Definition 0.1. Let k be an infinite field, $char(k) \neq 2$ and V be a finite-dimensional k-vector space. We denote

(1) by $R = Sym(V^{\vee})$ the graded ring of regular functions on V where linear functionals $v^{\vee} \in V^{\vee}$ have degree two.
(2) by $\mathcal{R}$ the monoidal category of $\mathbb{Z}$-graded R-bimodules which are finitely generated as both left and right R-modules where the tensor product in $\mathcal{R}$ is given by

$$(X, Y) \to X \otimes_R Y, X, Y \in Ob(\mathcal{R})$$

(3) by $< \mathcal{R} >$ the split Grothendick group of the category $\mathcal{R}$. That is $< \mathcal{R} >$ is the abelian group $< \mathcal{R} >$ is generated by elements $< X >, X \in Ob(\mathcal{R})$ and relations

$$< X > = < Y > + < Z >, X, Y, Z \in Ob(\mathcal{R})$$

for triples X, Y, Z such that X is isomorphic to $Y \oplus Z$.
(4) by $\circ$ the algebra structure on $< \mathcal{R} >$ induced by the tensor product in $\mathcal{R}$.
(5) For any $M = \oplus_i M_i \in Ob(\mathcal{R})$ and $n \in \mathbb{Z}$ we define an object $M[n] \in Ob(\mathcal{R})$ by $M[n]_i = M_{i+n}$.

Definition 0.2. Let (W, S) be a Coxeter system, $\mathcal{T} \subset W$ be the subset of reflections and $\rho : W \to Aut(V)$ be a faithful representation.

(1) We say that ρ is *reflection faithful* if for $x \in W$ we have $dim(V/V^x) = 1$ iff $x \in \mathcal{T}$.
(2) For any $x \in W$ we define

$$V^{-x} := \{v \in V | xv = -x\}$$

(3) We say that ρ is *reflection vector faithful* if $\rho(t)$ is a reflection for all $t \in \mathcal{T}$ and $V^{-x} \neq V^{-y}$ for $x \neq y \in \mathcal{T}$.

Lemma 0.3. *Any reflection faithful representation ρ is reflection vector faithful.*

Proof. Assume that $V^{-t} = V^{-r}$ and set $x := t^{-1}r$. By the construction x acts trivially on V^{-t}. Since $det(x) = 1$ we see that x is unipotent. On the other hand since ρ is reflection faithful x is an involution. By the assumption $char(k) \neq 2$. So $x = 1$. □

Claim 0.4. *Let V be a finite-dimensional k-vector space, $e_s, s \in S$ be a set of linearly independent vectors in V and $e_s^{\vee} \in V^{\vee}$ be a set of linearly independent linear forms such that*

$$e_s^{\vee}(e_t) = -2\cos(\frac{\pi}{m_{s,t}}), t, s \in S$$

(1) *The formula $s(v) = v - e_s^{\vee}(v)v, v \in V$ defines a faithful representation of W.*

(2) *If there is no proper subspace V' of V containing all the vectors $e_s, s \in S$ and such that the restriction of the functionals $e_s^\vee$ on V' are linearly independent then ρ is reflection faithful.*

Definition 0.5. Let $W \to Aut(V)$ be a reflection faithful representation of W and as before R be the graded k-algebra of regular functions on V. We identify $R \otimes_k R$ with the graded k-algebra of regular functions on $V \times V$.

