Let $R = k[t, t^{-1}]$. Any element $F \in R$ has a form $F = \sum_{n=-N}^{N} c_n(F)t^n$. We define $Res : R \to k$ by $Res(F) := c_1(F)$. [The correct way is to write $Res(Fdt) := c_1(F)$ but I'll ignore the distinction]. It is clear that $Res(dF/dt) \equiv 0$.

We denote by $sl_n(R)$ the Lie algebra of $n \times n$-matrices A with coefficients in R such that $Tr(A) = 0$ and define a new Lie algebra $\widetilde{sl}_n(R)$ which is equal to $sl_n(R) \oplus k$ as a vector space and

$$[(A, c), (B, d) := [A, B], Res(Tr(AB')), A, B \in sl_n(R), c, d \in k$$

where

$$Tr(AB') = \sum_{1 \leq p,q \leq n} (a_{pq} db_{qp}/dt)$$

Claim 1. $\widetilde{sl}_n(R)$ is a Lie algebra.

Remark. The Jakobi identity follows from the equality $Res(dF/dt) \equiv 0$.

Let $D : R \to R$ be the differentiation given by $D(F) := tdF/dt$. It defines a linear operators D of the Lie algebra $sl_n(R), D(a_{ij}) := (D(a_{ij}))$ and the Lie algebra $\widetilde{sl}_n(R), D(A, c) := (D(A), 0)$.

Claim 2 $D : \widetilde{sl}_n(R) \to \widetilde{sl}_n(R)$ is a differentiation of the Lie algebra $\widetilde{sl}_n(R)$.

We define $\hat{sl}_n(R) := D \ltimes \widetilde{sl}_n(R)$.

Lemma [Easy]. If $I \subset \hat{sl}_n(R)$ is an ideal then either $I = \{0\}$ or $I = \widetilde{sl}_n(R)$ or $I = \{(0, c)\}, c \in k$ of $I = \hat{sl}_n(R)$.

Let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The realization of A could be presented as $\mathfrak{h}, \alpha_1^\vee, \alpha_2^\vee \in \mathfrak{h}, \alpha_1, \alpha_2 \in \mathfrak{h}^\vee$ where

$$\mathfrak{h} = \mathfrak{h}^\vee = \mathbb{R}^3, \alpha_1^\vee = (1, 0, 0), \alpha_2^\vee = (0, 1, 0), \alpha_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

We choose $h'' = (0, 0, 1)$ and denote by $<, >: \mathfrak{h}^\vee \times \mathfrak{h} \to \mathbb{R}$ be the natural pairing.. Then the symmetric bilinear form $[,] : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ such that

$$[h'', h''] = 0, [\alpha_i^\vee, h] =< \alpha_i, h >$$

is given by the pairing $[,] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$[(x_1, x_2, x_3), (y_1, y_2, y_3)] := \sum_j c_{ij} x_i y_j$$

where

$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

By the definition $\tilde{\mathfrak{g}}_A$ is the Lie algebra generated by e_1, e_2, f_1, f_2, h where $h \in \mathfrak{h}$ and relations

$$[e_i, f_j]\delta_{ij}\alpha_i^\vee, [h, h'] = 0, [h, e_i] =< \alpha_i, h >, [h, f_i] = - < \alpha_i, h >$$

and $\mathfrak{g}_A = \tilde{\mathfrak{g}}_A / r$ where $r \subset \tilde{\mathfrak{g}}_A$ is the maximal ideal such that $r \cap \mathfrak{h} = \{0\}$. I want to give an explicit description of the Lie algebra $\mathfrak{g}_A$.

We construct now a surjective Lie algebra morphism $\phi : \tilde{\mathfrak{g}}_A \to \hat{sl}_2(R)$ by

$$e_1 \to (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0), f_1 \to (\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0)$$

$$e_2 \to (\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, 0), f_2 \to (\begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}, 0)$$

$$\alpha_1^\vee \to (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0), \alpha_2^\vee \to (\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 0), h'' \to D$$

[Please check the relations and correct signs if I made a mistake].

It is clear that ϕ defines an imbedding of ϕ into $\hat{sl}_2(R)$. It follows now from Lemma that ϕ defines an isomorphism $\mathfrak{g}_A \to \hat{sl}_2(R)$.

Problem 0.1. *a) Give an explict description of the Weyl group W and the Tits cone X for $\mathfrak{g}_A$ where $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.*

b) Find a Cartan metrix A_n such that $\mathfrak{g}_{A_n} = \hat{sl}_n(R)$.