Definition 0.1. Let $E$ be a finite-dimensional Euclidean vector space, with the standard Euclidean inner product denoted by $(\cdot, \cdot)$. A [reduced] root system in $E$ is a finite set $\Phi$ of vectors (called roots) that satisfy the following conditions:

1. The set $\Phi$ spans $E$.
2. The only scalar multiples of a root $\alpha \in \Phi$ that belong to $\Phi$ are $\alpha$ and $-\alpha$.
3. For every root $\alpha \in \Phi$, the set $\Phi$ is closed under reflection $\sigma_\alpha$ through the hyperplane perpendicular to $\alpha$ where
   \[ \sigma_\alpha(v) := v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha, \quad v \in E \]
4. For any roots $\alpha, \beta \in \Phi$ we have $a_{\alpha, \beta} := (\alpha^\vee, \beta) \in \mathbb{Z}$ where $\alpha^\vee := \frac{2}{(\alpha, \alpha)}\alpha \in E$ is the corresponding coroot. So $\sigma_\alpha(v) = v - (\alpha^\vee, v)\alpha$.

Definition 0.2. (1) A subset $\Phi^+$ of $\Phi$ is a set of positive roots if
   (a) for each root $\alpha \in \Phi$ exactly one of the roots $-\alpha, \alpha$ belongs to $\Phi^+$ and
   (b) for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Phi^+$.
2. The group of isometries of $V$ generated by reflections $\sigma_\alpha, \alpha \in \Phi$ is called the Weyl group $W_\Phi$ of $\Phi$. As the group $W_\Phi$ acts faithfully on the finite set $\Phi$ we see that it is finite.
3. If a subset of positive roots $\Phi^+$ is chosen, elements of $-\Phi^+$ are called negative roots.
4. A positive root is simple if it cannot be written as the sum of two positive roots. We denote by $\Delta \subset \Phi^+$ the set of simple roots.
5. The rank of root system $\Phi \subset E$ is the dimension of $E$.

Low rank examples.

There is only one root system of rank 1, consisting of two nonzero vectors $\{\alpha, -\alpha\}$ such that $(\alpha, \alpha) = 2$. This root system is called $A_1$.

In rank 2 there are four possibilities, which are called $A_1 \times A_1, A_2, B_2$ and $G_2$. The root system $A_1 \times A_1$ is of the form $\{\pm \alpha, \pm \beta\}$ where $\alpha, \beta \in V$ are orthogonal vectors. The set $\Delta$ of simple roots for systems $A_2, B_2$ and $G_2$ consists of vectors $\alpha, \beta \in V$ such that $a_{\alpha, \alpha} = 1, a_{\alpha, \beta} = -1, -2$ or $-3$ correspondingly, $(\alpha, \alpha) = 1, (\beta, \beta) = -a_{\alpha, \beta}$. Therefore $\sigma_\alpha(\beta) = \beta + n\alpha$, where $n = 1, 2, 3$ and $\sigma_\beta(\alpha) = \alpha + \beta$.

Whenever $\Phi$ is a root system in $E$, and $U$ is a subspace of $E$ spanned by $\Psi = \Phi \cap U$ then $\Psi$ is a root system in $U$. Thus, the exhaustive list of four root systems of rank 2 shows that any two roots meet at an angle of 0, 30, 45, 60, 90, 120, 135, 150 or 180 degrees.

Claim 0.3. (1) Given a root system $\Phi$ we can always choose a set $\Phi^+$ of positive roots.
2. The Weyl group $W_\Phi$ acts simply-transitively on the set $\{\Phi^+\}$ of subsets of positive roots.
The set $\Delta$ of simple roots is a basis of $E$ with the property that every vector in $\Phi^+$ is a linear combination of elements of $\Delta$ with integral non-negative coefficients.

(4) For any two different simple roots $\alpha, \beta \in \Phi^+$ we have $-(\alpha, \beta) \in \{0, 1, 2, 3\}$.