(1) For any $w \in W$ we define a subvariety $Gr(w)$ of $V \times V$ by

$$Gr(w) := \{(wv, v) | v \in V\}$$

(2) For any finite subset A of W we define

$$Gr(A) := \cup_{w \in A} Gr(w) \subset V \times V$$

(3) Since the subvarieties $Gr(A) \subset V \times V$ are homogeneous the ideal $I_A \subset R \otimes R$ of functions vanishing on $Gr(A)$ is graded and the quotient $R(A) := R \otimes R / I_A$ is a graded algebra. We consider $R(A)$ as a R-bimodule. It is easy to see that $R(A) \in Ob(\mathcal{R})$.
(4) We write R_x instead of $R(\{x\})$ and $R_{x,y}$ instead of $R(\{x, y\})$ for $x, y \in W$.
(5) For any $w \in W$ we write $R(\leq w)$ instead of $R(A)$ where $A =: \{x | x \leq w\}$.
(6) Given $f \in R, w \in W$ we define $f^w \in R$ by $f^w(v) := f(wv), v \in V$.

Claim 0.6. (1) $(Gr(x) + Gr(y)) \cap (V \times \{0\}) = Im(xy^{-1} - Id) \times \{0\}$
(2) *for any $u \neq w \in \mathcal{T} - \{e\}$ we have $Gr(u) + Gr(w) + Gr(e) \supset V \times \{0\}$*

Corollary 0.7. *If $|S| = 2$ and $x, y, z \in W$ are distinct elements not all of the same parity then $Gr(x) + Gr(y) + Gr(z) \supset V \times \{0\}$.*

Proof. We may assume that the parity of z is different from parities of x, y. Then $w := xz^{-1}, u := yz^{-1}$ have odd length and therefore are reflections [we use here that $|S| = 2$]. Now we can apply the previous result. □

Theorem 0.8. *Let (W, S) be a Coxeter system and that $\rho : W \to Aut(V)$ be a reflections vector faithful representation. Then there exists unique ring homomorphism $\mathcal{E} : \mathcal{H} \to < \mathcal{R} >$ such that $\mathcal{E}(v) = R[1]$ and*

$$\mathcal{E}(T_s + 1) = R \otimes_{R^s} R, s \in S$$

where $R^s := \{r \in R | s(r) = r\}$.

Proof. Consider first the case when $S = \{s\}$. In this case we may assume that $V = k$. Then $R = k[a], R^s = k[a^2]$ and the algebra $\mathcal{H}$ is generated over $\mathbb{Z}[v, v^{-1}]$ by $(T_s + 1)$ and the relation

$$(T_s + 1)^2 = (1 + v^{-2})(T_s + 1)$$

So it is sufficient to check that the bimodule $(R \otimes_{R^s} R) \otimes_R (R \otimes_{R^s} R)$ is isomorphic to $R \otimes_{R^s} R \oplus R \otimes_{R^s} R[-2]$.

Claim 0.9. *The map* $(f, g) \to f + tg, f, g \in R^s$ *defines an isomorphism between* R^s*- bimodules* $R^s \oplus R^s[-2]$ *and* R.

So we have

$$(R \otimes_{R^s} R) \otimes_R (R \otimes_{R^s} R) = R \otimes_{R^s} R \otimes_{R^s} R = R \otimes_{R^s} (R^s \oplus R^s[-2]) \otimes_{R^s} R = \otimes_{R^s} R \oplus R \otimes_{R^s} R[-2]$$

Now back to the general case. Since the Hecke algebra $\mathcal{H}$ is defined by quadratic relations

$$(T_s + 1)^2 = (1 + v^{-2})(T_s + 1), s \in S$$

and the braid relations

$$T_s T_t ... = T_t T_s ..., s, t \in S$$

it is sufficient to prove Theorem for Coxeter systems of rank 2.

For any $w \in W$ we write $C_w := \sum_{x \leq w} T_x \in \mathcal{H}$. It is clear that the following claim implies the validity of Theorem.

Proposition 0.10. *Let* $\mathcal{E} : \mathcal{H} \to < \mathcal{R} >$ *be the homomorphism of abelian groups given by*

$$\mathcal{E}(v^n C_x) := < R(\leq x) > [n + l(x)]$$

Then $\mathcal{E}$ *is a ring homomorphism.*

Proof. To show that $\mathcal{E}$ is a ring homomorphism it is sufficient to check the equality

$$\mathcal{E}((T_s + 1)C_w) = \mathcal{E}(T_s + 1) \circ \mathcal{E}(C_w), w \in W, s \in S$$

For any $x \in W$ we define $A(x) := \{y \in W | y \leq x\}$. We fix $w \in W$ and write $A = A(w)$.

