I$ of formulas in L such that $D = \cap_{i \in I} \psi_i$ then there exists a finite subset I' of I such that $D = \cap_{i \in I'} \psi_i$.*

Proof. We have $\mathbb{A}^n - D = \cup_{i \in I} V_{\neq \phi_i}$. Since D is L_M-definable the set $\mathbb{A}^n - D$ is compact and there exists a finite subset I' of I such that $\mathbb{A}^n - D = \cup_{i \in I'} V_{\neq \phi_i}$. But then $D = V_\psi, \psi = \wedge_{i \in I'} \phi_i$.□

7. Basics of the theory of valued fields

There are different ways to formalize the concept of a valued field. In 2.15 i) we used an axiomatization VF of the theory in a 2-sorted language (F, Γ) where F corresponds to field elements and Γ to elements of a valued group. But there are other axiomatizations of the theory.

Definition 7.1. D:V A subring $\mathcal{O}$ of field F is *a valuation ring* if $u \in \mathcal{O}$ of $u^{-1} \in \mathcal{O}$.

The following two Lemmas are well known [see for example [L] 12.4)

Lemma 7.2. *Let $\mathcal{O} \subset F$ be a valuation ring. Then*

a) $\mathcal{O}$ is a local ring.

b) The ordering $\leq$ on $\Gamma = F^\star/\mathcal{O}^\star$ defines on Γ a structure of an ordered abelian group.

c) The natural surjection $v : F^\star / to \Gamma$ defines a structure of a valued field on F.

d) For any valued field $(F.\Gamma, v)$ the ring $\mathcal{O} : \{a \in F | v(a) \leq 0\}$ is a valued ring.

Problem 7.3. *Let τ_{VR} be the type obtained by the augmentation of the type τ of the theory of fields (see 2.15 b)) by an 1-placed relation symbol o. Find an extension VR of the system TF of axioms of the theory of fields to a system of axioms in the language of τ_{VR} such models of VR correspond to valued fields.*

Remark 7.4. The theories VR and VF describe the same mathematical objects but they are very different. For example in the theory VR there is no sybols to talk about elements of the value group. We see later how to reconstruct elements of the value group as *imaginary elements* of VR.

Definition 7.5. Let $\mathcal{O} \subset F$ be a valuation ring.

a) We denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$, by $k = \mathcal{O}/\mathfrak{m}$ the quotient field, by $p : \mathcal{O} \to k$ the natural projection, by $\bar{k}$ an algebraic closure of k and by $\bar{F}$ an algebraic closure of F.

b) We denote by $\mathcal{R}$ the set of subrings of $\bar{F}$ containing $\mathcal{O}$ for which there exists an extention of $p : \mathcal{O} \to k$ to a ring homomorphsim $p' : \mathcal{O}' \to \bar{k}$ and by $\mathcal{R}'$ the set of maximal elelements of $\mathcal{R}$.

Lemma 7.6. *L:L a) Any $\mathcal{O}' \in \mathcal{R}'$ is a valuation subring of $\bar{F}$.*

b) Any valuation subring of $\bar{F}$ containing $\mathcal{O}$ belongs to $\mathcal{R}'$. Let $\mathcal{O}'$ be a maximal subring of $\bar{F}$ Then

c) For any two valuation subrings of $\mathcal{O}', \mathcal{O}'' \subset \bar{F}$ containing $\mathcal{O}$ there exists an automorphism of σ of the field $\bar{F}$ such that $\mathcal{O}'' = \sigma(\mathcal{O}')$.

Corollary 7.7. *C:L Let $F, v : F^\star \to \Gamma$ a valued field and $\bar{F}$ an algebraic closure of F. Then*

a) there exists an extension of v to a valuation $\bar{v} : F^\star \to \Gamma \otimes \mathbb{Q}$ and

b) if $\bar{v}' : F^\star \to \Gamma \otimes \mathbb{Q}$ is another such extension then there exists $\sigma \in Gal(\bar{F}/F)$ such that $\bar{v}' = \bar{v} \circ \sigma$.

One can ask when there is unique extension of v to a valuation $\bar{v} : F^\star \to \Gamma \otimes \mathbb{Q}$. In other words when there is unique maximal subring $\mathcal{O}'$ of $\bar{F}$ such that there exists an extention of $p : \mathcal{O} \to k$ to a ring homomorphsim $p' : \mathcal{O}' \to \bar{k}$.

Definition 7.8. D:hens A local ring $A \supset \mathfrak{m}$ is *Henselian* if for any monic polynomial $p(t) \in A[t]$ a decompostions of the reduction $\bar{p}(t) \in k[t], k = A/\mathfrak{m}$ in a a product of relatively prime factors $\bar{q}(t), \bar{r}(t) \in k[t]$ lifts to decompostions of $p(t)$ in $A[t]$.

Remark 7.9. R:hens a) It it is easy to show (see the first four pages of [R]) that a local ring A is Henselian iff any integral A-algebra B finitely generated as an A-module is local.

b) One can show (see [R] Proposition 7.3) that a local ring $A \supset /\mathfrak{m}$ is Henselian iff for any monic polynomial $p(t) \in A[t]$ and $a \in A$ such that $p(a) \in \mathfrak{m}, p'(a) \notin \mathfrak{m}$ there exists unique $b \in A$ such that $p(b) = 0$ and $a - b \in \mathfrak{m}$.

Lemma 7.10. *L:hens Let $\mathcal{O} \subset F$ be a Henselian valuation ring. Then there is unique extension of v to a valuation $\bar{v} : F^\star \to \Gamma \otimes \mathbb{Q}$.*

Proof. Let $\bar{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in $\bar{F}$. It is easy to see that $\mathcal{O}' \in \mathcal{R}'$ of $\bar{F}$ contains $\bar{\mathcal{O}}$. So it is sufficient to show that the ring $\bar{\mathcal{O}}$ is a maximal proper subring of $\bar{F}$.

Since $\bar{F}$ is a union of finite extension $E \supset F$ is sufficient to show that for any finite extension $E \supset F$ the integral closure B of $\mathcal{O}$ in E is a maximal proper subring of E. Since $\mathcal{O}$ is Henselian and B is integral ring finitely generated as a $\mathcal{O}$-module the ring B is local. It is now easy to see now that B is a maximal proper subring of E.

Remark 7.11. The converse statement is also true.

Since for any algebraically closed valued field its value group is divisible, we start the analyzis of the theory AVF with a result on $OADG$ (see 2.15 g).

Definition 7.12. D:simG a) Let $\Gamma \subset U$ be a pair of abelian divisible groups, $u \in U$. We denote by $\Gamma_u = \{ru + \gamma\}, r \in \mathbb{Q}, \gamma \in \Gamma$ the minimal abelian divisible subgroup of U containing Γ and u.

b) Let $U \supset \Gamma$ and $U' \supset \Gamma$ be ordered abelian divisible groups. Given $u \in U, u' \in U'$ we say that u, u' are *equivalent* and write $u \sim u'$ if for any $\gamma \in \Gamma$ we have

$u < \gamma \leftrightarrow u' < \gamma$ and $u > \gamma \leftrightarrow u' > \gamma$.

Remark 7.13. We don't know a priory that any two ordered abelian divisible groups containing $U \supset \Gamma, U' \supset \Gamma$ are imbeddable into a bigger model group U'' [in other words that any two models M, M' of the theory $OADG_\Gamma$ are imbeddable into a bigger model M'']. Therefore we can not assume that u, u' are elements of the same ordered abelian divisible group.

Lemma 7.14. *L:extG Let $U, U' \supset \Gamma$ be ordered abelian divisible groups and $u \in U, u' \in U'$ be equivalent elements. Then there exists an isomorphism $f : \Gamma_u \to \Gamma_{u'}$ of ordered abelian groups such that $f(u) = u'$ and $f(\gamma) = \gamma$ for all $\gamma \in \Gamma$.*

Proof. It is clear that if $u \sim u'$ then $u \in \Gamma \leftrightarrow u' \in \Gamma$. If both $u, u' \in \Gamma$ there is nothing to prove. So we assume that $u, u' \notin \Gamma$ and we can define an a group isomorphism $f : \Gamma_u \to \Gamma_{u'}$ by $f(ru + \gamma) = ru' + \gamma, r \in \mathbb{Q}, \gamma \in \Gamma$.

To prove the Lemma we have to check that f compatible with the orders on U, U'. In other words we have to show that an inequality $ru + \gamma < su + \delta$ implies that $ru' + \gamma < su' + \delta, r, s \in \mathbb{Q}, \gamma, \delta \in \Gamma$. But this follows immediately from the condition $u \sim u'$.$\square$

Let $F, v : F^\star \to \Gamma$ be algebraically closed valued field [that is a model of AVF (see 2.15].

Definition 7.15. D:balls a) Given $\alpha, \beta \in \Gamma$ we write $\alpha \leq \beta$ if either $\alpha < \beta$ or $\alpha = \beta$.

b) To any pair $a \in F, \gamma \in \Gamma^+$ we associate a definable sets $D(a,\gamma), \bar{D}(a,\gamma)$ and $R(a,\gamma)$ in the theory AVF_F which we call an open balls and closed *balls* and *rings* of radius $\gamma \in \Gamma^+$. These "sets" associate to any model M of the theory AVF_F sets

$$D(a,\gamma)(M) = \{u \in U(M) : v(u-a) < \gamma\}$$

$$\bar{D}(a,\gamma)(M) = \{u \in U(M) : v(u-a) \leq \gamma\}$$

and

$$R(a,\gamma)(M) = \{u \in U(M) : v(u-a) = \gamma\}$$

It is clear that $\mathcal{O} := \bar{D}(0,\bar{0})$ is the *valuation ring* of F and $\mathfrak{m} := D(0,\bar{0})$ is the maximal ideal of $\mathcal{O}$. We denote the residue field $\mathcal{O}/\mathfrak{m}$ by k.

Let $E \supset F$ be algebraically closed valued field extension of $F, e \in E$. We denote by $F(e) \subset E$ the subfield generated by e over F, by $\bar{F(e)}$ the algebraic the smallest closure of $F(e)$ in E and by Γ_e the image of $\bar{F(e)}^\star$ under the valuation v_E. Then $\bar{F}(e), v_E : \bar{F(e)}^\star \to \Gamma_e$ is an algebraically closed valued field.

Lemma 7.16. *L:Ge a) If $v_E(e-a) \in \Gamma$ for all $a \in F$ then $\Gamma_e = \Gamma$.*

b) If $\Gamma_e \neq \Gamma$ then there exists $a \in F$ such that $v_E(e-a) \notin \Gamma$ and $\Gamma_e = \mathbb{Q}v_E(e-a) + \Gamma$.

Proof. a) Assume that $v_E(e-a) \in \Gamma$ for all $a \in F$. Since the field F is algebraically closed any polynomial $p(x) \in F[x]$ is a is a product of linear factors and therefore $v_E(p(e)) \in \Gamma$. It follows now from Corollary 7.7 that $\Gamma_e = \Gamma \otimes \mathbb{Q} = \Gamma$.

b) Choose $a \in F$ such that $v_E(a-e) \notin \Gamma$. Let $\tilde{\Gamma} = \mathbb{Q}v_E(a-e) + \Gamma$. Since $\tilde{\Gamma}$ is divisible it is sufficient to show that for any $p(x) \in F[x]$ we have $v_E(p(a-e)) \in \tilde{\Gamma}$. Since the field F is algebraically closed we have

$$p(x) = c\prod_{i=1}^{n}(x-a_i), c, a_i \in F$$

Therefore it is sufficient to show that $v_E(a-e-a_i) \in \tilde{\Gamma}$ for any $a \in F$.

But $v_E(a-e) \notin \Gamma$. Therefore $v_E(a-e) \neq v(a_i)$ and either

$$v_E(a-e-a_i) = v_E(a-e)$$

or

$$v_E(a-e-a) = v(a_i) \in \Gamma.\square$$

Definition 7.17. D:imF Let $E \supset F$ and $E' \supset F$ be algebraically closed valued field extensions of F. Given $e \in E, e' \in E'$ we say that $e.e'$ are *equivalent* and write $e \sim e'$ if

$$e \in D(a,\gamma) \leftrightarrow e' \in D(a,\gamma), e \in R(a,\gamma) \leftrightarrow e' \in R(a,\gamma)$$

for any $a \in F, \gamma \in \Gamma^+$.

Proposition 7.18. *P:extv If $E \supset F$ and $E' \supset F$ are algebraically closed valued field extensions of F and elements $e \in E, e' \in E'$ are equivalent then there exists an isomorphism $\bar{g} : \bar{F(e)} \to \bar{F(e')}$ of valued fields.*

Proof. We start with the following result.

Lemma 7.19. *L:exGe a) If $e \in E, e' \in E'$ are equivalent then $v_E(e) \sim v_{E'}(e')$.*

b) If $\Gamma_e = \Gamma$ then $\Gamma_{e'} = \Gamma$.

c) If $\Gamma_e = \Gamma_{e'} = \Gamma$ then there exists a field isomorphism

$$\tilde{f} : F(e) \to F(e')$$

such that $\tilde{f}(e) = e'$ and $v_E(x) = v_{E'}(x)$ for any $x \in F(e)$.

d) If $\Gamma_e \neq \Gamma$ then there exists an isomorphism $f : \Gamma_e \to \Gamma_{e'}$ of ordered abelian groups and a field isomorphism

$$g : F(e) \to F(e')$$

such that $g(e) = e'$ and $f(v_E(x)) = v_{E'}(x)$ for any $x \in F(e)$

Proof of Lemma. a) We have to show that for any $\gamma \in \Gamma$

$$v_E(e) < \gamma \leftrightarrow v_{E'}(e') < \gamma, v_E(e) > \gamma \leftrightarrow v_{E'}(e') > \gamma$$

But $, v_E(e) < \gamma \leftrightarrow e \in D(0,\gamma)$ and $, v_E(e) > \gamma \leftrightarrow e \notin \bar{D}(0,\gamma)$. So the assumption $e \sim e'$ implies that $v_E(e) \sim v_{E'}(e')$.

b) Suppose that $\Gamma_{e'} \neq \Gamma$. By Lemma 7.16 there exists $a \in F$ such that $v_{E'}(e'-a) \notin \Gamma$. As follows from a), $v_E(e-a) \sim v_{E'}(e'-a)$. So $v_E(e-a) \notin \Gamma$.

c) Let $g : F(e) \to F(e')$ be the field isomorphism such that $g(e) = e'$. We have to show that for any polynomial $p(x) \in F[x]$ we have $v_E(p(e)) = v_{E'}(p(e'))$.