(5) The group $W_\Phi$ is generated by reflections $s_\alpha, \alpha \in \Delta$ and relations
\[
s_\alpha^2 = e, \alpha \in \Delta, (s_\alpha s_\beta)^{m_{\alpha, \beta}} = e, \alpha \neq \beta \in \Delta
\]
where $m_{\alpha, \beta} = 2, 3, 4$ or 6 in the case when $-(\alpha, \beta) = 0, 1, 2$ or 3.

(6) For any two different roots $\alpha, \beta \in \Phi$ the roots of the form $\alpha + k\beta, k \in \mathbb{Z}$ form an unbroken chain $\{\alpha - r\beta, \ldots, \alpha, \ldots, \alpha + s\beta\}$ where $r, s \in \mathbb{Z}^+$ and $r + s \leq 4$.

(7) For any $\beta \in \Phi^+$ there exists $\alpha \in \Delta$ such that $(\alpha^\vee, \beta) \in \mathbb{Z}^+$

(8) The set $\{\alpha^\vee\}, \alpha \in \Phi$ of coroots forms the dual root system $\Phi^\vee$ in $V$.

(9) $s_{\alpha^\vee} = s_\alpha$ for all $\alpha \in \Phi$. So both root systems $\Phi$ and $\Phi^\vee$ have the same Weyl group.

(10) Any reflection $s \in W$ is equal to $s_\alpha$ for some $\alpha \in \Phi$.

Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. The Killing form of $\mathfrak{g}$ induces the Euclidean inner product $(,)_{\mathfrak{h}}$ on the dual space $\mathfrak{h}^\vee$. For any $\alpha \in \mathfrak{h}^\vee$ we define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g}|[h, x] = \alpha(h)x, h \in \mathfrak{h}\}$. Let $\Phi_\mathfrak{g} \subset \mathfrak{h}^\vee$ be the the set of non-zero functionals $\alpha \in \mathfrak{h}^\vee$ such that $\mathfrak{g}_\alpha \neq \{0\}$.

**Theorem 0.4.**
1. $\dim(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Phi_\mathfrak{g}$.
2. $\Phi_\mathfrak{g} \subset \mathfrak{h}^\vee$ is a root system.
3. Let $\mathfrak{g}, \mathfrak{g}'$ be semi-simple Lie algebras $\mathfrak{g}, \mathfrak{g}'$ such that the root systems $\Phi_\mathfrak{g}, \Phi_\mathfrak{g}'$ are isomorphic. Then the Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are isomorphic.
4. For any root system $\Phi$ in a vector space $V$ there exists a semi-simple Lie algebra $\mathfrak{g}$ such that $\mathfrak{h} = V$ and $\Phi_\mathfrak{g} = \Phi$. Moreover the Lie algebra $\mathfrak{g}$ is generated by elements $h_i, e_i, f_i, i \in \Delta$ and the Chevalley-Serre relations:

\[
\begin{align*}
1. [h_i, h_j] &= 0 \\
2. [e_i, f_j] &= h_i \text{ if } [e_i, f_j] = 0 \text{ if } i \neq j \\
3. [h_i, e_j] &= a_{ij}e_j \\
4. [h_i, f_j] &= -a_{ij}f_j \\
5. ad(e_i)^{1-a_{ij}}(e_j) &= 0 \\
6. ad(f_i)^{1-a_{ij}}(f_j) &= 0.
\end{align*}
\]

**Definition 0.5.** Let $\Phi \subset E$ be a root system and $\Phi^+$ be a set of positive roots.

1. For any positive root $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ we define $ht(\beta) := \sum_{\alpha \in \Delta} c_\alpha$.
2. The root lattice $\Lambda_r$ is the span in $E$ of the set $\Phi$.
3. The weight lattice $\Lambda$ is defined by
   \[
   \Lambda := \{\lambda \in E|(\alpha^\vee, \lambda) \in \mathbb{Z}\} \text{ for all } \alpha \in \Delta.
   \]

