Coxeter systems.

Definition 0.1.

1. A Coxeter group is a group with the presentation $\langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1\rangle$ where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for $i \neq j$. The condition $m_{ij} = \infty$ means that no relation of the form $(s_i s_j)^m$ is imposed.
2. A pair (W, S) where W is a Coxeter group with generators S is called a Coxeter system.
3. Elements of S are *simple reflections* and elements of W conjugate to simple reflections are *reflections.*
4. For any $w \in W$ we denote by $l(w)$ the length of the shortest presentation of w as a product $w = s_1 ... s_{l(w)}$ of simple reflections and say that a presentation $s_1 ... s_{l(w)}$ is a *reduced expression of* w.
5. If $w = s_1 ... s_{l(w)}$ is a reduced expression of $w \in W$ we say that $w' \in W$ an a subexpression of $s_1 ... s_{l(w)}$ if $w' = s_{i_1} ... s_{i_r}$ for some sequence $1 \leq i_1 < i_2 ... < i_r \leq l(w)$.
6. Given $w', w \in W$ we say that $w' \to w$ if $l(w) > l(w')$ and $t := (w')^{-1} w$ is a reflection. We say that $w' < w$ if there exists a sequence $w' = w_0 \to w_1 \to ... \to w_n = w$ [the Bruhat order].

Note that in general S is not uniquely determined by W. For example, the Coxeter groups of type BC_3 and $A_1 \times A_3$ are isomorphic but the Coxeter systems are not equivalent.

The relation $m_{ii} = 1$ means that $s_i^2 = 1$ for all $i \in I$. If $m_{ij} = 2$ then the generators s_i and s_j commute.

Claim 0.2.

1. *If $w = s_1 ... s_r, s_i \in S$ and t is reflection such that $l(wt) < l(w)$ then there exists i such that $wt = s_1 ... \hat{s}_i ... s_r$ (omitting s_i). Moreover if $r = l(w)$ then such i is unique.*
2. *If $w = s_1 ... s_r, s_i \in S$ and $r > l(w)$ then there exists indices $i < j$ such that $w = s_1 ... \hat{s}_i ... \hat{s}_j ... s_r$. [Strong exchange condition]*
3. *If $w' \leq w$ and $s \in S$ then either $w's \leq w$ or $w's \leq ws$ or both.*
4. *$w' \leq w$ if w' is a subexpression of w. Conversely if $w' \leq w$ and $w = s_1 ... s_{l(w)}$ is a reduced expression then w' is a subexpression of the presentation $w = s_1 ... s_{l(w)}$.*
5. *For any $w' \leq w$ there exists a sequence exists $w' = w_0, w_1, ..., w_n = w$ such that $w_{i-1} < w_i$ and $l(w_i) - l(w_{i-1}) = 1$.*

Hecke algebras. Let (W, S) be a Coxeter system, A be a commutative ring with chosen elements $a_s, b_s, s \in S$ such that $a_s = a_{s'}, b_s = b_{s'}$ when s', s are conjugate in W. Let V be the free A-module with generators $T_w, w \in W$.

Theorem 0.3. *There exists unique A-algebra structure on V such that*
$T_s T_w = T_{sw}$ *if* $l(sw) > l(w)$ *and*
$T_s T_w = a_s T_w + b_s T_{sw}$ *if* $l(sw) < l(w)$.

PROOF. We start with the following general result. Let A be a commutative ring, V a free A-module with basis $v_i, i \in I, S$ be a subset of $I, e \in S$ and $\alpha_s, \beta_s \in \mathrm{End}_A(V), s \in S$ be such that

$$\alpha_s(v_e) = \beta_s(v_e) = v_s, [\alpha_s, \beta_t] = 0, s, t \in S$$

For any sequence $\sigma = \{s_1, ..., s_r\}, s_i \in S, 1 \leq i \leq r$ we write $\alpha_\sigma := \alpha_{s_1} ... \alpha_{s_r}, \beta_\sigma =: \beta_{s_1} ... \beta_{s_r}$.

LEMMA 0.4. *Assume that* $V = Span\{\alpha_\sigma(v_e)\} = Span\{\beta_\sigma(v_e)\}$. *Then there exists unique structure of an associate algebra on* V *such that* v_e *is the unit,* α_s *acts as the left multiplication by* v_s *and* β_s *acts as the right multiplication by* v_s.

PROOF. Let $H \subset \mathrm{End}_A(V)$ be the A-subalgebra generated by α_s and $f : H \to V$ be the A-linear map such that $f(h) := hv_e$. Since $V = Span\{\alpha_\sigma(v_e)\}$ we see that f is onto. On the other hand since $[\alpha_s, \beta_t] \equiv 0$ we see that $h\beta_\sigma v_e = \{0\}$ for all $h \in Ker(f)$ and all sequences σ. Since $V = Span\{\beta_\sigma(v_e)\}$ we see that f is injective. So f is an isomorphism which defines an A-algebra structure on V. Since $\alpha_s(v_e) = \beta_s(v_e) = v_s$ we see that α_s acts as the left multiplication by v_s and β_s acts as the right multiplication by v_s. □

Now we can prove the Theorem. We define V as a free A-module with basis $T_w, w \in W$ and define $\alpha_s, \beta_s \in \mathrm{End}_A(V), s \in S$ by

$\alpha_s(T_w) := T_{sw}$ if $l(sw) > l(w)$
$\alpha_s(T_w) := a_s T_w + b_s T_{sw}$ if $l(sw) < l(w)$
$\beta_s(T_w) := T_{ws}$ if and $l(ws) > l(w)$
$\beta_s(T_w) := a_s T_w + b_s T_{ws}$ if $l(ws) < l(w)$.

it is clear that $Span\{\alpha_\sigma(v_e)\} = Span\{\beta_\sigma(v_e)\} = V$. The equality $[\alpha_s, \beta_t] \equiv 0$ follows easily from the following result which I'll leave for you to prove.

