DEFINITION 0.1. (1) Let $\mathfrak{g}$ be a semi-simple Lie algebra of a connected algebraic $\mathbb{C}$-group $G$ which acts by conjugation on $\mathfrak{g}$. We denote by $\mathcal{B}$ the *flag variety* of Borel subalgebras of $\mathfrak{g}$.

(2) For any $\mathfrak{b} \in \mathcal{B}$ we denote by $\mathfrak{h}_{\mathfrak{b}}$ the quotient of $\mathfrak{b}$ by the unipotent radical.

(3) For any $\mathfrak{b} \in \mathcal{B}$ we denote by $B_{\mathfrak{b}} \subset G$ the stabilizer of $\mathfrak{b}$ in $G$. Then $B_{\mathfrak{b}}$ is a Borel subgroup of $G$.

(4) For any pair $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$ there exist $g \in G$ such that $Ad(g)(\mathfrak{b}') = \mathfrak{b}$ and such element $g$ defines an isomorphism $i_g$ between and $\mathfrak{h}_{\mathfrak{b}}$ and $\mathfrak{h}'_{\mathfrak{b}}$. Since $g$ is uniquely defined up to a left multiplication by an elements of the Borel subgroup $B_{\mathfrak{b}}$ which acts trivially on the quotient $\mathfrak{h}_{\mathfrak{b}}$ we see that the isomorphism $i_g$ does not depend on a choice of $g$. So we have a canonical identification of commutative Lie algebras $\mathfrak{h}_{\mathfrak{b}}, \mathfrak{b} \in \mathcal{B}$. We denote this Lie algebra $\tilde{\mathfrak{h}}$ and call it the *abstract Cartan algebra*. The Weyl group $W$ acts on $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}$ is isomorphic to any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the isomorphism is well defined up to a composition with $w \in W$.

(5) As well known the characteristic polynomial of $ad(x) \in \mathrm{End}(\mathfrak{g}), x \in \mathfrak{g}$ is divisible by $t^r, r := dim(\tilde{\mathfrak{h}})$ and so has a form $t^r D(x,t)$ where $D(x,t)$ is a polynomial in $t$ of degree $dim(\mathfrak{g}) - r$. Moreover [please check] $D(x,t)$ is polynomial function on $\mathfrak{g} \times \mathbb{A}^1$ and we define $D(x) := D(x,0)$.

(6) We define $\mathfrak{g}_{rs} := \{x \in \mathfrak{g} | D(x) \neq 0\}, \mathfrak{h}_{rs} := \mathfrak{g}_{rs} \cap \mathfrak{h}$ and use the identification of $\mathfrak{h}$ with the abstract Cartan algebra $\tilde{\mathfrak{h}}$ to define the open subset $\tilde{\mathfrak{h}}_{rs}$ of $\tilde{\mathfrak{h}}$. Please check that

$$\tilde{\mathfrak{h}}_{rs} = \{h \in \tilde{\mathfrak{h}} | St_W(h) = \{e\}\}$$

(7) We define $\tilde{\mathfrak{g}}$ as the subvariety in $\mathfrak{g} \times \mathcal{B}$ of pairs $(x, \mathfrak{b}) \subset \mathfrak{g} \times \mathcal{B}$ such $x \in \mathfrak{b}$ and denote by $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ and by $\tau : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{h}}$ the natural projections. Since the space $\mathcal{B}$ is compact we see that the morphism $\pi$ is proper.

(8) We define $\tilde{\mathfrak{g}}_{rs} := \pi^{-1}(\mathfrak{g}_{rs})$ and denote by $\pi_{rs}, \tau_{rs}$ the restrictions of $\pi$ and $\tau$ on $\tilde{\mathfrak{g}}_{rs}$.

CLAIM 0.2. (1) *The Weyl group $W$ acts freely on $\tilde{\mathfrak{g}}_{rs}$ and $\pi_{rs}$ is a $W$-torsor [that it $W$ acts simply-transitively on fibers of $\pi_{rs}$].*

(2) *The projection $\tau_{rs}$ is $W$-equivariant.*

Let $\mathfrak{g}$ be a semi-simple Lie algebra of an algebraic $\mathbb{C}$-group $G, \mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $W$ the Weyl group of $\mathfrak{g}, S(\mathfrak{g}^\vee)$ and $S(\mathfrak{h}^\vee)$ be the rings of polynomials functions on $\mathfrak{g}$ and $\mathfrak{h}$. We denote by $A \subset S(\mathfrak{g}^\vee)$ the subring of $Ad$-invariant polynomials and by $C_{\mathfrak{h}} \subset S(\mathfrak{h}^\vee)$ the subring of $W$-invariant polynomials. As we know we can identify the ring $C_{\mathfrak{h}}$ with the ring $C$ of polynomial functions on the abstract Cartan algebra $\tilde{\mathfrak{h}}$. It is clear [please check] that

(1) the restriction map defines a ring homomorphism $r_{\mathfrak{h}} : A \to C_{\mathfrak{h}}$.

(2) The induced ring homomorphism $r : A \to C$ does not depend on a choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

THEOREM 0.3 ( Chevalley). *The ring homomorphism $r : A \to C$ is an isomorphism.*

PROOF. Since the set of semi simple elements is dense in $\mathfrak{g}$ and any semi simple element is conjugate to one in $\mathfrak{h}$ we see that $r$ is injective. To prove the surjectivity consider the ring $\hat{A}$ of $Ad$-invariant regular on $\mathfrak{g}_{rs}$ and the ring $\hat{C}$ of $W$-invariant regular functions on $\tilde{\mathfrak{g}}_{rs}$. As before we have a ring homomorphism $\hat{r} : \hat{A} \to \hat{C}$.

LEMMA 0.4. *The ring homomorphism $\hat{r}$ is an isomorphism.*

PROOF. We want to show that any $f \in \hat{C}$ is of the form $\hat{r}(F)$ for $F \in \hat{A}$. Let $F' := \tau_{rs}^{\star}(f)$. As follows from Claim 2 the function $F'$ is $W$-invariant and therefore has a form $F' = \pi_{rs}^{\star}(F_f), F_f \in \hat{A}$. But this implies that $f = \hat{r}(F_f)$. □

To finish the proof of the theorem it is sufficient to show that the function $F_f$ on $\mathfrak{g}_{rs}$ extends to a regular function on $\mathfrak{g}$ in the case when $f$ extends to a regular function on $\tilde{\mathfrak{h}}$. But this follows immediately from the properness of $\pi$ and the following well known result.

CLAIM 0.5. *Let $X$ be a smooth algebraic variety, $Y \subset X$ a proper closed subvariety $F$ a regular function on $X - Y$ which is bounded [as an analytic function] near any point $y \in Y$. Then $F$ extends to a regular function on $X$.*

[Please check that you know a proof of this Claim.] □