Claim 0.11.
1. *If* $l(sw) < l(w)$ *then* $sA = A$
2. *If* $l(sw) > l(w)$ *then* $A \cup sA = A(sw)$
3. *If* $l(sw) > l(w)$ *and* $l(w) > 1$ *then* $A \cap sA = A(tw)$ *for some* $t \in \mathcal{T}, t \neq s$.
4. *If* $l(sw) > l(w)$ *and* $l(w) \leq 1$ *then* $A \cap sA = \emptyset$
5. $(T_s + 1) \sum_{x \in A} T_x = \sum_{x \in A \cup sA} T_x + \sum_{x \in A \cap sA} T_x$.

We see that for a proof of Theorem we have to construct an isomorphism between bimodules $R(A \cap sA)[-2] \oplus R(A \cup sA)$ and $(R \otimes_{R^s} R) \otimes_R R(A)$.

We start with the following general result. Let L be a k-vector space, $t \in Aut(L)$ be a reflection along a hyperplane given L^t given by the equation

$$\lambda(l) = 0, \lambda \in L^\vee - \{0\}$$

We define the Demazure operator ∂_t on $R(L)$ by

$$\partial_t(f) = \frac{f - f^t}{2\lambda}$$

If $X \subset L$ is a t-invariant Zarski closed subvariety then t induces an involution on X and we obtain direct sum decomposition $R(X) = R(X)^+ \oplus R(X)^-$.

Claim 0.12. *Assume that X does not have any irreducible components contained in L^t. Then*

(1) *∂_t stabilizes the kernel of the restriction $R(L) \to R(X)$ and therefore induces a map $\partial_t : R(X) \to R(X)$.*
(2) *∂_t and the multiplication by λ define mutually inverse isomorphism between graded $R(L)$-modules $R(X)^+$ and $R(X)^-[2]$.*

Lemma 0.13. *Let W be a group acting on a k-vector space V, A be a finite subset of W and $s \in W$ be an element acting on V as a reflection and such that $sA = A$. We denote by $R(A)^+ \subset R(A)$ the invariants under the action of $s \times id$. Then*

(1) *The graded bimodules $R \otimes_{R^s} R(A)$ and $R(A) \oplus R(A)[-2]$ are isomorphic.*
(2) *The multiplication induces an isomorphism $R \otimes_{R^s} R(A)^+ \to R(A)$ of bimodules.*

Proof. Take $L = V \times V, t = s \times id$ and $X = R(A)$. We obtain a decomposition $R(A) = R(A)^+ \oplus R(A)^-$ and an isomorphism $R(A)^+ \to R(A)^-[2]$ defined by the multiplication by $\lambda \otimes 1$ where $\lambda \in V^\vee$ is the equation defining V^s. Since $R = R^+ \oplus \lambda R^+$ we see that the multiplication induces an isomorphism $R \otimes_{R^s} R(A)^+ \to R(A)$. The existence of the decomposition $R(A) = R(A)^+ \oplus R(A)^-$ and an isomorphism between $R(A)^+$ and $R(A)^-[2]$ finishes the proof of Lemma. □

To prove the Proposition [and therefore the Theorem] it is sufficient to construct isomorphisms between bimodules

$(R \otimes_{R^s} R) \otimes_R R(A) = R \otimes_{R^s} R(A)$ and $R(A \cap sA)[-2] \oplus R(A \cup sA)$.