Since any polynomial in $F[x]$ is a product of linear factors it is sufficient to show that $v_E(a-e) = v_{E'}(a-e')$ for all $a \in F$.

Let $\gamma := v_E(a-e) \in \Gamma$. Then $e \in R(a,\gamma)$ and therefore $e' \in R(a,\gamma)$. So $v_{E'}(a-e') = \gamma$.

d) By replacing e with $e-b, b \in F$ we can assume [see Lemma 7.16] that $v_E(e) \notin \Gamma$. As follows from a) $v_{E'}(e') \sim v_E(e)$. Therefore, (see Lemma 7.14), there exists an isomorphism $f : \Gamma_u \to \Gamma_{u'}$ of ordered abelian groups such that $f(v_E(e)) = v_{E'}(e')$. We have to show that for any polynomial $p(x) \in F[x]$ we have $f(v_E(p(e))) = v_{E'}(p(e'))$. Since any polynomial in $F[x]$ is a product of linear factors it is sufficient to show that $f(v_E(a-e)) = v_{E'}(a-e')$ for all $a \in F$.

Since $v_E(e) \notin \Gamma$ we have $v_E(a-e) = min(v(a), v_E(e))$. More precisely $v_E(a-e)) = v(a)$ if $e \in D(0, v(a)$ and $v_E(a-e)) = v_E(e)$ if $e \notin D(0, v(a)$.

Analogously $v_E(a-e') = v(a)$ if $e' \in D(0, v(a)$ and $v_{E'}(a-e)) = v_E(e')$ if $e' \notin D(0, v(a)$.

So for all $a \in F$ we have $f(v_E(a-e)) = v_{E'}(a-e')$.□

Now we can finish the proof of Proposition 7.18. Let $\bar{g} : F(\bar{e}) \to F(\bar{e}')$ be an extension of a field isomorphism $g : F(e) \to F(e')$. Then v_E and $v_{E'} \circ \bar{g}$ define valuations of the field $\bar{F}(e)$ which coincide on $F(e)$. As follows from Lemma 7.6, there exists $\sigma \in Gal(F(\bar{e})/F(e))$ such that $v_E(x) = v_{E'}(\bar{g} \circ \sigma(x))$ for all $x \in F(\bar{e})$. Therefore $\bar{g} \circ \sigma : F(\bar{e}) \to F(\bar{e}')$ is an isomorphism of valued fields.□

Definition 7.20. We denote by $\mathcal{C}$ the collection of definable sets of AVF_M which are finite Boolean combinations of sets of the form $D(a,\gamma)$ and $\bar{D}(a,\gamma)$.

Proposition 7.21. *P:openv a) Any constructible subset $D \subset \mathbb{A}^1$ belongs to $\mathcal{C}$*

b) $X(M) \neq \emptyset$ for any non-empty set in $X \in \mathcal{C}$.

Proof. a) We say that $p, p' \in S_1^c(T)$ are equivalent and write $p \sim p'$ if for any realizations $u, u' \in U(M')$ of p, p' we have

$$u \in D(a,\gamma) \leftrightarrow u' \in D(a,\gamma), u \in \bar{D}(a,\gamma) \leftrightarrow u' \in \bar{D}(a,\gamma)$$

Let $S'(T)$ be the quotient of $S_1^c(T)$ by the relation $\sim$ and $\pi' : S_1^c(T) \to S'(T), p \to [p]$ be the natural projection.

The same arguments as in the proof of Corollary 6.14 show that the validity of Proposition 7.21 would follow from the injectivity of π'. In other words we have to show that for any valued field extensions E, E' of F and any $u \in E, u' \in E'$ the equivalences

$$u \in D(a,\gamma) \leftrightarrow u' \in D(a,\gamma), u \in \bar{D}(a,\gamma) \leftrightarrow u' \in \bar{D}(a,\gamma) \forall a \in F, \gamma \in \Gamma^+$$

imply that for any open formula ϕ of L_T with x as the only free variable we have

$$u \in V_\phi \leftrightarrow u' \in V_\phi$$

But the equivalence $u \in V_\phi \leftrightarrow u' \in V_\phi$ follows from Proposition 7.18.

b). It is clear that for any two balls B, B' either $B \subset B'$ or $B' \subset B$ or $B \cap B' = \emptyset$. Therefore any non-empty set $X \in \mathcal{C}$ is a union of non-empty sets Y of the form $X = B - \cup_{i=1}^n B_i$ where $B_i \subset B$ are balls of radii $\gamma_i \in \Gamma$ and centers $a_i \in V, 1 \leq i \leq n$ and either B is a ball or B is equal to $\mathbb{A}^1$. It is sufficient to show that $Y(M) \neq \emptyset$. I consider first the case when B is a ball of radius $\gamma \in \Gamma$.

After the shift by a and a multiplication by $b \in F = U(M)$ such that $v(b) = \gamma$ we can assume that B is a ball of radius $\bar{0} \in \Gamma$ with the center at 0. We consider separately the case when $B = \mathfrak{m}$ is an open ball and the case when $B = \mathcal{O}$ is a closed ball. We define $\delta_i = v(a_i)$.

i) Assume that $B = \mathfrak{m}$. Since $X \neq \emptyset$ we have $B_i \neq B$ for all $i, 1 \leq i \leq n$ and therefore $\gamma_i, \delta_i > \bar{0}$ for all $i, 1 \leq i \leq n$.

Choose $c \in V$ such that $v(c) = min_{i=1}^n(\gamma_i, \delta_i)$. Since the field V is algebraically closed we can find $d \in V$ such that $c = d^2$. Then $d \in X(M)$.

ii) Let $\pi : \mathcal{O} \to k$ be the natural projection. Since $X \neq \emptyset$ we have $B_i \neq B$ for all $i, 1 \leq i \leq n$ and therefore $\bar{b}_i = \pi(B_i) \subset k$ is a point. Since the field k is algebraically closed and therefore infinite we can find $\bar{e} \in k^\star$ such that $\bar{e} \neq \bar{b}_i$ for all $i, 1 \leq i \leq m$. Let $e \in V$ be an element such that $v(e) = \bar{e}$. Then $e \in X(M)$.

Consider now the case when $B = \mathbb{A}^1$. Since $\Gamma \neq \bar{\{0\}}$ it is easy to see that there exists $\gamma \in \Gamma, \gamma > 0$ such that $B_i \subset D(0, y)$. If $a \in F$ is such that $v(a) = \gamma$ then $a \in X(M)$.$\square$

Corollary 7.22. *For any polynomials $p(x), q(x) \in F[x]$ the definable sets*

$$M' \to \{u' \in U' : v(p(u')) = v(q(u'))\}$$

and

$$M' \to \{u' \in U' : v(p(u')) < v(q(u'))\}$$

belong to $\mathcal{C}$.

8. Elimination of quantifiers

Definition 8.1. D:QE a) A theory T *admits the quantifier-elimination [QE]* if for model M of T any definable "set" X of $T(M)$ is equivalent to a constructible set (see Definition 6.1 c)).

b) A formula ϕ is *simply existential* if it has a form $\phi = \exists x\psi$ where ψ is an open formula.

L:sim

Lemma 8.2. *If Let T is a theory such that for any simply existential formula ψ there exists an open formula ϕ equivalent to ψ. Then T admits quantifier-elimination.*

Remark 8.3. To simplify the exposition I will assume that in this section that the language τ contains constants in all it's sorts.

Proof. We have to show that for any formula ψ there exists an equivalent open formula ϕ. The proof is by the induction in the number of times the quantifier $\exists$ appearing in ψ. We decrease the number ofappearences by replacing the most inner subformula of ψ containing the quantifier $\exists$ by an open formula.□

Problem 8.4. *P:QE Let T be a theory which admits QE.*

a) Any monomorphism between models of T is elementary.

b) For any substructure A of a model M of T theories T_A and $T(M,A)$ coincide.

c) For any model M of T and extensions M', M'' there exists an extension $\tilde{M}$ of M containing both M' and M''.

d) If for any open sentence ϕ in the lanugage of T the validity $\phi(M)$ does not depend on a choice of a model M then T is complete.

Remark 8.5. a) Not every theory T which admits quantifier-elimination is complete. For example, as we will see, the theory ACF admits quantifier-elimination. But it is not complete since there are algebraically closed fields of different characteristics.

b)) If a theory T admits QE then for any substructure A of a model of T theory T_A also admits QE. Really any formula ψ_A of T_A is otained from a formula ψ of T after a substitution of some free variables by elements of A. Since ψ is equivalent to an open formula ϕ the formula ψ_A is equivalent to the open formula ϕ_A .

c) If a type τ does not have any constants it could happen that while a τ-theory T does not admit QE, the theory obtained from T by an addition of a constant admits QE.

Really, consider the case when in addition to $x_1 = x_2$ the type τ consists of a one-places predicate P, does not contain any functions and the theory T is generated by a sentence $\forall x P(x) \vee \forall x \neg P(x)$. Then the sentence $\exists x P(x)$ is not equivalent to any open sentence.

But if we augment τ by a constant k then the sentence $\exists x P(x)$ will be equivalent to an open sentence $P(k)$. One can show that the theory $T[k]$ admits QE.$\square$

Given any theory T we can construct a theory $T^{\mathcal{S}}$ which admits QE and is essentially equivalent to T. We start with a definition.

Definition 8.6. D:Sk We say that a theory T in a language L is *Skolem* if for any $n \geq 0$ and any formula $\psi(x_1, \dots, x_{n+1})$ in L with free variables $x_1, \dots, x_{n+1}$ there exists a functions symbol $f(x_1, \dots, x_n)$ such that

$$T \models [\forall \{x_1, \dots, x_n\} \exists x_{n+1} \leftrightarrow \psi(x_1, \dots, x_n, f(x_1, \dots, x_n))]$$

[In particular for formula $\psi(x)$ there exists a constant $k \in K$ such that

$$T \models \exists x \psi(x) \leftrightarrow \psi(k)]$$

Lemma 8.7. *L:Sk a) Any Skolem theory andmits QE.*

b) For any theory T in a language L admits a Skolem extension $(T^{\mathcal{S}}, LT^{\mathcal{S}})$ such that any model M of T can be expanded to a model $M^{\mathcal{S}}$ of $T^{\mathcal{S}}$.

Proof. a) The proof is by induction in the number of appearence of $\exists$ in a formula ψ. Suppose that $\psi(x_1, \dots, x_n)$ has a form

$$\exists x_{n+1} \psi'(x_1, \dots, x_{n+1})$$

Since the theory T is *Skolem* there exists a functions symbol $f(x_1, \dots, x_n)$ such that

$$T \models [\forall \{x_1, \dots, x_n\} \exists x_{n+1} \leftrightarrow \psi'(x_1, \dots, x_n, f(x_1, \dots, x_n))]$$

By induction there exists an open formula $\phi(x_1, \dots, x_n)$ such that

$$T \models \psi'(x_1, \dots, x_n, f(x_1, \dots, x_n)) \leftrightarrow \phi'(x_1, \dots, x_n)$$

But then

$$T \models \psi(x_1, \dots, x_n) \leftrightarrow \phi(x_1, \dots, x_n)$$

b) Give a language L we define an extension L' of L by adding a new n-placed symbol f_ψ for any formula $\psi(x_1, \dots, x_{n+1})$ in L with free variables $x_1, \dots, x_{n+1}$ and define $L^{\mathcal{S}}$ by

$$L_0 = L, Lm + 1 = L^{\star}_{M}, L^{\mathcal{S}} = \cup_{m=0}^{\infty} L_m$$

Let $T^{\mathcal{S}}$ a theory in the language $L^{\mathcal{S}}$ obtained by adding to T the axioms

$$\forall\{x_1, \dots, x_n\}\exists x_{n+1} \leftrightarrow \psi(x_1, \dots, x_n, f_\psi dotsc, x_n))]$$

for any formula $\psi(x_1, \dots, x_{n+1}), n \geq 0$ of $L^{\mathcal{S}}$.

If M is a model of T we can, using the axiom of choice, extend M to a model $M^{\mathcal{S}}$ of $T^{\mathcal{S}}$ with the same universe. □

Remark 8.8. We see that for any theory T it is possible to eliminate the quantifiers if we expand the language of T. So when we ask about the possibility to eliminate quantifiers in an theory we talk about the elimination in it's *natural* language.

The following result is called the downward Skolem-Lowenheim theorem.

