DEFINITION 0.1. Let $A$ be a unital finite-dimensional algebra over an algebraically closed field $k$ and $M_i, 1 \leq i \leq r$ be representatives of non-isomorphic simple $A$-modules.

(1) We define the radical of $A$ by

$$Rad(A) := \{ a \in A | \pi(a) = 0 \}$$

for all irreducible representations $\pi$ of $A$.

(2) A finite-dimensional algebra $A$ is semisimple iff $Rad(A) = \{0\}$.

CLAIM 0.2. (1) $Rad(A) \subset A$ is a two-sided nilpotent ideal.

(2) If the algebra $A$ is semisimple then the maps $A \rightarrow End_k(M_i), 1 \leq i \leq r$ induce an isomorphism $A \rightarrow \oplus_{i=1}^{r} End_k(M_i)$.

LEMMA 0.3. Let $A$ be a ring, $I \subset A$ a two-sided nilpotent ideal, $\tilde{A} := A/I$ and $\tilde{e} \in \tilde{A}$ an idempotent. Then $\tilde{e}^2 = \tilde{e}$. Then

(1) There exists a lift of $\tilde{e}$ to an idempotent $e \in A$.

(2) Any two such lifts are conjugate by an element in $Id + I$.

PROOF. By induction it is easy to reduce the proof to the case when $I^2 = \{0\}$. So we assume that $I^2 = \{0\}$. In this case $I$ has a structure of a two-sided $A/I$-module. Let $\tilde{e} \in A$ be any lift of $\tilde{e}$ and $a := \tilde{e}^2 - \tilde{e} \in I$. Any lift $e$ of $\tilde{e}$ has the form $e = b + \tilde{e}, b \in I$ and the condition $e^2 = e$ is equivalent to the condition $eb + \tilde{e}b - b = a$. For the proof of the first claim it is sufficient to note that $b := (2\tilde{e} - 1)a$ satisfies this condition. Let $e'$ be another lift of $\tilde{e}$ such that $e'^2 = e'$. Then $e' = e + c$ where $c \in I$ is such that $ec + ce = c$. Since $e^2 = e$ this equation implies that $ece = 0$ and that $(1 - e)c(1 - e) = 0$. So $c = ec(1 - e) + (1 - e)ce = [e, [e, c]]$. Hence [since $I^2 = \{0\}$] we have $e'(1 + [c, e])e(1 + [c, e])^{-1}$.

\[ \square \]

DEFINITION 0.4. A complete system of orthogonal idempotents in a unital algebra $B$ is a collection of elements $e_1, \ldots, e_n \in B$ such that

$$e_ie_j = e_i\delta_{i,j}, 1 \leq i, j \leq n$$

COROLLARY 0.5. Given a complete system of orthogonal idempotents $\tilde{e}_1, \ldots, \tilde{e}_n \in A/I$ there exists lift $e_1, \ldots, e_n \in A$ of $\tilde{e}_1, \ldots, \tilde{e}_n$ to a complete system of orthogonal idempotents in $A$.

PROOF. The proof is by induction in $m$. If $m = 2$ then this Corollary is a restatement of the previous Lemma. For $m > 2$ we choose a lift $e_1$ of $\tilde{e}_1$ and apply the inductive assumption to the algebra $A_1 = A(e_1)$.

\[ \square \]

THEOREM 0.6. (1) For any $i, 1 \leq i \leq r$ there exists unique indecomposable finitely generated projective $A$-module $P_i$ such that

$$\dim(\text{Hom}_A(P_i, M_j)) = \delta_{i,j}$$

(2) $A = \oplus_{i=1}^{r} d_i P_i$ where $d_i := \dim_k(M_i)$. 

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Any indecomposable finitely generated projective $A$-module is isomorphic to $P_i$ for some $i, 1 \leq i \leq r$.

**Proof.** For any $i, 1 \leq i \leq r$ choose a basis $\{m^i_t\}, 1 \leq t \leq d_i$ of $M_i$ and denote by $\bar{e}^i_t \subset \text{End}(M_i)$ the projection on the line $km^i_t$ along the hyperplane generated by vectors $m^i_s, s \neq t$. As we know $\text{Rad}(A) \subset A$ is a two-sided nilpotent ideal and $A/\text{Rad}(A) = \oplus_{i=1}^r \text{End}_k(M_i)$ and it is clear that $\{\bar{e}^i_t\}$ is a complete system of orthogonal idempotents in $A/\text{Rad}(A) = \oplus_{i=1}^r \text{End}_k(M_i)$.