The case when $sA = A$ follows from the previous Lemma. The case when $w = e$ is also clear. So may assume that $A = A(w), w \neq e$ and $l(sw) > l(w)$. It is easy to see that in this case $A - sA \cap A = \{(tw, w)\}$ for some reflection $t \neq s$. As before we construct a decomposition of $R(A) = M \oplus N$ as $R^s \times R$-bimodule such that $R(A \cap sA)[-2] = R \otimes_{R^s} M$ and $R(A \cup sA) = R \otimes_{R^s} N$.

Since ρ is a reflections vector faithful representation the -1-eigenspaces of reflections s and t span a 2-dimensional subspace U of V. As follows from Claim 6 there exists a linear functional $\lambda \in (V \times V)^\vee$ trivial on $Gr(w)$ and on $Gr(tw)$ whose restriction on U is non-trivial.

Lemma 0.14. *For any $x \in A \cap sA$ the restriction of λ on $Gr(x) \cup Gr(sx)$ is not s-invariant.*

Proof. If the restriction of λ on $Gr(x) \cup Gr(sx)$ were s-invariant then the restriction of λ on $Gr(x) + Gr(sx)$ is also s-invariant. But then λ vanishes on V^{-s}. Since $V^{-s} \neq V^{-t}$ this would imply that λ vanishes on U. □

We define M and N as $R^s \times R$-subbimodules of $R(A)$ generated by the images $\bar{1}, \bar{\lambda}$ of $1, \lambda \in R \otimes R$ in $R(A)$. It follows from the previous Lemma that Proposition is implied by the following result.

Lemma 0.15. (1) *The graded bimodules $R^s \times R$-bimodules M and $R(A \cap sA)^+[-2]$ are isomorphic.*
(2) *The graded bimodules $R^s \times R$-bimodules N and $R(A \cup sA)^+$ are isomorphic.*
(3) $R(A) = M \oplus N$

Proof. As follows from Corollary 7 the restriction of λ on $Gr(x)$ is not zero for any $x \in A \cap sA$. So the kernel of the multiplication by λ in $R(A)$ is equal to the kernel of the surjection $p : R(A) \to R(A \cap sA)$. Therefore the multiplication by λ defines an imbedding $R(A \cap sA) \hookrightarrow R(A)$ which is an isomorphism on the image $\lambda R(A) \subset R(A)$. Since the image of $R^s \otimes R$ in $R(A)$ consists of $s \times Id$-invariants we obtain an isomorphism $R(A \cap sA)^+[-2] \to M$.

The same observation [the image of $R^s \otimes R$ in $R(A)$ consists of $s \times Id$-invariants] provides an isomorphism $R \cup sA)^+ \to N$.

To prove the equality $R(A) = M + N$ it is sufficient to show that as an (R^s, R)-bimodule $R \otimes R$ is generated by 1 and λ. Let $\mu \in V^\vee$ be the equation of the hyperplane $V^s \subset V$. It is clear that $R \otimes R$ is generated as an (R^s, R)-bimodule by 1 and $(\mu, 0)$. Since λ is not s-invariant [otherwise λ would vanish on $Gr(sx)$ and therefore by Corollary 7 on the whole U] we have

$$(V \oplus V)^\vee = V^{s\vee} \oplus V^\vee \oplus k\lambda$$

We see that $(\mu, 0)$ belongs to (R^s, R)-bimodule $R \otimes R$ generated by 1 and λ and therefore $R \otimes R$ is generated as an (R^s, R)-bimodule by 1 and λ.

To finish the proof of the Theorem we have to show only that $M \cap N = \{0\}$. For any $n \in N$ and any $x \in A \cap sA$ the restriction of n onto $Gr(x) \cup Gr(sx)$ is s-invariant. On the other hand as follows from Lemma 14 the restriction of λ on $Gr(x) \cup Gr(sx)$ is not s-invariant. This implies that there is no non-zero $m \in M$ such that restrictions of m on $Gr(x) \cup Gr(sx)$ are s-invariant for all $x \in A \cap sA$. So $M \cap N = \{0\}$.

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