Corollary 8.9. *Given any theory T in a language L we can decompose any monomorphism $a : M' \to M$ of T-models as a composition $a = b \circ c$ where $c : M' \to M''$ is a monomorphism of T-models, $b : M'' \to M$ is an elementary monoimorphism, and $\kappa(M'') \leq max(\kappa(M'), \kappa(L))$*

Proof. Let $M^{\mathcal{S}}$ be an extension of M to a $T^{\mathcal{S}}$-model as in Lemma 8.7 and $M''^{\mathcal{S}} \subset M^{\mathcal{S}}$ be the minimal $T^{\mathcal{S}}$-submodel containing $a(U(M')$. As follows from Lemma 8.7 a) and Problem 8.4 a) the monomorphism $b^{\mathcal{S}} : M''^{\mathcal{S}} \to M^{\mathcal{S}}$ given by the natural inclusion is elementary. Therefore the natural inclusion induces an elementary monomorphism $b : M'' \to M$.

On the other hand it is clear that $\kappa(M'') \leq max(\kappa(M'), \kappa(L))$ and that a induces a monomorphism $c : M' \to M''$ such that $a = b \circ c$.□

One of important achievements of the Model theory is the discovery of a simple criterion for the existence of quantifier-elimination.

Definition 8.10. D:cond Let T be a theory. We say that

a) T satisfies the *isomorphism condition* if for any models M, M' of T, any substructures $A \subset M, A' \subset M'$ and an isomorphism $f : A \to A'$ we can find submodels $B \subset M, B' \subset M'$ containing A and A' respectively and an extension of $f : A \to A'$ to an isomorphism $\tilde{f} : B \to B'$.

b) T satisfies the *submodel condition* if for any pair $M \subset M'$ of models of T and every simply existential sentence ϕ of L_M we have $\exists x\phi(M) \leftrightarrow \exists x\phi(M')$.

Theorem 8.11. *T:QE If a theory T satisfies both the isomorphism and the submodel conditions, then T admits the quantifier-elimination.*

Proof of Theorem 8.11 . We start the proof with the following definition.

Definition 8.12. D:alop Let T be a theory in the language L. A sentence ψ in L is *almost open* if for any two models M, M' of T such that $\phi(M) = \phi(M')$ for any open sentence ϕ in L the equality $\psi(M) = \psi(M')$ is also true.

Lemma 8.13. *L:alop Any almost open sentence ψ in L is equivalent to an open one.*

Proof of Lemma 8.13. Let Γ be the set of all open sentences which are theorems in $T(\psi)$ (see Definition 2.11). It is sufficient to show that ψ is a theorem in $T(\Gamma)$. Really if ψ is a theorem in $T(\Gamma)$ there exists $\phi_1, .., \phi_r \in \Gamma$ such that

$$\vdash_T \phi_1 \wedge \cdots \wedge \phi_r \rightarrow \psi.$$

On the other hand $\vdash_T \psi \rightarrow \phi_i, 1 \leq i \leq r$. So $\psi \leftrightarrow \phi_1 \wedge \cdots \wedge \phi_r$.

We see that it is sufficient to show that an assumption that ψ is not a theorem in $T(\Gamma)$ leads to a contradiction.

Assume that ψ is not a theorem in $T(\Gamma)$. Then, as follows from Theorem 4.1, there exists a model M of $T(\Gamma)$ such that ψ is false in M. Let Δ be the set of all open sentences which are true in M.

I claim that for any model M' of $T(\Delta)$ the sentence ψ is false in M'. Indead, by the definition of Δ, any open formula ϕ is true in M iff it is true in M'. On the other hand the formula $\neg\psi$ is almost open and true in M. So it is true in M'.

Since the sentence $\neg\psi$ is true in M' for any model M' of $T(\Delta)$ it is a theorem in $T(\Delta)$ (see Theorem 4.1). In other words there exists $\phi_1, .., \phi_r \in \Delta$ such that $\vdash_T \phi_1 \wedge, .., \phi_r \rightarrow \neg\psi$. So $\vdash_T \psi \rightarrow \neg(\phi_1 \wedge \cdots \wedge \phi_r)$ and therefore $\neg(\phi_1 \wedge, .., \phi_r) \in \Delta$. We see that $\neg(\phi_1 \wedge \cdots \wedge \phi_r)$ is true in M. On the other hand for all $i, 1 \leq i \leq r$ the sentence ϕ_i is true in M. This contradiction proves Lemma 8.13.□

Lemma 8.14. *L:opset*

Let M', M'' be structures of a type τ such that for any open sentence ϕ in L_τ we have $\phi(M') = \phi(M'')$. Then there exists an isomorphism $\pi : U_0(M') \to U_0(M'')$ where $U_0(M') \subset M'$ and $U_0(M'') \subset M''$ are the minimal substructures [see Definition 2.1].

Proof of Lemma 8.14. For any $u \in U_0(M')$ there exists a term $t = t_u$ without free variables such that $u = t^{M'}$ [see Problem 2.8]. We define $\pi(u) := t^{M''}$. I claim that $\pi(u) \in U_0(M'')$ does not depend on a choice of a term t such that $u = t^{M'}$. Really if s is an other term such that $u = s^{M'}$ then the open sentence $t = s$ is true in M'. Therefore this sentence is also true in M'' and we have $t^{M''} = s^{M''}$.

The same arguments show that for any $i \in I, u_1, ..., u_{n(i)} \in U_0(M')$ we have

$R_i^{M'}(u_1, ..., u_{m(j)})$ iff $R_i^{M''}(\pi(u_1), ..., \pi(u_{m(j)}))$

and that for any $j \in J, u, u_1, ..., u_{m(j)} \in U_0'$ we have

$f_j^{M'}(u_1, ..., u_{m(j)}) = u$ iff $f_j^{M''}((\pi(u_1), ..., (\pi(u_{m(j)})) = \pi(u)$

Therefore the map $\pi : U_0(M') \to U_0(M'')$ is an isomorphism of substructures.□

Lemma 8.15. *L:both Assume that a theory T satisfies both the isomorphism and the submodel conditions and any sort of T contains a constant. Then any simply existential sentence in the language L of T is equivalent to an open sentence.*

Proof of Lemma 8.15 Let ψ be a simply existential sentence. By Lemma 8.13 it is sufficient to show that the sentence ψ is almost open. In other words we have to show that for any pair $M.M'$ of models of T such that $\phi(M) = \phi(M')$ for all open sentences ϕ of L we also have $\psi(M) = \psi(M')$.

By Lemma 8.14 there exists an isomorphism $\pi : U_0(M) \to U_0(M')$ between the minimal substructures of M, M'. By the isomorphism condition we can extend π to an isomorphism of submodels $N \subset M, N' \subset M'$. Since the sentence ψ is simply existential, the submodel condition implies that $\psi(N) = \psi(M)$ and $\psi(N') = \psi(M')$. On the other hand the models N, N" of T are isomorphic and therefore $\psi(M) = \psi(M')$.□

Lemma 8.16. *L:adcon Let T be a theory satisfying the isomorphism and the submodel conditions and T' be the theory obtained from T by adding a new constant k [see Problem 2.2]. Then*

a) T' satisfies the isomorphism condition, *b) T' satisfies the submodel condition.*

Proof of Lemma 8.16 a) Let M, M' be models of T' and $f : A \to A'$ be an isomorphism between substructures A, A' of M, M'. We want to construct an extension h of f to an isomorphism of submodels C, C' of M, M'.

Let $\bar{M}, \bar{M}'$ be models of T obtained by the restrictions of M, M' to T. Since T satisfies the isomorphism condition we can find submodels $\bar{D}, \bar{D}'$ of $\bar{M}, \bar{M}'$ and an isomorphism $\bar{g} : \bar{D} \to \bar{D}'$. We expand $\bar{D}$ to a structure D of T' by $k^D = k^A$. Analogously we expand $\bar{D}'$ to a structure D' of T'. Then D, D' are models of T' and we can define the extension of $\bar{g}$ to a monomorphism $g : D \to D'$ by $g(k^D) = k^{D'}$. It is clear that $g : D \to D'$ is an isomorphism.

b) Let $M \subset M'$ be models of T' and ψ' a simply existential sentence of $L(T')$. We want to show that $\psi'(M) = \psi'(M')$.

Let ψ be the sentence of $L_M(T)$ obtained from ψ' by the replacement of the free variable x by k^M. Since T satisfies the submodel condition we have $\psi(M) = \psi(M')$.

On the other hand it follows from Problem 2.13 a) that $\psi'(M) = \psi(M)$ and $\psi'(M') = \psi(M')\Box$

Now we can prove Theorem 8.11. Let T be a theory satisfying both the isomorphism and the submodel conditions. We want to show that for any simply existential formula ψ in the language L there exist an equivalent open formula ϕ. Let $x_1, ..., x_r$ be all the free variables of ψ. Consider the theory T' be the theory obtained from T by adding new constants $k_1, ..., k_r$. As follows from Lemma 8.16 the theory T' satisfies the isomorphism condition and the submodel condition. As follows from Lemma 8.15 the sentence $\psi' = \psi[k_1, ..., k_r]$ is equivalent to an open sentence ϕ' of T'. Therefore, (see Problem 2.13) the open formula ϕ obtained from ϕ' be the substitution of constants k_i by variables $x_i, 1 \le i \le r$ is equivalent to ψ.$\Box$

Corollary 8.17. *The theory ACF admits quantifier-elimination.*

Proof. It is sufficient to show that ACF satisfies the isomorphism and the submodel conditions.

a) For the proof of the isomorphism condition we have to show that for any pair K, K' of algebraically closed fields, a pair of subrings $A \subset K, A' \subset K'$ and a ring isomorphism $f : A \to A'$ there exist subfields $E \subset K, E' \subset K'$ containing A, A' and an isomorphism $\tilde{f} : E \to E'$ extending $f : A \to A'$.

Let $F \subset K, F' \subset K'$ be subfields generated by A, A'. It is clear that F, F' are the fraction fields of A, A' and the ring isomorphism $f : A \to A'$ extends to an isomorphism $f_1 : F \to F'$. Let E, E' be the algebraic closures of E, E' in K, K'. Since K, K' are algebraically closed it follows from the uniqueness of the algebraic closures that one can extend an isomorphism $f_1 : F \to F'$ to an isomorphism $\tilde{f} : E \to E'$.

b) For the proof of the submodel condition we have to show that for any pair $K \subset K'$ of algebraically closed fields and any simply existential sentence ϕ in ACF_K we have $\phi(K) = \phi(K')$. This result follows immediately from Corollary 5.8. But we give a different proof, the one which admis a generalization.

Any simply existential sentence ψ in ACF_K is equivalent to a $\exists x\psi$ where ψ is a Boolean combinations of formulas of the form

$$p(x) = 0, p(x) \in K[x]$$

(see Problem 6.6 b)). Since the field K is algebraically closed, a formula $p(x) = 0$ is equivalent to a finite union formulas of the form $x = a, a \in K$. Therefore any non-empty Boolean combinations of formulas of the form $p(x) = 0$ is either equivalent to a formula

$$[x = a_1] \wedge \ldots \wedge [x = a_n], a_l \in K$$

or to a formula

$$[x \neq a_1] \vee \ldots \vee [x \neq a_n], a_l \in K, 1 \leq l \leq n$$

It is clear now that either $\psi = \perp$ or $\exists x\psi(K) \neq \emptyset$.□

Corollary 8.18. *C:aclacf Let F be an algebraically closed field, A a subring of F containing 1 and $E \subset F$ be the field of fractions of A. Let $dcl(A), acl(A)$ be the definable and the algebraic closures of the structure A in the model F of ACF. Then*

a) $acl(A)$ is the algebraic closure $\bar{E}$ of E in F.

b) $dcl(A)$ is the perfect closure of E in F.

Proof. a) It is clear that $acl(A)$ contains the algebraic closure of E. So it is sufficient to show for any algebraically closed subfield E of F we have $acl(E) = E$. Since the theory ACF admits quantifier elimination, $E \subset F$ is an elementary submodle. Therefore [see Problem 6.2 c)] $acl(E) = E$.

b) It is clear that $dcl(A)$ contains the perfect closure of E. So it is sufficient to show for any perfect subfield E of F we have $dcl(E) = E$.

The group $Gal(\bar{E}/E)$ acts on $acl(E) = \bar{E}$ and it is clear that $dcl(E) \subset E)^{Gal(\bar{E}/E)}$. But the standard Galois theory of fields implies that $dcl(E) \subset E^{Gal(\bar{E}/E)}$ is the perfect closure of E.$\square$

Our next goal is to show that the theory RCF of real closed fields also admits QE. I start with a short reminder of the theory of real closed fields.

Definition 8.19. Let F be an ordered field. A *real closure* of F is a real closed field $E \supset F$ such that the ordering on E induces a given ordering on F and such that every element of E is algebraic over F.

Lemma 8.20. *L:or Let F be an ordered field. Then*
a) there exists a real closure $E \supset F$ of F,
b) if E, E' are two real closures of F, then there exist an order field isomorphism $f : E \to E'$ trivial on F.

A proof of this Lemma can be found on pages 452-457 of [L].

Lemma 8.21. *Any simple existential formula ψ of RCF_K is equivalent to a union of points $x = a$ and intervals $\{a < x\}, \{a < x < b\}, \{x < b\}, a, b \in K$.*

Proof of Lemma 8.21. As follows from the Problem 6.6 d) ψ is equivalent to a Boolean combinations of sentences of either of the form $p(x) = 0$ or of the form $a < p(x)$ where $p(x) \in K[x], a \in K$. Since any Boolean combinations of a union of intervals and points is itself a union of intervals and points, it is sufficient to show that sets V_ϕ where ϕ are formulas of the form $p(x) = 0$ or $a < p(x), p(x) \in K[x], a \in K$ are equivalent to unions of intervals and points.

Consider first the set $\psi = [p(x) = 0]$. I claim that in this case V_ψ is equal to the union of points which are the roots of $p(x)$ in K. For this we have to show that for any real closed extension $K' \supset K$ we have $V_\psi(K') = V_\psi(K)$. But it is clear that all elements of $V_\psi(K')$ are algebraic over K. So the result follows from Lemma 8.20.