CLAIM 0.5. *If* $w \in W, s', s'' \in S$ *are such that* $l(s'ws'') = l(w)$ *and* $l(s'w) = l(ws'')$ *then* $s'w = ws''$.

Let $\circ$ be the multiplication on V the construction in Lemma 4. To finish the proof of the Theorem we have to check that

$T_s \circ T_w = T_{sw}$ if $l(sw) > l(w)$ and that
$T_s \circ T_w = a_s T_w + b_s T_{sw}$ if $l(sw) < l(w)$.

The first equality is immediate and to prove the second we observe that the inequality $l(sw) < l(w)$ implies that $w = sw'$ where $l(sw') > l(w')$ so it is sufficient to check the equality $T_s^2 = a_s T_s + b_s T_e$ which is immediate. □

DEFINITION 0.6. We define

1. We denote the algebra constructed in the Theorem 3 by $\mathcal{H}_A(W, a_s, b_s)$ and omit $\circ$ writing the product.
2. We write $\mathcal{H}$ or $\mathcal{H}(W)$ for

$$\mathcal{H}_{\mathbb{Z}[v,v^{-1}]}(W, a_s, b_s), a_s \equiv v^{-2} - 1, b_s \equiv v^{-2}$$

(3) We denote by ι the involution on $\mathcal{H}$ such that $\iota(T_s) := T_s^{-1} = v^2T_s - (1-v^2)T_e$ and $\iota(v) = v^{-1}$ and say that an element $H \in \mathcal{H}$ is self-dual if $\iota(H) = H$.
(4) For $P \in A \subset \mathcal{H}$ we write $\bar{P}(v) = P(v^{-1})$.
(5) We write $H_x := q^{-l(x)/2}T_x, x \in W$.

Proposition 0.7. *For any $w \in W$ there exists unique self-dual element $C_w \in \mathcal{H}$ which has a form $C_w = H_w + \sum_x h_x H_x, h_x \in v\mathbb{Z}[v]$.*

Proof. By the defintion we have $C_e = H_e$. Consider first the case when $w = s \in S$. Since $\iota(H_s) := H_s + (v - v^{-1})$ we see that we can take $C_s = H_s + v$. It is easy to check that the right multiplication on C_s in $\mathcal{H}$ is given by
$H_xC_s = H_{xs} + vH_x$ if $xs > x$ and
$H_xC_s = H_{xs} + v^{-1}H_x$ if $xs < x$
To show the existence of a self-dual element $C_w \in \mathcal{H}$ of the form $C_w = H_w + \sum_x v\mathbb{Z}[v]H_x$ we prove the following stronger result.

Lemma 0.8. *For any $w \in W$ there exists a self-dual element $C_w \in \mathcal{H}$ of the form $C_w = H_w + \sum_{x<w} h_x H_x, h_x \in v\mathbb{Z}[v]$.*

Proof. The proof is by induction in $l(x)$. The statement is clear if $x = e$. So assume that the existence of C_y is known for all $y < x$ and $x \neq e$. Choose $s \in S$ such that $xs < x$. By the induction assumption we have $C_{xs}C_s = H_x + \sum_{y<x} h_y H_y, h_y \in \mathbb{Z}[v]$ and we can take $C_x = C_{xs}C_s - \sum_{y<x} h_y(0)C_y$. □

The unicity of C_x follows immediately from the following result.

Lemma 0.9. *Any is self-dual element $H \in \mathcal{H}$ of the form*

$$H = h_x H_x, h_x \in v\mathbb{Z}[v]$$

is equal to 0.

Proof. As follows from the previous Lemma we have

$$H_w = C_w + \sum_{x<w} r_x C_x, r_x \in A$$

and therefore

$$\iota(H_w) = H_w + \sum_{x<w} r'_x C_x, r'_x \in A.$$

If $H \neq 0$ choose a maximal z such that $h_x \neq 0$. The equality $\iota(H) = H$ implies that $\bar{h}_x = h_x$. But his contradicts the inclusion $h_x \in v\mathbb{Z}[v]$. □

□

Examples 0.10. (1) Consider the case $S = (s,t), m_{s,t} = m$. In this case for any $l, 0 < l < m$ there exists two elements s'_l, s''_l of length l and $x < w \equiv l(x) < l(w)$. Let's show by induction in $l(w)$ that elements $\tilde{C}_w := v^{l(w)} \sum_{x \leq w} T_x$ are self-dual. We can assume that $l(w) > 1$ and write w in the form $w = sx$ where $l(x) = l(w) - 1$. Then $v(T_s + 1)\tilde{C}_x = \tilde{C}_w + \tilde{C}_x$ and we see by induction that $\tilde{C}_w$ is self-dual. It is clear now that $C_w = \tilde{C}_w = v^{l(w)} \sum_{x \leq w} T_x$.

(2) Assume that W is finite and that $w_0 \in W$ is the longest element of W. I claim that $C_{w_0} = \sum_{x \in W} v^{l(w_0) - l(x)} H_x$.

Let $\tilde{C} := \sum_{x \in W} v^{l(w_0) - l(x)} H_x$. To check the equality $C_{w_0} = \tilde{C}$ it is sufficient to show that $\tilde{C}$ is self-dual. Form the explicit formulas for the multiplication by C_s it is easy to derive that

$$A\tilde{C} = \{H \in \mathcal{H} | HC_s = (v + v^{-1})H, s \in S\}$$

But this equality implies that $\iota(\tilde{C}) = a\tilde{C}, a \in A$. Then $a\bar{a} = 1$. Which implies that $a = \pm 1$. It is easy to see then that $a = 1$.