**Remark 0.6.** It is clear that $\Lambda_r \subset \Lambda$. 

Example 0.7 (The root system $A_{n-1}$). Let $\mathbb{R}^n$ be the standard $n$-dimensional Euclidean vector space with the basis $e_i, 1 \leq i \leq n, E \subset \mathbb{R}^n$ be the subspace of vectors $\sum c_ie_i, c_i \in \mathbb{R}$ such that $\sum c_i = 0; \Phi \subset E$ be the set of vectors of the form $\alpha_{i,j} := e_i - e_j, 1 \leq i \neq j \leq n$ and $\Phi^+ \subset \Phi$ be the subset of vectors of the form $\alpha_{i,j}, 1 \leq i < j \leq n$. Then $\Phi$ is a root system in $E, \Phi^+ \subset \Phi$ is a set of positive roots, $\Delta = \{ \alpha_i \}, 1 \leq i < n$ where $\alpha_i := e_i - e_{i+1}, W_\Phi = S_n, \Lambda_r = \mathbb{Z}^n \cap E$ and $\Lambda$ is the span of $\Lambda_r$ and any one of vectors $v_k := e_k - 1/n \sum e_i, 1 \leq k \leq n$. So $\Lambda/\Lambda_r = \mathbb{Z}/n\mathbb{Z}$. We have $m_{\alpha_i, \alpha_j} = 2$ if $|i - j| > 1$ and $m_{\alpha_i, \alpha_j} = 3$ if $|i - j| = 1$. The rank of the root system $A_{n-1}$ is equal to $n - 1$ and $ht(\alpha_{i,j}) = j - i$

Problem 0.8. Let $S_n$ the symmetric group on $n$ elements. The Weyl group $W$ of the Lie algebra of $n \times n$-matrices is equal to $S_n$.

1. For any permutation $w \in S_n$ and a pair $(i, j), 1 \leq i, j \leq n$ we define $w[i; j]$ as the number of elements $k, 1 \leq k \leq i$ such that $w(i) \leq j$.

   Show that $w' \leq w; w', w \in W$ iff $w[i; j] \leq w'[i; j]$ for all pairs $(i, j), 1 \leq i, j \leq n$.

2. Let $\kappa : W \hookrightarrow SL_n(\mathbb{C})$ be the imbedding of $S_n$ as the subgroup of permutation matrices. For any $w \in W$ we define $O_w := B_wB \subset SL_n(\mathbb{C})$ where $B \subset SL_n(\mathbb{C})$ is the subgroup of upper-triangular matrices. Show that $w' \leq w; w', w \in W$ iff $O_{w'} \subset O_w$ where $O_w$ is the closure of $O_w$ in $SL_n(\mathbb{C})$.

Definition 0.9.  

1. Let $\mathfrak{g}$ be a semi-simple Lie algebra of a connected algebraic $\mathbb{C}$-group $G$ which acts by conjugation on $\mathfrak{g}$. We denote by $\mathcal{B}$ the flag variety of Borel subalgebras of $\mathfrak{g}$.

2. For any $\mathfrak{b} \in \mathcal{B}$ we denote by $\mathfrak{h}_\mathfrak{b}$ the quotient of $\mathfrak{b}$ by the unipotent radical and by $q_\mathfrak{b}$ the projection $q_\mathfrak{b} : \mathfrak{b} \to \mathfrak{h}_\mathfrak{b}$.

3. For any $\mathfrak{b} \in \mathcal{B}$ we denote by $B_\mathfrak{b} \subset G$ the stabilizer of $\mathfrak{b}$ in $G$. As well known $B_\mathfrak{b}$ is a Borel subgroup of $G$.

4. For any pair $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$ there exist $g \in G$ such that $Ad(g)(\mathfrak{b}') = \mathfrak{b}$ and such element $g$ defines an isomorphism $i_g$ between and $\mathfrak{h}_\mathfrak{b}$ and $\mathfrak{h}_\mathfrak{b}'$. Since $g$ is uniquely defined up to a left multiplication by an elements of the Borel subgroup $B_\mathfrak{b}$ which acts trivially on the quotient $\mathfrak{h}_\mathfrak{b}$ we see that the isomorphism $i_g$ does not depend on a choice of $g$. So we have a canonical identification of commutative Lie algebras $\mathfrak{h}_\mathfrak{b}, \mathfrak{b} \in \mathcal{B}$. We denote this Lie algebra $\mathfrak{h}$ and call it the abstract Cartan algebra.