Consider now the set $\psi = [a < p(x)]$. Let $b_1 < \ldots < b_n$ be roots of $p(x) - a$ in K. We can write $p(x) - a$ in the form

$$p(x) - a = c \prod_{1 \leq l \leq n} (x - b_l)^{n_l} q(x)$$

where $q(x)$ is a monic polynomial which does not have roots in K. The same arguments as before show that for any real closed extension $K' \supset K$ and any $t \in K'$ we have $0 < q(t)$. In other words the formula

$a < p(x)$ is equivalent to the formula $\phi' = [0 < c\prod_{1\leq l\leq n}(x-b_l)^{n_l}]$. But it is clear that $V_{\phi'}$ equivalent to a union of points and intervals.□

Theorem 8.22. *The theory RCF of real closed fields admits QE.*

As follows from the Theorem 8.11 it is sufficient to check now that the theory RCF satisfies the isomorphism and the submodel conditions .

a) For the proof of the isomorphism condition we have to show that for any pair K, K' of real closed fields, a pair of subrings $A \subset K, A' \subset K'$ and a ring isomorphism $f_0 : A \to A'$ there exist real closed subfields $E \subset K, E' \subset K'$ containing A, A' and an isomorphism $f : E \to E'$ extending $f : A \to A'$.

Let $F \subset K, F' \subset K'$ be the quotient rings of A, A'. It is clear that f_0 extends uniquely to an isomorphism $f_1 : F \to F'$ of ordered fields.

Let $E \subset K, E' \subset K'$ be subfields of elements algebraic over F, F'. It is clear that E, E' are real closed fields and therefore real closures of F. It follows now from Lemma 8.20 that there exists an isomorphism $f : E \to E'$ of ordered fields such that $f_F = f_1$.

b) let $K \subset K'$ be real closed fields. We have to show that for any simple existential sentence ϕ of L_K such that $V_\phi(K) = \emptyset$ we have $V_\phi(K') = \emptyset$.

As follows from Lemma 8.21, the sentence ϕ is equivalent to a formula $\exists x \in C$ where C is a union of points $x = a, a \in K$ and intervals with endpoints in K. It is clear then that $V_\phi(K) = \emptyset$ iff $C = \emptyset$. But then $V_\phi(K) = \emptyset$.□

Definition 8.23. We say that a set $X \subset \mathbb{R}^n$ is *semialgebraic* if $X = V_\phi(\mathbb{R})$ where ϕ is an open sentence in $OF_{\mathbb{R}}$. [In other words X is if a Boolean combination of subsets of $\mathbb{R}^n$ of the form $p(x) = 0$ or $p(x) > 0$ for $p(x) \in \mathbb{R}[x]$.]

Corollary 8.24. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a function with a semialgebraic graph and $X_f \subset \mathbb{R}^n$ be the set of points $x \in \mathbb{R}^n$ such that the limit $\lim_{y\to+\infty} f(x, y)$ exists. Then the set X_f is semialgebraic.*

Proof. Let

$$\psi = [x \in \mathbb{A}^n | \exists a \forall \epsilon > 0 \exists r \forall y > r \exists z [f(x, y) = z \wedge (z - a)^2 < \epsilon]$$

Then $X = \psi(\mathbb{R})$.

Since RCF admits QE there exists an open formula ϕ in the language of $OF_{\mathbb{R}}$ equivalent to ψ. Therefore the set $V_\psi(\mathbb{R})$ is equal to $V_\phi(\mathbb{R})$□.

Corollary 8.25. *Let V be a finite-dimensional real vector space, $P(v)$ a real-valued polynomial function on V. Define a function $y_P(x)$ for $x > 0$ by*

$$y_P(x) := min_{\|v\|^2=x} P(v)^2$$

Then there exists a non-zero $q(x,y) \in \mathbb{R}[x,y]$ such that $q(r, y_P(r)) \equiv 0$.

Proof. Let

$$\psi(x,y) = [\exists v : \|v\|^2 = x \wedge P(v)^2 = y]$$

Since RCF admits QE there exists an open formula $\phi(x,y)$ equivalent to $\psi(x,y)$. But any such $\phi(x,y)$ has a form

$$\phi(x,y) = \phi_1(x,y) \vee \ldots \vee \phi_n(x,y)$$

where $\phi_i(x,y), 1 \leq i \leq n$ is a system S_i of polynomial equalities and inequalities. Since $y_P(x)$ is function, we see that for any $i, 1 \leq i \leq n$ the system S_i contains a non-trivial equality $q_i(x,y) = 0$. Now we can take $q = \prod_{1 \leq i \leq n} q_i$.$\square$

Problem 8.26. *a) In the set-up of the Corollary 8.25 show the existence of a rational number r and a real number a such that $y_P(x) = ax^r$ for small $x > 0$,*

b) let X be the set of zeros of P in the ball $\|v\| \leq 1$. Show that if $X \neq \emptyset$ then there exist $A \in \mathbb{R}, r \in \mathbb{Q}$ such that $|P(v)| \geq Ad^r(X,v)$ where $d(X,v)$ is the distance from v to X,

c) For any polynomial $P(x) \in \mathbb{R}[x_1, ..., x_n]$ there are positive constants c, r such that

$$|P(x)| \geq c|x - X|^r, \forall x \in \mathbb{R}^n, |x| \leq 1$$

where $X \subset \mathbb{R}^n$ is the set of zeros of P and $|x| = \sqrt{x_1^2 + ... + x_n^2}$.

Remark 8.27. You can find other applications of Theorem 2 in [G]

Theorem 8.28. *T:QEV The theory AVF admits QE.*

Proof. As before we have to check that AVF satisfies the isomorphism and the submodel conditions.

a) To prove the isomorphism condition we have to show that for any pair $v : F \to \Gamma, v' : F' \to \Gamma'$ of valued fields, valued subdomains $(A, \Pi), (A', \Pi')$ and an isomorphism $f : (A, \Pi) \to (A', \Pi')$ we can extend f to an isomorphism $\tilde{f} : E \to E'$ of valued subfields of F, F' containing A, A' and such that $\Pi \subset v(E^\star), \Pi' \subset v'(E'^\star)$. By enlarging Π, Π' we can assume that Π, Π' are ordered divisible groups.

We consider first the case when $\Pi \subset v(A-\{0\})o\times\mathbb{Q}$. The existence of an isomorphism $f:(A,\Pi)\to(A',\Pi')$ shows that in this case $\Pi'\subset v'(A'-\{0\})\otimes\mathbb{Q}$. Let $E\subset F, E'\subset F'$ be the fraction fields of the rings A,A'. It is clear that f extends f to an isomorphism $f':E\to E'$ of substructures E,E' of F,F'. Let $\tilde{E},\tilde{E}'$ be the algebraic closures of E,E' in F,F'. As follows from the uniqueness of the algebraic closure [see Corollary 7.7] the isomorphism f_E extends to an isomorphism $\tilde{f}:\tilde{E}\to\tilde{E}'$. Since $\Pi\subset v(A-\{0\})\otimes\mathbb{Q}, \Pi'\subset v'(A'-\{0\})\otimes\mathbb{Q}$ valued fields E,E' are extensions of thesubstructures $(A,\Pi),(A',\Pi')$.

It is sufficient now to show that any pair E,E' of algebraically closed subfields of F,F', divisible subgroups $\Pi\subset\Gamma,\Pi'\subset\Gamma'$ containing $v(\tilde{E}^\star), v'(\tilde{E}'^\star)$ and an isomorphism $f:(E,\Pi)\to(E',\Pi')$ we can extend it to an isomorphism $\tilde{f}:(A,\Pi)\to(A',\Pi')$ where $(A,\Pi),(A',\Pi')$ are substructures of F,F' such that $\Pi=v(A-\{0\})o\times\mathbb{Q},\Pi'=v'(A'-\{0\})\otimes\mathbb{Q}$. By induction in $dim_{\mathbb{Q}}(\Pi/v(E^\star))$ we can assume that

$$\Pi=v(\tilde{E}^\star)\oplus\mathbb{Q}e, \Pi=v'(\tilde{E}'^\star)\oplus\mathbb{Q}e'$$

Choose now any $x\in F, x'\in F'$ such that $v(x)=e, v'(x')=e'$. Since x,x' do not belong to algebraically closed subfields $A.A'$ of F,F' there exists an extension of field isomorphism $f:E\to E'$ to a field isomorphism $\tilde{f}:E(e)\to E'(e)$. The same arguments as in the proof of Lemma 7.16 show that $\tilde{f}:A(e)\to A'(e)$ is an isomorphism of valued fields. So we can take $A=E(e), A'=E'(e')$.

b) The submodel condition. Let $F\subset F'$ be two models and ϕ a simply existential sentence of T. We have to show that if $V_\phi(M')\neq\emptyset$ then $V_\phi(M)\neq\emptyset$.

As follows from Proposition 7.21 b) we can assume that V_ϕ belongs to $\mathcal{C}$. But then the claim follows from Proposition 7.21 a) .□

Corollary 8.29. *L:aclacvf Let F be an algebraically closed valued field [that is a model M of AVF] and $E\subset F$ a subfield. Then*

a) $acl(E)$ is the algebraic closure $\bar{E}$ of E in F.

b) $dcl(E)=E$ iff E is perfect and the ring $\mathcal{O}_E$ is Henselian.

Proof. a) We have to show that for any algebraically closed subfield E of F we have $acl(E)=E$. If the restriction of the valuation $v:F-\{0\}\to\Gamma$ to E is not trivial then E is a model of AVF and the claim follows from Problem 6.2 c).

Consider now the case when the restriction of the valuation $v:F-\{0\}\to\Gamma$ to E is trivial. By replacing F by a bigger algebraically closed valued field we can assume the existence of elements t', t'' in F which are algebriacally independent over E and such that $v(t') \neq \bar{0}$ and $v(t'') \neq \bar{0}$. Let $E', E'' \subset F$ be the algebraic closures of $E(t'), E(t'')$. Since the restrictions of the valuation v on E' and E'' are not trivial we see that $acl(E) \subset E' \cap E''$. On the other hand since elements t', t'' are algebriacally independent over E we have $E' \cap E'' = E$. So $acl(E) = E$.

b) Assume that the field E is perfect and the ring $\mathcal{O}_E$ is Henselian. We want to show that $dcl(E) = E$. Let $\bar{E}$ be the algebraic closure of E in F. Since $\mathcal{O}_E$ is Henselian the there exists unique extension of the valuation $v : E^\star \to \Gamma$ to a valuation $\bar{v} : \bar{E}^\star \to \Gamma \times \mathbb{Q}$ on $\bar{E}$. Therefore $dcl(E) \subset E^{Gal(\bar{E}/E)} = E$.

Now we have to show that $dcl(E) \neq E$ in the case when $\mathcal{O}_E$ is not Henselian. In this case we can find a monic polynomial $p(t) \in \mathcal{O}_E[t]$ such that the reduction $\bar{p}(t) \in k[t]$ of $p(t)$ mod $\mathfrak{m}$ is a product of relatively prime factors $\bar{q}(t), \bar{r}(t)$ which does not lift to a decomposition $p(t) = q(t) \times r(t), q(t), r(t) \in \mathcal{O}_E[t]$.

Since any algebraically closed field is Henselian there exists unique lift of the decomposition $\bar{p}(t) = \bar{q}(t) \times \bar{r}(t)$ to a decomposition $p(t) = q(t) \times r(t), q(t)$ in a product of monic polynomials in $\mathcal{O}_F[t]$. But then of the polynomials $q(t), r(t)$ belong to $dcl(E)$.□

9. Galois theory and saturated models

As we have seen [Problem 6.6] for any model M of the theory ACF and a substructure $A \subset U = U(M)$ the rational closure $dcl(A)$ of A in U coincides with the set of $Gal(M, A)$-fixed points of U where $Gal(M, A)$ is the subgroup of the group of automorphisms σ of the model M such that $\sigma(a) = a$ for all $a \in A$.

Unfortunately for other theories we can not describe the rational closure $dcl(A)$ as the set of fixed points of $Gal(M, A)$.

Consider for example valued field $\bar{\mathbb{Q}}_p$ as a model of the theory AVF and a substructure [even a submodel] A of of elements of $\bar{\mathbb{Q}}_p$ which are algebraic over $\mathbb{Q}$. Then $Gal(M, A) = \{e\}$ but $A \neq U(M)$.

It is natural to ask about the existence of models M for which the Galois theory holds.

Definition 9.1. D:satur Let M be an infinite model of a theory T and A a substructure of M. We say that

a) M is *A-saturated* if any type $p \in S_1(M, A)$ of the theory $T(M, A)$ is realizable in M.

b) M is *κ-saturated* if for any substructure A of cardinality less then κ the model M is A-saturated.

c) M is *saturated* if it is $\kappa(M)$-saturated.

Remark 9.2. In the case when T is a complete theory and M is a saturated model of T we will often identify L_M -definable sets D with the subsets $D(M) \subset M^n$.

Problem 9.3. *P:satur a) Let M be a saturated model of $T, U = U(M)$ Show that*

a) for any $n \in \mathbb{N}$ the topological space $(U^n(M), \mathcal{T}(M))$ is quasi-compact.

b) Let $X_i, i \in I$ be a set of closed subsets of $U^n(M)$ such that $\kappa(I) < \kappa(U)$ and for any finite subset $I' \in I$ the intersection $\cap_{i\in I'} X_i$ is not empty. Then $\cap_{i\in I} X_i \neq \emptyset$.

c) An algebraically closed field [considered as a model of ACF] is saturated iff it has infinite trancendence degree over it's prime subfield.

d) If M a saturated model of T then any type $p \in S_n(T(M,A))$ is realized.

e) If M a saturated model of T then topological spaces $(U(M)^n, \mathcal{T}(M)), n \in \mathbb{N}$ [see Definition 6.7 l)] are quasi-compcat.