Let $\{e^i_t\} \in A$ be a lift of $\{\bar{e}^i_t\}$ to a complete system of orthogonal idempotents in $A$. We define $P_{i,t} := Ae^i_t \subset A$ for $1 \leq i \leq r, 1 \leq t \leq d_i$. Then $A = \bigoplus_{1 \leq i \leq r, 1 \leq t \leq d_i} P_{i,t}$ and we see that the $A$-modules $P_{i,t}$ are projective.

By the construction $\text{Hom}_A(P_{i,t}, M_j) = e^i_t M_j$. So we see that $\dim(\text{Hom}_A(P_{i,t}, M_j)) = \delta_{i,j}$. Since for a fixed $i$ the elements $\bar{e}^i_t \in \text{End}(M_i)$ are conjugated by an element of $\text{End}^*(M_i)$ it follows form Lemma 1 that the elements $e^i_t \in A, 1 \leq t \leq d_i$ are conjugated by an element of $A^*$ and therefore the $A$-modules $P_{i,t}, 1 \leq t \leq d_i$ are isomorphic. We will write $P_i$ instead of $P_{i,t}, 1 \leq t \leq d_i$.

I’ll leave for you to check that the modules $P_i$ are indecomposable and that any indecomposable finitely generated projective $A$-module is isomorphic to $P_i$ for some $i, 1 \leq i \leq r$. \qed

**Definition 0.7.** Let $C$ be an abelian $k$-category

1. We say that $C$ is finite if
   a. It has a finite number of equivalence classes of simple objects $M_i, 1 \leq i \leq r$ and $\text{End}_C(M_i) = k$ for all $i, 1 \leq i \leq r$.
   b. Every object of $C$ has finite length and
   c. For any simple object $M \in \text{Ob}(C)$ there exists a projective object $P \in \text{Ob}(C)$ such that $\text{Hom}_C(P, M) \neq \{0\}$.

2. We say that a projective object $P \in \text{Ob}(C)$ is a progenerator if any object of $C$ is a quotient of some finite multiple of $P$.

**Problem 0.8.** Let $C$ be a finite abelian $k$-category. Then

1. A projective object $P \in \text{Ob}(C)$ is a progenerator if $\text{Hom}_C(P, M) \neq \{0\}$ for any simple object $M \in \text{Ob}(C)$
2. There exists a progenerator $P \in \text{Ob}(C)$

**Definition 0.9.** Let $C$ be a finite abelian $k$-category, $P \in \text{Ob}(C)$ be a progenerator. We denote

1. by $A_P$ the ring $\text{End}_C(P)^{op}$
2. by $\text{A}_P - \text{fmodules}$ the category of finitely generated $A_P$-modules
3. by $F_P$ the functor from $C$ to the category $A_P - \text{fmodules}$ given by

   $$F_P(M) := \text{Hom}_C(P, M)$$

**Theorem 0.10.** The functor $F_P(M)$ defines an equivalence between $C$ and the category $A_P - \text{fmodules}$.
Proof. The action of $A_P$ on $P$ defines a map $\alpha_P \in \text{Hom}_C(P \otimes_k A_P, P)$. For any finitely generated $A_P$-module $X$ we define $\alpha_X \in \text{Hom}_{A_P}(A_P \otimes_k X, X)$ as the action map from $(a \otimes \rightarrow ax, a \in A_P, x \in X$. We denote by $G$ the functor from the category $A_P - \text{fmodules}$ to $C$ given by $G(X) =: \text{Coker}(\psi_X)$ where $\psi_X \in \text{Hom}_C(P \otimes_k A_P \otimes_k X, P \otimes_k X)$ is given by

$$\psi_X := \alpha_P \otimes \text{Id}_X - \text{Id}_P \otimes \alpha_X : P \otimes_k A_P \otimes_k X \rightarrow P \otimes_k X$$

I’ll leave for you to construct isomorphisms $F \circ G \rightarrow \text{Id}_{A_P - \text{fmodules}}$ and $G \circ F \rightarrow \text{Id}_C$. \qed