Let $A$ be the algebra generated by $p, q, z$ with the relations $[p, q] = z, [p, z] = [q, z] = 0$.

**Claim 0.1.**

1. $p^n q^b = \sum_{k=0}^{a} \frac{(a)}{k} \prod_{j=0}^{a-1} (b - j) q^{b-k} p^k z^{a-k}$
2. $p^{2b-a} q^a q^{2a-b} = \sum_{i=0}^{b+a} c(a, b, i) p^{a+b-i} q^{a+b-i} z^i$ where $c(a, b, i)$ are polynomials of degree $i$.

**Definition 0.2.** Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a semi-simple Lie algebra.

1. We fix a complete order on the set $\alpha \in \Phi^+$ of positive roots.
2. We denote by $\sigma$ the anti-involution of $\mathfrak{g}$ such that $\sigma|_h = Id, \sigma(x) = y, \sigma(y) = x \in \Phi$.
3. We denote by $q : U(\mathfrak{g}) \to U(\mathfrak{h})$ the projection defined by the isomorphism $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n})$.
4. Let $A$ be the bilinear form on $U(\mathfrak{g})$ with values in $U(\mathfrak{h}) = S(\mathfrak{h})$ given by $A(x, y) := q(\sigma(x)y), x, y \in U(\mathfrak{g})$.
5. For any $\eta \in \Lambda_+$ we denote by $A_\eta$ the restriction of the form $A$ to $U(\mathfrak{n}^-)_\eta$ and by $D_\eta \in S(\mathfrak{h})$ the discriminant of the form $A_\eta$ which is defined uniquely up to a multiplication by $c \in \mathbb{C}^*$.
6. A partition $\omega$ of $\eta \in \Lambda_+$ is an presentation of $\eta$ as a sum $\eta = \alpha_1 + \alpha_2 + ... + \alpha_d(\omega)$ where $\alpha_i$ are positive roots such that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_d(\omega)$. We denote by $\bar{\mathcal{P}}(\eta)$ the set of partition of $\eta$.
7. For any partition $\omega$ of $\eta \in \Lambda_+$ we define $y_\omega := y_{\alpha_1} y_{\alpha_2} ... y_{\alpha_d} \in U(\mathfrak{n}^-)_\eta$.

**Claim 0.3.**

1. $|\bar{\mathcal{P}}(\eta)| = \mathcal{P}(\eta)$.
2. $\deg(A_\eta(y_{\omega'}, y_{\omega''})) \leq \min(d(\omega'), d(\omega''))$.
3. If $\omega' \neq \omega''$ and $d(\omega') = d(\omega'')$ then $\deg(A_\eta(y_{\omega'}, y_{\omega''})) < d(\omega')$.
4. If $\omega' = \omega''$ then $\deg(A_\eta(y_{\omega'}, y_{\omega''})) = d(\omega)$.
5. $\deg(D_\eta) = \sum_{\omega \in \mathcal{P}(\eta)} d(\omega)$.

**Theorem 0.4 (Shapovalov).** $D_\eta = \prod_{\alpha \in \Phi^+} \prod_{r > 0}(h_\alpha + \rho(\alpha) - r)^{\mathcal{P}(\eta - \rho(\alpha)r)}$

**Proof.** Let $\tilde{D}_\eta := \prod_{\alpha \in \Phi^+} \prod_{r > 0}(h_\alpha + \rho(\alpha) - r)^{\mathcal{P}(\eta - \rho(\alpha)r)}$. As follows from the previous Claim we have $\deg(D_\eta) = \deg(\tilde{D}_\eta)$. So it is sufficient to prove that $D_\eta$ is divisible by $(h_\alpha + \rho(\alpha) - r)^{\mathcal{P}(\eta - \rho(\alpha)r)}$ for any $\alpha \in \Phi^+, r > 0$.