Proposition 9.4. *P:count*
A theory T with a countable language L has a countable saturated model iff sets $S_n(T)$ are countable for all $n \in \mathbb{N}$.

We start with the folloing result.

Lemma 9.5. *Let T be theory such that all the sets $S_n(T), n \in \mathbb{N}$ are countable. Then for any model M of T and any finite subset $A \subset U(M)$ all the sets $S_n(T(M,A)), n \in \mathbb{N}$ are countable.*

Proof of Lemma. By the definition any formula ψ of L_A has a form

$$\psi = \tilde{\psi}_{x_1,\dots,x_m}[u_1,\dots,u_m]$$

for some formula $\tilde{\psi}$ of L. Therefore the set $S_m(T(M,A)), m \in \mathbb{N}$ is a subset of $\cup_{n=1}^{\infty} S_n(T)$.$\square$

Proof of Proposition 9.4. a) Suppose that T has a countable saturated model M. Then any type $p \in S_n(T)$ is realizable by some $\bar{u} \in U(M)^n$. Since $U(M)$ is countable the set $S_n(T)$ is also countable.

b) Assume that all the sets $S_n(T)$ are countable. Then (see Theorem 4.1) there exists a countable model M_0 of T. Now it follows from Problem 6.11 a) that there exists an elementary monomorphism $M_0 \prec M_1$ such that for any finite set $A \subset U(M)$ all the types $S_n(T(M, A)), n \in \mathbb{N}$ are realizable in M_1 and $U(M_1)$ is countable.

By induction we can define elementary monomorphism $M_n \prec M_{n+1}$ such that all the types $T(M_n)$ are realized in M_{n+1} and the sets $U(M_n)$ are countable. Let M be the direct limit of the direct system $M_0 \prec M_1 \dots \prec M_n \dots$. Since any type of M comes from a type of M_n for some $n \in \mathbb{N}$ we see that M is saturated. Since the set $U(M)$ is a direct limit of countable sets $U(M_n)$ it is countable.□

The validity of the next result Proposition is conditional - it depends on the acceptence of the Generalized continuum hypothetisis.

Proposition 9.6. *P:satur If T is a theory with countable language then for any regular uncountable cardinality κ there exists a saturated model M of T of cardinality κ.*

Let N be a model of T of cardinality κ. As follows from the GCH the cardinality of the set of all subsets of $U(N)$ of cardinality $< \kappa$ is also equal to κ. As follows from Problem 6.11 a) there exists an elementary monomorphism $N \prec N'$ such that $\kappa(N') = \kappa$ and for any subset $A \subset U = U(N)$ of cardinality κ all the types $p \in S_1(T, A)$ are realized in N'.

We choose a model M_0 of T of cardinality κ and construct by the transfinite induction an elementary directed system of models $M_\delta, \delta \leq \kappa, \kappa(M_\delta) = \kappa$ such that for any $\delta < \kappa$ and any subset $A \subset U = U(M_\delta)$ of cardinality κ all the types $p \in S_1(T, A)$ are realized in $M_{\delta+1}$.

If $\delta = \gamma + 1$ we define $M_\delta = M'_\gamma$. On the other hand if δ is a limit ordinal we define $M_\delta = \varinjlim M_\gamma, \gamma < \delta$. It is easy to see that for any $\delta \leq \kappa$ we have $\kappa(M_\delta) \leq \kappa$.

I claim that $M = M_\kappa$ is a saturated model of T. We have to show that for any subset $A \subset U = U(M)$ of cardinality $< \kappa$ any type $p \in S_1(T, A)$ is realizable in M. Since the cardianl κ is regular there exists $\delta < \kappa$ such that $A \subset U(M_\delta)$. But then any $p \in S_1(T, A)$ is realizable in $M_{\delta+1} \prec M$.□

To show that the Galois theory holds for saturated models we start with the following definition.

Definition 9.7. D:homog Let T be a theory with the language L and M be a model of T.

a) If $f : A \to B$ is a bijection between subsets of $U = U(M)$ and ψ is a sentence of L_A we denote by $f(\psi)$ the sentence in L_B obtained by the substitution of every occurence of $a \in A$ in ψ by $f(a) \in B$. [In other words, if $\psi = \tilde{\psi}_{x_1,...,x_r}[a_1, ..., a_r]$ where $\tilde{\psi}$ is formula in L with free variables $x_1, ..., x_r$ then $f(\psi) = \tilde{\psi}_{x_1,...,x_r}[f(a_1), ..., f(a_r)]$].

b) We say that a bijection $f : A \to B$ between subsets and A, B of $U = U(M)$ is an *elementary partial automorphism* of M if a sentence ψ of L_A is true in M iff the sentence $f(\psi)$ is true in M. In this case we write

$$< M, a >_{a \in A} \equiv < M, f(a) >_{a \in A}$$

c) We say that an elementary partial automorphism $f : A \to B$ is *immediately extendable* if for any $u \in U - A$ there exists an element $u' \in U - B$ such that the extention of f to a bijection $f' : A \cup \{u\} \to B \cup \{u'\}$ is an elementary partial automorphism.

d) We say that the model M is *homogeneous* if any elementary partial automorphism $f : A \to B$ such that $\kappa(A) < \kappa(U)$ is immediately extendable.□

Lemma 9.8. *L:homog If M a homogeneous and A a subset of U of cardinality less then $\kappa := \kappa(U)$. Then every elementary partial automorphism $f : A \to B$ of M of can be extended to an elementary automorphism of M.*

Proof Using the transfinite induction it is easy to extend f to an elementary partial automorphism $f' : U \to C \subset U$. On the other hand if one works with $f^{-1} : B \to A$ it is easy to extend f to an elementary partial automorphism $f'' : C \to U$. So it is not surprising that the construction of an extensions of f to an elementary automorphism of M is based on the back-and-forth technique when on even steps we build extensions of f and on odd steps extensions of f^{-1}.

By the assumption $\kappa(U - A) = \kappa(U - B) = \kappa$. We can write

$$U - A = \{x_\delta | \delta < \kappa\}, U - B = \{y_\delta | \delta < \kappa\}$$

Now we define elements $c_\delta \in U - A, d_\delta \in U - B, \delta < \kappa\}$ by induction in δ in such a way that the extensions f_δ of f to a map

$$f_\delta : A \cup \cup_{\gamma<\delta} c_\gamma \to B \cup \cup_{\gamma<\delta} d_\gamma, f_\delta(c_\gamma) = d_\gamma$$

are elementary partial automorphisms.

If δ is a limit ordinal and f_γ is defined for all $\gamma < \delta$ then f_δ is also defined. So we have only to explain how to define $f_{\gamma+1}$ if f_γ is already defined.

Case 1. γ is even [see Definition 3.1 e)] .

Let c_γ be the element in $U - (A \cup \cup_{\epsilon<\gamma} c_\epsilon)$ with least subscript. Since

$$f_\gamma : A \cup \cup_{\epsilon<\gamma} c_\epsilon \to B \cup \cup_{\epsilon<\gamma} d_\epsilon$$

is an elementary partial automorphism and M is homogeneous we can find $b \in U - (B \cup \cup_{\epsilon<\gamma} d_\epsilon)$ such that

$$< M, a, c_\epsilon, c_\gamma >_{a\in A, \epsilon<\gamma} \equiv < M, f(a), d_\epsilon, b >_{a\in A, \epsilon<\gamma}$$

and define $d_\gamma = b$

Case 2. γ is odd.

Apply the same procedure to f^{-1}. Let d_γ be the element in $U - (B \cup \cup_{\epsilon<\gamma} d_\epsilon)$ with least subscript. Since M is homogeneous we can find $\tilde{a} \in U - (A \cup \cup_{\epsilon<\gamma} c_\epsilon)$ such that

$$< M, a, c_\epsilon, \tilde{a} >_{a\in A, \epsilon<\gamma} \equiv < M, f(a), d_\epsilon, d_\gamma >_{a\in A, \epsilon<\gamma}$$

and define $c_\gamma = \tilde{a}$.

It is easy to see that the maps f_δ are elementary partial automorphisms and the map f_κ is a bijection which defines an elementary automorphism of M extending f.□

Problem 9.9. *P:iso a) Let M, M' be two saturated elementary equivalent models of the same cardinality. Then M and M' are isomorphic.*

A hint. Use the back-and-forth arguments as in the proof of Lemma 9.8.

b) Let M, M' be two homogeneous models of a theory T of the same cardinality which realize the same set of n-types for all $n \in \mathbb{N}$. Then M and M' are isomorphic.

c) Let ω be the countable cardinal, $\omega_1 = \omega^+, \omega_{n+1} = \omega_n^+, \omega_\omega = \cup_{n\in\mathbb{N}} \omega_n$. Show that the theory LO of linear ordering does not have a saturated model of the cardinality ω_ω.

Now we develope the Galois theory for saturated models.

Definition 9.10. a) Let M be a saturated model of a theory T with a countable language, $U = U(M), Gal(M)$ be the group of automorphisms of M and D be an M-definable subset of U^n. We define

a) $Gal(M, D) := \{\sigma \in Gal(M) | \sigma(x) = x \forall x \in D(M)\}$.

b) $St(D) = \{\sigma \in Gal(M) | \sigma(D(M)) = D(M)\}$.

c) $\widetilde{dcl}(D) = \{\bar{u} \in U^\star | \sigma(\bar{u}) = \bar{u}, \forall \sigma \in St(D)\}$.

d) $\widetilde{acl}(D))$ is the set of all points $\bar{u} \in U^\star$ such that there exists a finite $St(D)$-invariant definable set D containing $\bar{u}$.

e) We say that $\bar{u} \in U^m$ is a *canonical parameter* of D if $Gal(M, \bar{u}) = St(D)$.

f) We say that $\bar{u} \in U^m$ is an *alomost canonical parameter* of D if $Gal(M, \bar{u})$ is a subgroup of finite index of $St(D)$.

Lemma 9.11. *L:homsat Let M be a saturated model of a theory T with a countable language, $A \subset U(M)$ be a finite subset and $Gal(M)$ be the group of automorphisms of M.*

a) M is homogeneous.

b) Let $D \subset U^n$ be a L_M-definable subset such that the subset $D(M) \subset M^n$ is $Gal(M)$-invariant. Then D is T-definable.

c) If $\bar{v} = (v_1, \dots, v_m \in U^m$ is a canonical parameter of an M-definable subset $D \subset U^n$ then there exists a formula $\psi(x_1, \dots, x_n; y_1, \dots, y_m)$ such that

$$\{\bar{v}\} = \{\bar{w} \in U^m | D(M) = V_{\psi_{y_1, \dots, y_m}[\bar{v}]}\}$$

Proof of Lemma 9.11. a) Let $f : A \to B$ an elementary partial automorphism of M and u an elelment of U. Let $p_A(u) \in S_1(T_A)$ be the type of u and $q \subset B_1(T_B)$ be the set of formulas of the form $f(\psi)$ for $[\psi] \in p$. Since $f : A \to B$ is an elementary partial automorphism of M we have $q \in S_1(T_K)$. But then the extension $f' : A \cup \{u\} \to B \cup \{u'\}$ of f is an elementary partial automorphism.

b) Let $B_n(D)$ be the set of formulas ψ such that $D(M) \subset V_\psi := V_\psi(M)$ and consider an ∞-definable set

$$\tilde{D} := \cap_{\psi \in B_n(D)} V_\psi$$

of U^n. It is clear that $D(M) \subseteq \tilde{D}(M)$. We first show that an assumption that $\tilde{D} \nsubseteq D$ leads to a contrudiction.

Suppose that $\tilde{D} \nsubseteq D$. Since the model M is saturated there exists $u \in \tilde{D}(M) - D(M)$.

Let $B_n(u)$ be the set of formulas ϕ of L such that $u \subset V_\phi$. I claim that

$$D(M) \cap V_\phi \neq \emptyset$$

for any $\phi \in B_n(u)$. Really if

$$D(M) \cap V_\phi = \emptyset$$

then $\neg\phi \in B_n(D)$. But this would contrudict the assumption that $u \in \tilde{D}(M)$.

Since $D(M) \cap V_\phi \neq \emptyset$ for any ϕ in the countable set $B_n(u)$ and the model M is saturated we have (see Problem 9.6)

$$D(M) \cap \cap_{\phi \in B_n(u)} V_\phi(M) \neq \emptyset$$

Choose any $v \in D(M) \cap \cap_{\phi \in B_n(u)} V_\phi(M)$. By the constuction the types of u and v coincide and the the map $u \to v$ is an elementary partial automorphism. Since M is homogeneous it extends to an automorphism $\sigma \in Gal(M)$ such that $\sigma(u) = v$. Since $v \notin D(M)$ such an equality would contrudict the assumption that $D(M)$ is $Gal(M)$-invariant.

