**Definition 0.1.**  
(1) Let \( \mathfrak{g} \) be a semi-simple Lie algebra of a connected algebraic \( \mathbb{C} \)-group \( G \) which acts by conjugation on \( \mathfrak{g} \). We denote by \( \mathcal{B} \) the flag variety of Borel subalgebras of \( \mathfrak{g} \).

(2) For any \( b \in \mathcal{B} \) we denote by \( \mathfrak{h}_b \) the quotient of \( b \) by the unipotent radical.

(3) For any \( b \in \mathcal{B} \) we denote by \( B_b \subset G \) the stabilizer of \( b \) in \( G \). Then \( B_b \) is a Borel subgroup of \( G \).

(4) For any pair \( b, b' \in \mathcal{B} \) there exist \( g \in G \) such that \( Ad(g)(b') = b \) and such element \( g \) defines an isomorphism \( i_g \) between and \( \mathfrak{h}_b \) and \( \mathfrak{h}'_b \).

Since \( g \) is uniquely defined up to a left multiplication by an elements of the Borel subgroup \( B_b \) which acts trivially on the quotient \( \mathfrak{h}_b \) we see that the isomorphism \( i_g \) does not depend on a choice of \( g \).

So we have a canonical identification of commutative Lie algebras \( \mathfrak{h}_b, b \in \mathcal{B} \). We denote this Lie algebra \( \mathfrak{h} \) and call it the abstract Cartan algebra. The Weyl group \( W \) acts on \( \mathfrak{h} \) and \( \mathfrak{h} \) is isomorphic to any Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) and the isomorphism is well defined up to a composition with \( w \in W \).

(5) As well known the characteristic polynomial of \( \text{ad}(x) \in \text{End}(\mathfrak{g}), x \in \mathfrak{g} \) is divisible by \( t^r, r := \text{dim}(\mathfrak{h}) \) and so has a form \( t'D(x,t) \) where \( D(x,t) \) is a polynomial in \( t \) of degree \( \text{dim}(\mathfrak{g}) - r \). Moreover [please check] \( D(x,t) \) is polynomial function on \( \mathfrak{g} \times \mathbb{A}^1 \) and we define \( D(x) := D(x,0) \).

(6) We define \( \mathfrak{g}_{rs} := \{ x \in \mathfrak{g} | D(x) \neq 0 \} \), \( h_{rs} := \mathfrak{g}_{rs} \cap \mathfrak{h} \) and use the identification of \( \mathfrak{h} \) with the abstract Cartan algebra \( \tilde{\mathfrak{h}} \) to define the open subset \( \tilde{h}_{rs} \) of \( \tilde{\mathfrak{h}} \). Please check that

\[
\tilde{h}_{rs} = \{ h \in \tilde{\mathfrak{h}} | \text{St}_W(h) = \{ e \} \}
\]

(7) We define \( \tilde{\mathfrak{g}} \) as the subvariety in \( \mathfrak{g} \times \mathcal{B} \) of pairs \( (x, b) \subset \mathfrak{g} \times \mathcal{B} \) such \( x \in b \) and denote by \( \pi : \tilde{\mathfrak{g}} \to \mathfrak{g} \) and by \( \tau : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{h}} \) the natural projections. Since the space \( \mathcal{B} \) is compact we see that the morphism \( \pi \) is proper.

(8) We define \( \tilde{\mathfrak{g}}_{rs} := \pi^{-1}(\mathfrak{g}_{rs}) \) and denote by \( \pi_{rs}, \tau_{rs} \) the restrictions of \( \pi \) and \( \tau \) on \( \tilde{\mathfrak{g}}_{rs} \).

**Claim 0.2.**  
(1) The Weyl group \( W \) acts freely on \( \tilde{\mathfrak{g}}_{rs} \) and \( \pi_{rs} \) is a \( W \)-torsor [that it \( W \) acts simply-transitively on fibers of \( \pi_{rs} \)].

(2) The projection \( \tau_{rs} \) is \( W \)-equivariant.

Let \( \mathfrak{g} \) be a semi-simple Lie algebra of an algebraic \( \mathbb{C} \)-group \( G, \mathfrak{h} \subset \mathfrak{g} \) a Cartan subalgebra, \( W \) the Weyl group of \( \mathfrak{g}, S(\mathfrak{g}^\vee) \) and \( S(\mathfrak{h}^\vee) \) be the rings of polynomials functions on \( \mathfrak{g} \) and \( \mathfrak{h} \). We denote by \( A \subset S(\mathfrak{g}^\vee) \) the subring of \( Ad \)-invariant polynomials and by \( C_\mathfrak{h} \subset S(\mathfrak{h}^\vee) \) the subring of \( W \)-invariant polynomials. As we know we can identify the ring \( C_\mathfrak{h} \) with the ring \( C \) of polynomial functions on the abstract Cartan algebra \( \mathfrak{h} \). It is clear [please check] that

(1) the restriction map defines a ring homomorphism \( r_\mathfrak{h} : A \to C_\mathfrak{h} \).
(2) The induced ring homomorphism $r : A \to C$ does not depend on a choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

**Theorem 0.3 (Chevalley).** The ring homomorphism $r : A \to C$ is an isomorphism.

**Proof.** Since the set of semi simple elements is dense in $\mathfrak{g}$ and any semi simple element is conjugate to one in $\mathfrak{h}$ we see that $r$ is injective. To prove the surjectivity consider the ring $\hat{A}$ of $Ad$-invariant regular on $\mathfrak{g}_{rs}$ and the ring $\hat{C}$ of $W$-invariant regular functions on $\mathfrak{g}_{rs}$. As before we have a ring homomorphism $\hat{r} : \hat{A} \to \hat{C}$.

**Lemma 0.4.** The ring homomorphism $\hat{r}$ is an isomorphism.

**Proof.** We want to show that any $f \in \hat{C}$ is of the form $\hat{r}(F)$ for $F \in \hat{A}$. Let $F' := \tau_{rs}^*(f)$. As follows from Claim 2 the function $F'$ is $W$-invariant and therefore has a form $F' = \pi_{rs}^*(F_f), F_f \in \hat{A}$. But this implies that $f = \hat{r}(F_f)$.

To finish the proof of the theorem it is sufficient to show that the function $F_f$ on $\mathfrak{g}_{rs}$ extends to a regular function on $\mathfrak{g}$ in the case when $f$ extends to a regular function on $\mathfrak{h}$. But this follows immediately from the properness of $\pi$ and the following well known result.

**Claim 0.5.** Let $X$ be a smooth algebraic variety, $Y \subset X$ a proper closed subvariety $F$ a regular function on $X - Y$ which is bounded [as an analytic function] near any point $y \in Y$. Then $F$ extends to a regular function on $X$.

[Please check that you know a proof of this Claim.]