5. One can define the Weyl group $W$ as the set of $G$-orbits on $\mathcal{B} \times \mathcal{B}$. Given a representative $(\mathfrak{b}, \mathfrak{b}')$ of $w \in W$ the restrictions of the projections $q_\mathfrak{b}, q_{\mathfrak{b}'}$ to the intersection $\mathfrak{b} \cap \mathfrak{b}'$ defines isomorphisms $t : (\mathfrak{b} \cap \mathfrak{b}')/\mathfrak{n} \to \mathfrak{h}_\mathfrak{b}$ and $t' : (\mathfrak{b} \cap \mathfrak{b}')/\mathfrak{n} \to \mathfrak{h}_{\mathfrak{b}'}$ where $\mathfrak{n}$ is the nilpotent radical of the Lie algebra $\mathfrak{b} \cap \mathfrak{b}'$ and therefore an isomorphism $h \to h_w := t' \circ t(h), h \in \mathfrak{h}$. There exists unique group structure on $W$ such that the map $w \to \hat{w}$ is an action of the group $W$ on $\mathfrak{h}$. 

(6) For any Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) a choice of a Borel subalgebra \( \mathfrak{b} \) defines an isomorphism \( \mathfrak{h} \rightarrow \mathfrak{b} \) well defined up to a composition with \( w \in W \).

(7) As well known the characteristic polynomial of \( ad(x) \in \text{End}(\mathfrak{g}) \), \( x \in \mathfrak{g} \) is divisible by \( t^r \), \( r := \dim(\mathfrak{h}) \) and so has a form \( t^r D(x, t) \) where \( D(x, t) \) is a polynomial in \( t \) of degree \( \dim(\mathfrak{g}) - r \). Moreover [please check] \( D(x, t) \) is polynomial function on \( \mathfrak{g} \times \mathbb{A}^1 \) and we define \( D(x) := D(x, 0) \).

(8) We define \( \mathfrak{g}_{rs} := \{ x \in \mathfrak{g} \mid D(x) \neq 0 \} \), \( \mathfrak{h}_{rs} := \mathfrak{g}_{rs} \cap \mathfrak{h} \) and use the identification of \( \mathfrak{h} \) with the abstract Cartan algebra \( \mathfrak{h} \) to define the open subset \( \mathfrak{h}_{rs} \) of \( \mathfrak{h} \). Please check that
\[
\mathfrak{h}_{rs} = \{ h \in \mathfrak{h} \mid St_W(h) = \{ e \} \}
\]

(9) We define \( \mathfrak{g} \) as the subvariety in \( \mathfrak{g} \times \mathcal{B} \) of pairs \( (x, b) \subset \mathfrak{g} \times \mathcal{B} \) such \( x \in \mathfrak{b} \) and denote by \( \pi : \mathfrak{g} \rightarrow \mathfrak{g} \) and by \( \tau : \mathfrak{g} \rightarrow \mathfrak{h} \) the natural projections. Since the space \( \mathcal{B} \) is compact we see that the morphism \( \pi \) is proper.

(10) We define \( \tilde{\mathfrak{h}}_{rs} := \pi^{-1}(\mathfrak{g}_{rs}) \) and denote by \( \pi_{rs}, \tau_{rs} \) the restrictions of \( \pi \) and \( \tau \) on \( \tilde{\mathfrak{h}}_{rs} \).

Claim 0.10.  
(1) The Weyl group \( W \) acts freely on \( \tilde{\mathfrak{g}}_{rs} \) and \( \pi_{rs} \) is a \( W \)-torsor [that it \( W \) acts simply transitively on fibers of \( \pi_{rs} \)].

(2) The projection \( \tau_{rs} \) is \( W \)-equivariant.