Now it follows from Lemma 6.16 that D is a T-definable set.

c) Let $\bar{v} = (v_1, \ldots, v_m) \in U^m$ be a canonical parameter of an M-definable subset $D \subset U^n$ and $T_{\bar{v}} = T(M, \bar{v})$ [see Definition 2.21 d)]. Since $Gal(T, \bar{v}) \subset St(X)$ and $Gal(T_{\bar{v}}) = Gal(T, \bar{v})$ it follows from the part b) that there eixsts a formula $\psi_0'(x_1, \ldots, x_n)$ in the language $L_{\bar{v}}$ such that $D(M) = V_{\psi_0'}(M)$. By the definition of the language $L_{\bar{v}}$ there exists a formula $\psi_0(x_1, \ldots, x_n; y_1, \ldots, y_m)$ in the language L of T such that

$$\psi_0'(x_1, \ldots, x_n) = \psi_{0, y_1, \ldots, y_m}[v_1, \ldots, v_m]$$

Let Ψ be the set of all the formulas $\psi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ such that

$$D(M) = V_{\psi_{y_1, \ldots, y_m}[v_1, \ldots, v_m]}(M)$$

For any $\psi \in \Psi$ we define

$$W_\psi = \{(w_1, \ldots, w_m) \in U^m | D(M) = V_{\psi_{y_1, \ldots, y_m}[w_1, \ldots, w_m]}(M)\}$$

Since $Gal(T, \bar{v}) \supset St(X)$, the same arguments as in the proof of b) show that

$$\{\bar{v}\} = \cap_{\psi \in \Psi} W_\psi$$

Now the same arguments as in the proof of Corollary 6.14 show the existence of a formula $\phi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ such that

$$\{\bar{v}\} = \{(w_1, \ldots, w_m) \in U^m | D(M) = V_{\phi_{y_1, \ldots, y_m}[w_1, \ldots, w_m]}(M) \square$$

Corollary 9.12. *C:Gal a) Any subset A of U of cardinality smaller then $\kappa(U)$ we have $dcl(A) = U^{Gal(M,A)}$ where*

$$Gal(M, A) = \{\sigma \in Gal(M) | \sigma(a) = a \forall a \in A\}$$

Proof. By replacing the theory T with $T(M, A)$ we can assume that $A = \emptyset$. But in this case Corollary 9.12 follows from Lemma 9.11 b) if $D = \{u\}$.$\square$

Definition 9.13. D:addrel Let T be theory of the type $\tau =< I, J, K, n, m >$ and let M be a saturated model of $T, U = U(M), X \subset U^n$ be a definable set. We define

a) $\tau_X =< I, J \cup \star, K, n_\star, m >$ where $n_\star$ is the extension of n such that $n_\star(\star) = n$.

b) M_X is the extension of M to a τ_X such that $\star^{M_X} = X$.

c) $T_X = T(M_X)$

Corollary 9.14. *C:addrel For any $u \in U^{St(X)}$ there exists a formula ψ in the language L_{τ_X} such that $V_\psi = \{u\}$*

Proof. Apply Lemma 9.11 b) for the theory is T_X to the case when $D = \{u\}$.

10. Imaginary elements

In this section we will assume that T is a complete theory with a countable language L, M is a saturated model of T and write $U = U(M)$.

Definition 10.1. D:parameter a) A subset $D \subset M^n$ is M-definable if there exists a finite subset $A \subset M$ and a formula ψ in the language L_A such that $D = V_\psi(M)$.

b) A subset $D \subset M^n$ is definable if there exists a formula ψ in the language L such that $D = V_\psi(M)$.

c) If X is an M-definable subset a *canonical parameter* of X is a point $\bar{u} \in M^\star$ such that $St(X) = Gal(M, \bar{u})$.

d) An *almost canonical parameter* of X is a point $\bar{u} \in M^r$ such that $St(X) \subset Gal(M, \bar{u})$ is a subgroup of finite index.

Example 10.2. E:parameter Consider a subset $\{(a, b) \cup (b, a)\} \subset M^2$. Then

a) in the theory TF the point $(a + b, ab)$ is a canonical parameter of the set (a, b).

b) in the theory of AG of abelian groups the set (a, b) does not have a canonical parameter.

There is a class of theories for which a M-definable subset $D \subset M^n$ admits an almost canonical parameter.

Definition 10.3. D:str a) A theory T is strongly minimal if form any model M of any M-definable subset of U in $T(M)$ is either a finite set or a complement to a finite set.

Example 10.4. E:str The theory and ACF are strongly minimal but the theory TF is not strongly minimal since it has an extension RCF which is not strongly minimal.

Lemma 10.5. *L:wea Let T be a strongly minimal theory such that for any model N of T the set $acl(N^0)$ is infinite [see Definition 2.1 e) and Definition 6.1], M be a saturated model of T and $\sim$ a definable equivalence relation of on M^n. Then any equivalence calss $D \subset M^n$ admits an almost canonical parameter.*

Proof. Since the equivalence relation $\sim$ is T-definable any point $\bar{u} \in D$ such that the orbit $St(D)(\bar{u})$ is finite is an almost canonical parameter for D. So is sufficien to show that any M-definable $D \subset M^n$ contains a point $\bar{u}$ such that the orbit $St(D)(\bar{u})$ is finite. We prove the existence of such a point by induction in n.

Consider the projection $X \subset U$ of D onto the n-th component. Since the theory T strongly minimal either X is finite or $U - X$ is finite.

If X is finite and choose a point $u \in X$. Since $St(D)(u) \subset X$ we see that the orbit $St(D)(u)$ is finite. Let $St'(D)$ be the stabilizer of u in $St(D)(u)$. Then

$$D' := D \cap U^{n-1} \times \{u\} \neq \emptyset$$

and $St(D') \supset St'(D)$. By the inductive assumption there exists $\bar{u} \in D'$ such that the orbit $St(D')(\bar{u})$ is finite. Therefore the orbit $St(D)(\bar{u})$ is also finite.

Assume now that $U - X$ is finite. Since the set $acl(M^0)$ is infinite we can find $u \in acl(M^0) \cap X$. Let $St'(D)$ be the stabilizer of u in $St(D)(u)$. Then

$$D' := D \cap U^{n-1} \times \{u\} \neq \emptyset$$

and $St(D') \supset St(D')$. By the inductive assumption there exists $\bar{u} \in D'$ such that the orbit $St(D')(\bar{u})$ is finite. Therefore the orbit $St(D)(\bar{u})$ is also finite.□

Definition 10.6. D:EI A theory T admits an *elimination of imaginaries (EI)* if for any model M of T all M-definable subsets admit a canonical parameter.

Lemma 10.7. *L:EI Let T be a theory of a 1-sorted similarity type. Then the following three conditions are equivalent.*

(1) A theory T admits EI.

(2) For any equivalence relation $\sim$ on U^r there exists a definable map $p : U^r \to U^l$ such that $x \sim y \leftrightarrow p(x) = p(y)$.

(3) For any equivalence relation $\sim$ on U^r and any model M of T all equivalence classes $X \subset M^r$ of $\sim$ admit canonical parameters.

Proof. (3) $\Longrightarrow$ (1). Let D be an M-definable subset of U^r. Then there exists a formula $\psi(x_1, \dots, x_r; y_1, \dots, y_n)$ in L and $\bar{u}^0 = (u_1^0, \dots, u_n^0) \in U^n$ such that $D = V_{\psi(\bar{u}^0)}$ where $\psi(\bar{u}^0) = \psi_{y_1,\dots,y_n}[u_1^0, \dots, u_n^0]$. We define an eqivalence relation of U^n by $\bar{u} \sim \bar{v} \leftrightarrow V_{\psi(\bar{u})} = V_{\psi(\bar{v})}$. Let $X \subset M^n$ be the equivalence class of $\bar{u}^0$. By (3) X admits a canonical parameter $\bar{w} \in U^m$. But it follows now from Lemma 9.11 b) that $\bar{w}$ is a canonical parameter of D.

(1) $\Longrightarrow$ (3). Clear.

(2) $\Longrightarrow$ (3). Let $X \subset M^r$ be an equivalence class of an equivalence relation $\sim$ on U^r and $p : U^r \to U^l$ be a definable map such that $x \sim y \leftrightarrow p(x) = p(y)$. Then $\bar{w} = p(X)$ is a canonical parameter of X.

(3) $\Longrightarrow$ (2). For any $\bar{v} \in U^r$ we denote by $\bar{u}_{\bar{v}} \in U^{m(v)}$ a canonical parameter of the equivalence class $D_{\bar{v}}$ of $\bar{v}$. As follows from Lemma 9.11 c) for any $\bar{v} \in U^r$ there exists a formula $\psi^{\bar{v}}(x_1, \dots, x_n; y_1, \dots, y_{m(\bar{v})})$ such that

$$D_{\bar{v}}(M) = \{(u_1, \dots, u_n) \in U^n | \psi^{\bar{v}}_{y_1,\dots,y_{m(\bar{v})}}[u_1, \dots, u_{m(\bar{v})}] = \{\bar{u}\}\}$$

Let $\mathcal{U}_{\bar{v}}$ be the set of $\bar{w} \in U^{m(\bar{v})}$ such that the set

$$\{\bar{w} \in U^{m(v)} | D_{\bar{w}}(M) = \{(u_1, \dots, u_n) \in U^n | \psi^{\bar{v}}_{y_1,\dots,y_{m(\bar{v})}}[\bar{w}]$$

By the definiton $\mathcal{U}_v$ is an open subset of $U^{m(v)}$ containing $\bar{v}$. It follows now from Problem 9.6 there exists a finite subset $(\bar{v}_1, \dots, \bar{v}_l)$ of U^r such that $U^r = \cup_{i=1}^l \mathcal{U}_{\bar{v}_i}$. Let $V_i = \mathcal{U}_{\bar{v}_i} - (\cup_{j<i} \mathcal{U}_{\bar{v}_j} \cap \mathcal{U}_{\bar{v}_i})$. Then U^r is a disjoint union of $V_i, 1 \leq i \leq l$. For any $i, 1 \leq i \leq l$ the formula the restriction of the formula $\psi^{\bar{v}}_{y_1,\dots,y_{m(v)}}[u_1, \dots, u_{m(v)}]$ to V_i is a definable p_i map from V_i to $U^{m(\bar{v}_i)}$. Since U^r is a disjoint union of $V_i, 1 \leq i \leq l$ we can define $p : U^r \to \oplus U^{m(\bar{v}_i)}, 1 \leq i \leq l$ in such a way that the restriction of p on V_i is given by p_i.$\square$

Problem 10.8. *P:EI Formulate the analog of Lemma 10.7 for many sorted theories.*

Lemma 10.9. *L:EI1 The following two conditions are equivalent*

(1) A theory T admits EI.

(2) For any equivalence relation $\sim$ on $\mathbb{A}^r$ there exists a definable map $p : \mathbb{A}^r \to \mathbb{A}^l$ such that $x \sim y \leftrightarrow p(x) = p(y)$.

(3) Any equivalence class $X \subset M^r$ of an equivalence relation $\sim$ on $\mathbb{A}^r$ has a canonical parameter.

Proof. (3) $\implies$ (1). Let D be an M-definable subset of $\mathbb{A}^r$. Then there exists a formula $\psi(x_1,\dots,x_r;y_1,\dots,y_n)$ in L and $\bar{u}^0=(u_1^0,\dots,u_n^0)\in U^n$ such that $D=V_{\psi(\bar{u}^0)}$ where $\psi(\bar{u}^0)=\psi_{y_1,\dots,y_n}[u_1^0,\dots,u_n^0]$. We define an eqivalence relation of $\mathbb{A}^n$ by $\bar{u}\sim\bar{v}\leftrightarrow V_{\psi(\bar{u})}=V_{\psi(\bar{v})}$. Let $X\subset M^n$ be the equivalence class of $\bar{u}^0$. By (3) X admits a canonical parameter $\bar{w}\in U^m$. But it follows now from Lemma 9.11 b) that $\bar{w}$ is a canonical parameter of D.

(1) $\implies$ (3). Clear.

(2) $\implies$ (3). Let $X\subset M^r$ be an equivalence class of an equivalence relation $\sim$ on $\mathbb{A}^r$ and $p:\mathbb{A}^r\to\mathbb{A}^l$ be a definable map such that $x\sim y\leftrightarrow p(x)=p(y)$. Then $\bar{w}=p(X)$ is a canonical parameter of X.

(3) $\implies$ (2). For any $\bar{v}\in U^r$ we denote by $\bar{u}_{\bar{v}}\in U^{m(v)}$ a canonical parameter of the equivalence class $D_{\bar{v}}$ of $\bar{v}$. As follows from Lemma 9.11 c) for any $\bar{v}\in U^r$ there exists a formula $\psi^{\bar{v}}(x_1,\dots,x_n;y_1,\dots,y_{m(\bar{v})})$ such that

$$D_{\bar{v}}(M)=\{(u_1,\dots,u_n)\in U^n|\psi^{\bar{v}}_{y_1,\dots,y_{m(\bar{v})}}[u_1,\dots,u_{m(\bar{v})}]=\{\bar{u}\}\}$$

Let $\mathcal{U}_{\bar{v}}$ be the set of $\bar{w}\in U^{m(\bar{v})}$ such that the set

$$\{barw\in U^{m(v)}|D_{\bar{w}}(M)=\{(u_1,\dots,u_n)\in U^n|\psi^{\bar{v}}_{y_1,\dots,y_{m(\bar{v})}}[\bar{w}]$$

By the definiton $\mathcal{U}_v$ is an open subset of $U^{m(v)}$ containing $\bar{v}$. It follows now from Problem 9.6 there exists a finite subset $(\bar{v}_1,\dots,\bar{v}_l)$ of U^r such that $\mathbb{A}^r=\cup_{i=1}^l\mathcal{U}_{\bar{v}_i}$. Let $V_i=\mathcal{U}_{\bar{v}_i}-(\cup_{j<i}\mathcal{U}_{\bar{v}_j}\cap\mathcal{U}_{\bar{v}_i})$. Then $\mathbb{A}^r$ is a disjoint union of $V_i, 1\le i\le l$. For any $i,1\le i\le l$ the formula the restriction of the formula $\psi^{\bar{v}}_{y_1,\dots,y_{m(v)}}[u_1,\dots,u_{m(v)}]$ to V_i is a definable p_i map from V_i to $\mathbb{A}^{m(\bar{v}_i)}$. Since $\mathbb{A}^r$ is a disjoint union of $V_i, 1\le i\le l$ we can define $p:\mathbb{A}^r\to\oplus\mathbb{A}^{m(\bar{v}_i)}, 1\le i\le l$ in such a way that the restriction of p on V_i is given by p_i.□

Corollary 10.10. *C:EIACF The theory ACF admits the elimination of imaginaries.*

Proof. We have to show that for any equivalence relation $\sim$ on $\mathbb{A}^r$ any equivalence class $X\subset M^r$ of $\sim$ on $\mathbb{A}^r$ has a canonical parameter. Since the theory ACF is strongly minmal it follows from rlwea that we can find $\bar{u}=(u_1,\dots,u_n)$ such that the orbit $St(X)(\bar{u})$ is finite. So we can write

$$St(X)(\bar{u})=\cup_{i=1}^m\bar{u}^i,\bar{u}^i=(u_1^i,\dots,u_n^i),1\le i\le m$$

Then coefficents $c_r, r \in R$ of the polynomial

$$P(Z, T_1, \dots, T_n) = \prod_{i=1}^{m} (Z + u_1^i T_1 + \dots u_n^i T_n)$$

are $St(X)$-invariant. It is clear that the point $(c_1, \dots, c_R) \in U^\star$ is a canonical base of $X.\square$.