Given $\alpha \in \Phi^+, r > 0$ we choose a generic $\lambda$ on the hyperplane $< \alpha, \lambda + \rho > = r$. Since we know the existence of an imbedding $M(\lambda - \rho\alpha) \hookrightarrow M(\lambda)$ we see that the kernel of the canonical map $\phi : M(\lambda) \to M(\lambda)^\vee$ contains $M(\lambda - \rho\alpha)$. Therefore $D_\eta(\lambda + t\rho)$ is divisible by $t^{\mathcal{P}(\eta - \rho(\alpha)r)}$ in the ring $\mathbb{C}[\mathbb{C}[t]]$. ☐

**Remark 0.5.** One can prove the Shapovalov's theorem and therefore the Jantzen's formula without the knowledge of the existence of imbeddings $M(\lambda - \rho\alpha) \hookrightarrow M(\lambda)$ [see Kac, V. G.; Kazhdan, D. A. Structure of representations with highest weight of infinite-dimensional Lie algebras. Adv. in Math. 34 (1979), no. 1, 97108. ] and then deduce the existence of imbeddings $M(\lambda - \rho\alpha) \hookrightarrow M(\lambda)$ from this formula.
The relative Lie algebra cohomology. Let \( g \) be a Lie algebra and \( b \) a Lie subalgebra and \( y \rightarrow \bar{y} \) be the projection from \( g \) to \( g/b \) We denote by \( D_k \) the \( U(g) \)-modules \( U(g) \otimes U(b) \Lambda^k(g/b) \) and define differentials \( d_k : D_k \rightarrow D_{k-1} \) for \( k > 0 \) by

\[
d_k(u \otimes x_1 \wedge ... \wedge x_k) = \sum_{i=1}^{k} (-1)^{i+1} u y_i \otimes x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge x_k \]

\[+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} u \otimes [y_i, y_j] \wedge x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge x_j \wedge ... \wedge x_k\]

where \( y_i = x_i \) and define \( d_0 : D_0 \rightarrow \mathbb{C} \) by \( d_0(u \otimes 1) = \epsilon(u) \) where \( \epsilon : U(g) \rightarrow \mathbb{C} \) is the counit.

**Lemma 0.6.**
1. The differential \( d_k \) is well defined [that is the right side does not depend on a choice of preimages \( y_i \) of \( x_i \)].
2. \( d_{k-1} \circ d_k = 0 \) for all \( k > 0 \)
3. The complex ... \( D_k \rightarrow ... \rightarrow D_0 \rightarrow \mathbb{C} \) is exact. In other words the complex ... \( D_k \rightarrow ... \rightarrow D_0 \) is a resolution of the \( U(g) \)-module \( \mathbb{C} \).

**Remark 0.7.** If \( \text{dim}(g) = n < \infty \) then \( D_k = \{0\} \) for \( k > n \).

**Proof.** The proof is pretty standard. I’ll outline a proof only in the case when \( g \) is the Lie algebra of a Lie group \( G \) and \( b \) is the Lie algebra of a subgroup \( B \) of \( G \). Let \( \Omega^* \) be the De-Rham complex on \( G/B \) and \( \Omega^* \) be the completion of \( \Omega^* \) at \( e \in G/B \). The natural action of \( G \) on \( G/B \) induces an action of the Lie algebra \( g \) on \( \Omega^* \). I’ll leave for you to construct an identification of \( D_k \) with continuous linear functional on \( \Omega^k \) in such a way that the differential \( d_k \) is dual to the De-Rham differential on \( \Omega^k \). The Lemma follows now from the exactness of the De-Rham complex. \( \square \)

**Definition 0.8.** Let \( g = n \oplus h \oplus n^- \) be a semi-simple Lie algebra.

1. We denote by \( -R_k \subset h^\vee \) the set of weights in the decomposition of \( h \)-module \( \Lambda^k(n^-) \). So

\[
\Lambda^k(n^-) = \sum_{\mu \in R_k} \Lambda^k(n^-)_{-\mu}
\]

2. For any \( w \in W \) we denote by \( \Phi(w) \subset \Phi^+ \) the subset of positive roots \( \gamma \) such that \( w(\gamma) \in \Phi^- \) and write \( \mu(w) := \sum_{\gamma \in \Phi(w)} \gamma \).

**Claim 0.9.**
1. \( \mu(w) \in R_{l(w)} \) for any \( w \in W \) and the space \( \Lambda^l(w)(n^-)_{-\mu(w)} \) is one-dimensional.
2. For any \( w', w'' \in W \) such that \( l(w') = l(w'') \) and any \( \gamma \in \Phi^+ \) we have \( \mu(w') - \mu(w'') - \gamma \notin \Lambda^+ \).