Let \( \mathfrak{g} \) be a semi-simple Lie algebra of an algebraic \( \mathbb{C} \)-group \( G, \mathfrak{h} \subset \mathfrak{g} \) a Cartan subalgebra, \( W \) the Weyl group of \( \mathfrak{g} \). Any semi-simple element \( x \in \mathfrak{g} \) is conjugate to an element \( h_x \in \mathfrak{h} \) under the adjoint action of \( G \) on \( \mathfrak{g} \) and the \( W \) orbit \( h_x^W \subset \mathfrak{h} \) of \( h_x \) does not depend on a choice of \( h_x \in \mathfrak{h} \). In other words we have a well defined map \( \tilde{r}_C : \mathfrak{g}(\mathbb{C}) \rightarrow \mathfrak{h}/W(\mathbb{C}) \) of sets. On the other hand any element \( x \in \mathfrak{g} \) can be written uniquely as the sum \( x = x_s + x_u \) where \( x_s, x_u \) are commuting semi-simple and unipotent elements. So we can define a map \( r_C : \mathfrak{g}(\mathbb{C}) \rightarrow \tilde{\mathfrak{h}}/W(\mathbb{C}) \) of sets where \( r_C(x) := \tilde{r}_C(x_s) \). The Chevalley theorem says that this map is algebraic. To be more precise let \( S(\mathfrak{g}^\vee) \) and \( S(\mathfrak{h}^\vee) \) be the rings of polynomials functions on \( \mathfrak{g} \) and \( \mathfrak{h} \). We denote by \( A \subset S(\mathfrak{g}^\vee) \) the subring of \( Ad \)-invariant polynomials and by \( C_\mathfrak{h} \subset S(\mathfrak{h}^\vee) \) the subring of \( W \)-invariant polynomials. As we know we can identify the ring \( C_\mathfrak{h} \) with the ring \( C \) of polynomial functions on the abstract Cartan algebra \( \tilde{\mathfrak{h}} \). It is clear [please check] that

1. the restriction map defines a ring homomorphism \( r_\mathfrak{h} : A \rightarrow C_\mathfrak{h} \).

2. The induced ring homomorphism \( r : A \rightarrow C \) does not depend on a choice of a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \).

Theorem 0.11 (Chevalley). The ring homomorphism \( r : A \rightarrow C \) is an isomorphism.

Proof. Since the set of semi simple elements is dense in \( \mathfrak{g} \) and any semi simple element is conjugate to one in \( \mathfrak{h} \) we see that \( r \) is injective. To prove
the surjectivity consider the ring $\hat{A}$ of $Ad$-invariant regular on $\mathfrak{g}_{rs}$ and the ring $\hat{C}$ of $W$-invariant regular functions on $\tilde{\mathfrak{g}}_{rs}$. As before we have a ring homomorphism $\hat{r} : \hat{A} \to \hat{C}$.

**Lemma 0.12.** The ring homomorphism $\hat{r}$ is an isomorphism.

**Proof.** We want to show that any $f \in \hat{C}$ is of the form $\hat{r}(F)$ for $F \in \hat{A}$. Let $F' := \tau_{rs}^*(f)$. As follows from Claim 2 the function $F'$ is $W$-invariant and therefore has a form $F' = \pi_{rs}^*(F_f), F_f \in \hat{A}$. But this implies that $f = \hat{r}(F_f)$.

To finish the proof of the theorem it is sufficient to show that the function $F_f$ on $\mathfrak{g}_{rs}$ extends to a regular function on $\mathfrak{g}$ in the case when $f$ extends to a regular function on $\mathfrak{h}$. But this follows immediately from the properness of $\pi$ and the following well known result.

**Claim 0.13.** Let $X$ be a smooth algebraic variety, $Y \subset X$ a proper closed subvariety $F$ a regular function on $X - Y$ which is bounded [as an analytic function] near any point $y \in Y$. Then $F$ extends to a regular function on $X$.

[Please check that you know a proof of this Claim.]