If the theory T does not admit EI then one can extend it to a theory T^{eq} which already admit EI by adjoining *imaginary elements*. For the simplicity I assume that the theory T is complete and that the type $\tau =< I, J, K, m, n >$ of T has only one sort λ_0.

Definition 10.11. D:eq We define T^{eq} as a many-sorted theory where the set Λ of sorts as the union $\Lambda = 0 \cup \cup_{E \in \mathcal{E}}$ where $\mathcal{E}$ is the set of all the definable equivalence realtions $E \subset \mathbb{A}^{m(E)} \times \mathbb{A}^{m(E)}$ in such a way that

a) The type $\tau^{eq} =< I^{eq}, J^{eq}, K^{eq}, n^{eq}, m^{eq} >$ is the type with sorts Λ such that

i) $I^{eq} = I, J^{eq} = J \cup \cup_{E \in \mathcal{E}} j_E, K^{eq} = K$

ii) the restriction of n^{eq} and m^{eq} on I coincides with n and m.

iii) $m^{eq}(j_E) = (0^{r(E)}, E)$.

b) The structure M^{eq} of the type τ^{eq} as the extension of the model M of T with

i) the universe $U_0 = U(M)$ and U_E be the set of equivalence classes of $M^{r(E)}$ under the equivalence relation $E(M)$.

ii) the functions $f_{j_E} : U(M)^{r(E)} \to U_E$ be the natural projections.

c) The theory $T^{eq}(M)$ is the complete theory $T(M^{eq})$ of the structure M^{eq}.

Problem 10.12. *P:image Show that a) The assignment $M \to M^{eq}$ is a functor from the category $\mathcal{M}(\tau)$ to the category $\mathcal{M}(\tau^{eq})$ [see Definition 6.1] which defines an equivalence of categories.*

b) If the theory T is complete then the theory $T^{eq}(M)$ does not depent on a choice of a model M.

Remark 10.13. a) Since for a complete theory T all the theories $T^{eq}(M)$ are isomorphic we denote the corresponding theories as T^{eq}.

b) Often people identify the type 0 with the type =.

Problem 10.14. *P:eq If M a model of a complete theory T and $\bar{u}, \bar{u}' \in U^n(M)$ are such that $p_{\bar{u}}(T) = p_{\bar{u}'}(T)$ then $p_{\bar{u}}(T^{eq}) = p_{\bar{u}'}(T^{eq})$ [see Definition 6.7 c)].*

A hint. The proof is analogous to the proof of Lemma 9.11.

Lemma 10.15. *L:eq For any complete theory theory T the theory T^{eq} admits EI.*

Proof. Let M be a saturated model of T. We have to show that for any T^{eq}-definable equivalence relation $\sim$ on a $(U_{E_1} \cup \ldots \cup U_{E_k})^n$ there exists a T^{eq}-definable set D and T^{eq}-definable map $f : U_{E_1} \cup \ldots \cup U_{E_k} \to D$ such that $\bar{u} \sim \bar{u}'$ iff $f(\bar{u}) = f(\bar{u}')$. For simplicity of notations I assume that $n = 1$.

Let $\sim_i$ be the equivalence relations on U^{r_i} corresponding to $E_i, 1 \leq i \leq k$ and m be the maximum of r_i. Let $A \subset U^{m+k+1}$ be the set of $(m+k+1)$-tuples $(a_0, a_1, \ldots, a_k; b_1, \ldots, b_m), b_i \in U^{r_i}$ such that there exists unique $i, 1 \leq i \leq k$ such that $a_0 = a_i$. We consider an equivalence relation $\sim'$ on A such that

$(a_0, a_1, \ldots, a_k; b_1, \ldots, b_m) \sim' (a'_0, a'_1, \ldots, a'_k; b'_1, \ldots, b'_m)$ iff
$(b_1, \ldots, b_{r_i}) \sim (b'_1, \ldots, b'_{r_j})$ where i, j are such that $a_0 = a_i, a'_0 = a'_j$

and extend it to an equivalence relation E on U^{m+k+1} by declairing $U^{m+k+1} - A$ as one equivalence class. By the construction the equivalence relation E is T^{eq}-definable. It follows now from Problem 10.14 [see the proof of Lemma 9.11] that it is T-definable and we have a T^{eq}-definable map $f_E : U^{m+k+1} \to U_E$. Consider now the T^{eq}-definable map $f : U_{E_1} \cup \ldots \cup U_{E_k} \to U_E$ such that the restriction of $f(b_1, \ldots, b_{r_i}), (b_1, \ldots, b_{r_i}) \in U_{E_i}, 1 \leq i \leq k$ is equal to $f_E(a_0, a_1, \ldots, a_k; b_1, \ldots, b_m)$ where $(a_0, a_1, \ldots, a_k; b_1, \ldots, b_m) \in A$ is any points such that $a_0 = a_i$ and $b_j \in U, r_i < j \leq m$ are arbitrary. It is clear that the T^{eq}-definable map f is well defined and fibers of f coincide with equivalence classes of $\sim$.□

Definition 10.16. D:VF We denote by VF' the extension of the VF by elements of the residue field which we interpret as imaginary elements corresponding to the equivalence on $\mathcal{O} \subset \mathbb{A}$ corresponding to cosets of the addition by $\mathfrak{m}$.

Remark 10.17. The theory VF' has three sorts F, Γ, k which correspond to elements of the valued field, the value group and the residue field.

If F is a valued field which is a local field [so either the field F is a finite extension of $\mathbb{Q}_p$ or it is isomorphic to the field $\mathbb{F}_q((t))$] then the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ is principle and one can chose a generator π of $\mathfrak{m}$. In the proof of the Ax-Kochen theorem we will use an extension $\to VF$ of VF' which axiomatizes the idea of fixing a generator of $\mathfrak{m}$. The corresponding language is called *Denef -Pas* language.

Definition 10.18. D:tVF We denote by $\tilde{V}F$ the extension of the VF by a functional symbol ac from the sort F to the sort k which is the semigroup homomorphism from the multiplicative group of F to the multiplicative group of k such that the restriction of ac on $\mathcal{O}^\star$ coincides with the restriction of the natural map from $\mathcal{O} \to k$ on $\mathcal{O}^\star$.

11. Ultrafilters

Definition 11.1. D:ultraf Let I be a non-empty set.

a) We denote by $\Lambda(I)$ the set of subsets of I.

b) A *filter* D over I is a subset $D \subset \Lambda(I)$ such that

i) $I \in D$

ii) if $X, Y \in D$ then $X \cap Y \in D$

iii) if $X \in D$ and $X \subset Y \subset$ then $Y \in D$

iv) $\emptyset \notin D$

c) An *ultrafilter* is a maximal filter.

d) An ultrafilter is ω *-regular* if there exists a countable decreasing chain

$$I = I_0 \supset I_1 \supset \dots$$

of elements $I_n \in D$ such that $\cap_n I_n = \emptyset$.

e) An ultrafilter D is principal if there exists $i \in$ such that $D = \{X \subset I | i \in X\}$.

Definition 11.2. D:ultrap a) If D is an ultrafilter over I and $U_i, i \in I$ are sets we define an equivalence relation $\sim_D$ on $\prod U_i$ by

$\{u_i\} \sim_D \{u_i'\}$ if the set $X = \{i \in I | u_i = u_i'\}$ belongs to D.

b) If D is a filter over I and $U_i, i \in I$ are sets we define $\prod_D U_i$ as the set of equivalence classes by the equivalence relation $\sim_D$ and for any $u_i \in \prod U_i$ we denote by $< u_i > \in \prod_D U_i$ it's equivalence class.

c) if $M_i, i \in I$ are structures of the same type τ we denote by $\prod_D M_i$ the structure of the type τ such that

i) $U(M) = \prod_D U(M_i)$

ii) for any n-placed relation symbol R and $\bar{u}^l \in U(M), 1 \leq l \leq n$ we have $R^M(\bar{u}^1, \ldots, \bar{u}^n)$ iff the set $X = \{i \in I | u_i R^{M_i}(u_i^1, \ldots, u_i^n)\}$ belongs to D where $\{u_i^l\} \in \prod U(M_i)$ are representatives of $\bar{u}^l, 1 \leq l \leq n$

iii) for any m-placed function symbol f of L_τ we have

$$f^M(\bar{u}^1, \ldots, \bar{u}^n) = \{f^{M_i}(u_i^1, \ldots, u_i^m)$$

iv) if c is a constant of L_τ then $c^M\{c^{M_i}\}$

d) if all the models M_i coincide with a fixed model M we denote $\prod_D M_i$ by $\prod_D M$ and call it the ultrapower of M.

Problem 11.3. *P:ultra a) If $E \subset \Lambda(I)$ is a set such that for any finite set $X_1, , \ldots, X_n$ of elemtns in E we have $\cap_{i=1}^n X_i \neq \emptyset$ then there exists an ultrafilter D over I containing E.*

b) If $M = \prod_D M_i$ then

i) for any term $t(x_1, \ldots, x_n)$ of L_τ and $\bar{u}^l \in U(M), 1 \leq l \leq n$ we have

$$t_M(\bar{u}^l) =< t_{M_i}(u_i^l) >$$

where $\{u_i^l\} \in \prod U(M_i)$ are representatives of $\bar{u}^l, 1 \leq l \leq n$.

ii) for any formula $\psi(x_1, \ldots, x_n)$ of L_tau and $\bar{u}^l \in U(M), 1 \leq l \leq n$ $M \models \psi(\bar{u}^1, \ldots, \bar{u}^n)$ if $\{i \in I | M_i \models \psi(u_i^1, \ldots, u_i^n)\}$ belongs to D.

iii) for any sentence ψ of LL_τ

$M \models \psi$ iff $\{i \in I : M_i \models \psi\} \in D$.

Lemma 11.4. *L:usat Let T be a theory with a countable language, D be a ω-regular ultrafilter on a countable set I and $M_i, i \in I$ models of T such that $\kappa(M_i) \leq \omega_1$ for all $i \in I$. Then the model $M = \prod_D M_i$ of T is saturated.*

Proof. Since I is countable and $\kappa(M_i) \leq \omega_1$ we have

$$\kappa(U(M)) \leq \kappa(\prod_D U(M_i)), i \in I \leq 2^\omega = \omega_1$$

Therefore it is sufficient to show that for any countable sequence $\bar{a}_n \in U(M_n), n \in \mathbb{N}$ and every set S of formulas $\psi(x)$ of $L[k_1, \ldots, k_n, \ldots]$ such that any finite subset of S is satisfiable in M the whole set S is satisfiable in M. Since the language $L[k_1, \ldots, k_n, \ldots]$ is also countable we can assume that $L[k_1, \ldots, k_n, \ldots] = L$.

Since the set S is countable we can write $S = \{\psi_n(x)\}, n \in \mathbb{N}$ an define $\phi_n = \psi_1 \wedge \ldots \wedge \psi_n$. Since D is a ω-regular ultrafilter we can find a sequence

$$I = I_0 \supset I_1 \supset \ldots \supset I_n \supset$$

of elelments in D such that $\cap_n I_n = \emptyset$. Let $X_0 = I$ and for any positive n we define

$$X_n = I_n \cap \{i \in I | M_i \models \exists \phi_n\}$$

Since any finite subset of S is satisfiable in M it follows from Problem 11.3 that $X_n \in D$. Since $X_{n+1} \subset X_n$ and $\cap_n X_n = \emptyset$ we see that for any $i \in I$ there exists a greatest $n(i) \in \mathbb{N}$ such that $i \in X_{n(i)}$.

For any $i \in I$ such that $n(i) > 0$ we choose $u_i \in U(M_i)$ in such a way that

$$M_i \models \phi_{n(i)}[u_i]$$

and for any $i \in I$ such that $n(i) = 0$ we choose some element $u_i \in U(M_i)$.

By the construction, for any $i \in X_n$ we have $n \leq n(i)$ and therefore $M_i \models \phi_{n(i)}[u_i]$. By the construction, for any $i \in X_n$ we have $n \leq n(i)$ and therefore for any $i \in X_n, M_i \models \phi_n[u_i]$. Let $\bar{u} =< u_i >$. Then it follows from Problem 11.3 that $M \models \phi_n(\bar{u})$ for all $n \in \mathbb{N}$.□

Definition 11.5. D:nat Let I be a non-empty set, D an ultrafilter over I and M be a model of a theory T. We denote by d the imbedding $M \to \prod_D M$ given by $d(u) :=< u >$.

Problem 11.6. *P:nat For any theory T, a model M and an ultrafilterD over I the imbedding is an elementary monomorphism.*

We show now how one can use ultraproducts to reprove and simplify some of the results of the Model theory.

The natural imbedding of M into $\prod_D M$
To start with we reprove Corollary 4.2.

Proof. Let S be a set of sentences in a language L such that any finite subset i of S has a model. We want to show that S has a model. Of course we can assume that S is infinite.

Let I be the set of finite subsets X of S. For each sentence s in S we define $\hat{s} \subset I$

$$\hat{s} := \{X \subset S | s \in X$$

For any finite set $i \subset S$ of sentences we have $i \in\in \cap_{s \in i} \hat{s}$. So the set $E := \{\hat{s}\}, s \in S$ is a filter over I and as follows from Problem 11.3 a) there exists an untrafilter D over I containing E. By our assumptions for any finite subset $i \subset S$ we can find a model M_i of the theory i. But then it follws from Problem 11.3 that $M := \prod_D M_i$ is a model of S.□

We now reprove a slightly weaker form of Corollary 4.3. We will show that for any model M l of theory T and a cardinality bigger then $\kappa(M)$

there exists model M' of T of a cardinality $\geq \kappa$ and an elementary monomorphism $M \to M'$.

Proof. Let I be a any set of cardinality κ and D an ultrafilter on I which contains $I-J$ for any subset J of I of cardinality $< \kappa$. As follows from Problem 11.6 the imbedding $d : M \to \prod_D M$ is an elementary monomorphism. It is easy to see that $\kappa(\prod_D M) \geq \kappa$.□

As the last application of the theory of ultrafilters we reprove Proposition 9.6. We want to show that for any countable theory T and any cardinality κ there exists a saturated model of T of cardinality κ^+. Let M be any countable model of T and D a regular ultrafilter on a contable set I. Then it follows from Lemma 11.4 that the ultrapower $\prod_D M$ is a a saturated model of T of cardinality $\omega_1 = \omega^+$.□

Remark 11.7. There many resuts of the Model theory require more delicate constructions then unltraproducts. Therefore I decided to present in the beginnig proofs which do not to use ultraproducts.

12. The theorem of Ax-Kochen

Let k be a field and Γ be an ordered abelian group. If (F, ac) is a valued field with the the value group Γ the residue field k and the semigroup homomorphism $ac : F \to k$ as in Definition 10.18 the we can consider (F, ac) as a model of the theory $\tilde{V}F$.

Theorem 12.1. *T:Ax If k is a field of characteristic zero and Γ an ordered abelian group. Then all the models (F, ac) of the theory $\tilde{V}F$ where F is a Henselian valued field with the residue field k and the value group Γ are elementary equivalent.*

In our treatments of the theory $\widetilde{VF}$ we denote by x, y, z elements of F, by $\bar{x}, \bar{y}, \bar{z}$ elements of F^m, by Greek letters such as μ, ν, ξ elements of Γ and by α, β elements of k. We start with the following definition.

Definition 12.2. D:cell a) A function $h : F^m \times k^n \to F$ is *strongly definable* if for any open formula $\phi(z, \bar{y}, \rho, a)$ there exists an open formula $\psi(\bar{x}, \nu, \bar{y}, \rho, a)$ such that

$$\phi(h(\bar{x}, \bar{\nu}), \bar{y}, \rho, \bar{\alpha}) = \psi(\bar{x}, \bar{\nu}, \bar{y}, \rho, \bar{\alpha})$$

b) A *cell* is a subset X of F^m such that there exists

a $C \subset \mathbb{A}_F^m \times k^n$ such that there exists

i) a subset $D \subset \mathbb{A}_F^m \times k^n$ defined by an open formula,

ii) strongly definable functions b_1, b_2, c from D to F,

iii) a positive integer r and

iv) a choice $<_1, <_2$ where $<_i$ is either $<$ or $\leq$ such that X is a disjoint union of the sets

$$X(\bar{\nu}), \bar{\nu} \in k^n$$

where

$$X(\bar{\nu}) = \{(\bar{x}, t) \in D(\bar{\nu}) | v(b_1((\bar{x}, \bar{\nu}) <_1 rv(t - c(\bar{x}, \bar{\nu}) <_2 v(b_1((\bar{x}, \bar{\nu}), ac(t - c(\bar{x}, \bar{\nu}) = \nu_1\}$$

where $\bar{\nu} = (\nu_1, \ldots, \nu_n), D(\bar{\nu}) = p^{-1}(\bar{\nu})$ and

c) The function c is the *center* of the cell C and subsets $X(\bar{\nu}) \subset X$ are *fibers* of the cell.

The Proof of Theorem 12.1 is based on the following result of Denef[see ?].

Proposition 12.3. *P:cell Let $f_i(\bar{x}, t), 1 \leq i \leq l, \bar{x} = (x_1, \ldots, x_m)$ be polynomials in t with coefficients in the ring of strongly definable functions on F^m. Then there exists a finite partition of $F^m \times F$ in a disjoint union of cells X and for any cell X*

i) strongly definable functions $h_i : F^m \times k^n \to F$, non-negative integers $r_i, 1 \leq i \leq l$ and a map $q : [1, l] \to [1, n]$ such that for any $i, 1 \leq i \leq l$ we have

$$v(f_i(\bar{x}, t)) = v(h_i(\bar{x}, p(\bar{x}, t))(t - c(\bar{x}, p(\bar{x}, t))^{r_i}), ac(f_i(\bar{x}, t)) = \nu_{q(i)}$$

Corollary 12.4. *C:cell Any formula in $\widetilde{VF}$ is equivalent to a formula without quantifiers over F variables.*

Proof. It is sufficient to prove that or any formula $\psi(t, \bar{x}, \nu, a), \bar{x} = (x_1, \ldots, x_m)$ without quantifiers over F variables the formula $\exists t \psi(t, \bar{x}, \nu, \bar{\alpha})$ is equivalent to a formula without quantifiers over F variables.

The formula $\psi(t, \bar{x}, \nu, \bar{\alpha})$ is constructed from atomic formulas. In any atomic formula the variables $t, \bar{x}$ appear either through expressions of the form $h(t, \bar{x}) = 0$, or through expressions of the form $v(h(t, \bar{x}))$ or through expressions of the form $ac(h(t, \bar{x}))$ where h is a polynomial with integral coefficients. By replacing the formula $h(t, \bar{x}) = 0$ by an equivalent formula $ac(h(t, \bar{x})) = 0$ we can disregard the first possibility. In other words the variables $t, \bar{x}$ appears in ψ either through k-variables $ac(f_i(t, \bar{x})), 1 \leq i \leq M$ or through Γ-variables $v(g_j(t, \bar{x})), 1 \leq j \leq N$ where $f_i(t, \bar{x}), g_j(t, \bar{x})$ are polynomials with integral coefficients.

Let ϕ be the formula obtained from ψ when we replace $f_i(t, \bar{x})$ by k-variables $\nu_i, 1 \leq i \leq M$ and $g_j(t, \bar{x}))$ by Γ-variables $\alpha_j, 1 \leq j \leq N$.

Then the formula $\psi(t,\bar{x},\nu,\bar{\alpha})$ is equivalent to the formula

$$(1)\exists\nu_i\exists\alpha_j|\phi(\bar{x},\nu,\bar{\alpha},\nu_i,\alpha_j)\wedge(\bigwedge_{i=1}^{M}ac(f_i(t,\bar{x})=\nu_i)$$

$$\wedge(\bigwedge_{j=1}^{N}v(g_j(t,\bar{x}))=\alpha_j$$

Since ϕ does not contain t it is sufficient to show that one can eliminate the quantifier $\exists$ over the F variable t in the formula

$$\exists t(\bigwedge_{i=1}^{M}ac(f_i(t,\bar{x})=\nu_i)\wedge(\bigwedge_{j=1}^{N}v(g_j(t,\bar{x}))=\alpha_j)$$

As follows from Proposition 12.3 there exists a partition of $F\times F^m$ into a finite number of cells X with parameters $c(\bar{x},\bar{\mu},\bar{\mu}\in k^n)$ such that for any for any $(t,\bar{x})\in X(\bar{m}u)$ we have

$$ac((f_i(t,\bar{x})=\mu_{q(i)},v(g_j(t,\bar{x})=v(h_j(\bar{x},\bar{\mu})(t-c(\bar{x},\mu)^{r_j}),1\leq i\leq M,1\leq j\leq N$$

where $h_j(\bar{x},\bar{\mu}),r_j,q(i)$ are as in Proposition 12.3.

Therefore (2) is equivalent to the formula

$$(2)\exists t\exists\mu_1\ldots\exists\mu_n|(t,\bar{x})\in X(\bar{\mu})\wedge(\bigwedge_{i=1}^{M}\mu_{q(i)}=\nu_i)$$

$$\wedge(\bigwedge_{j=1}^{N}v(h_j(\bar{x},\bar{\mu})+r_jv((t-c(\bar{x},\mu))=\alpha_j)))$$

By the defintion of a cell the condition $(t,\bar{x})\in X(\bar{\mu})$ has a form

$$\theta(\bar{x},\bar{\mu},v(t-c(\bar{x},\mu))\wedge ac(t-c(\bar{x},\mu))=\mu_1$$

where $\theta(\bar{x},\bar{\mu},\beta)$ is an open formula. If we introduce a Γ variable β then we can rewrite (2) in the form

$$(3)\exists t\exists\mu_1\ldots\exists\mu_n|\theta(\bar{x},\bar{\mu},\beta)\wedge(\bigwedge_{i=1}^{M}\mu_{q(i)}=\nu_i)$$

$$\wedge(\bigwedge_{j=1}^{N}v(h_j(\bar{x},\bar{\mu})+r_jv((t-c(\bar{x},\mu))=\alpha_j)))\wedge v(t-c(\bar{x},\mu))=\beta\wedge ac(t-c(\bar{x},\mu))=\mu_1$$

So it is sufficent to eleminate t in the formula

$$\exists t[v(t-c(\bar{x},\mu))=\beta\wedge ac(t-c(\bar{x},\mu))=\mu_1$$

But it is clear that the last formula is equivalent to $\neg[\mu_1=0]$.□

Now we can prove Theorem 12.1. Let F, F' be Henselian valued fields with such that the value groups $\Gamma, \Gamma'$and residue fields k, k' of F, F' are isomorphic and $ch(k) = 0$. We want to show that F, F' are elementary equivalent as models of $\widetilde{VF}$. In other words we want to show that for any sentence ψ in the language $\tilde{L}$ of $\widetilde{VF}$ we have $F \models \psi \leftrightarrow F' \models \psi$. As follows from Corollary 12.4 we can assume that the formula ψ does not have quantifiers over F variables. Since there is no functional symbols in $\tilde{L}$ relating Γ and k variables we see that the sentence ψ is equivalent to the sentence of the form

$$(\psi_1 \wedge \phi_1) \vee \ldots \vee \psi_m \wedge \phi_m)$$

where ψ_i are formulas in L_Γ and ϕ_i are formulas in L_k. Therefore $F \models \psi \leftrightarrow F' \models \psi$.□

Corollary 12.5. *Ax For any $n \in \mathbb{N}$ there exists $s(n) \in \mathbb{N}$ such that for any prime number $p > s(n)$ any homogeneous polynomial equation $P(x_0, ..., x_{n^2}) = 0$, where $P \in \mathbb{Q}_p[x_0, ..., x_{n^2}]$ is a polynomial of degree n, has a non-zero solution.*

Proof. We assume that there exists a number $n \in \mathbb{N}$ such that there exists an infinite set S of prime numbers such that there exists a homogeneous polynomial of degree n $P_p(x_0, ..., x_{n^2}) \in \mathbb{Q}_p$ such that the equation $P_p(x_0, ..., x_{n^2}) = 0$ does not have a non-zero solution and show that this assumption leads to a contrudiction.

Let ψ be the sentence in $\tilde{L}$ saying that there exists a a homogeneous polynomial of degree n $P_p(x_0, ..., x_{n^2})$ such that the equation $P_p(x_0, ..., x_{n^2}) = 0$ does not have a non-zero solution. Our assumption implies that ψ is true in $\mathbb{Q}_p$ for $p \in S$. D be any ω-regular ultrafilter over the set $\mathcal{P}$ of prime numbers containing S and $F = \prod_D \mathbb{Q}_p$. As follows from Problem 11.3 b) the sentence ψ is true in F.

On the other hand it follows from the theorem of Tsen, for any $n \in \mathbb{N}$ and any homogeneous polynomial $Q(x_0, ..., x_{n^2}) \in \mathbb{F}_p((u))$ the equation $Q(x_0, ..., x_{n^2}) = 0$ has a non-zero solution. So the sentence ψ is false in $\mathbb{F}_p((u))$ for all prime numbers p.Let $F' = \prod_D \mathbb{F}_p((u))$. As follows from Problem 11.3 b) the sentence ψ is false in F'.

But it is clear that F and F' are Henselian valued fields with the residue fields equal to the field $k = \prod_D \mathbb{F}_p$ of characteristic zero and valued groups equal to $\Gamma == \prod_D \mathbb{Z}$. Therefore it follows from Theorem 12.1 that the models F, F' of $\widetilde{VF}$ are elementary equivalent. This contrudiction shows that our assumption the existence of a number $n \in \mathbb{N}$ such that there exists an infinite set S of prime numbers such that there exists a homogeneous polynomial of degree n $P_p(x_0, ..., x_{n^2}) \in \mathbb{Q}_p$ such that the equation $P_p(x_0, ..., x_{n^2}) = 0$ does not have a non-zero solution leads to a contrudiction.□

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