DEFINITION 0.1. Let E be a finite-dimensional Euclidean vector space, with the standard Euclidean inner product denoted by $(\cdot,\cdot)$. A [reduced] *root system* in E is a finite set Φ of vectors (called roots) that satisfy the following conditions:

(1) The set Φ spans E.
(2) The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α and -α.
(3) For every root $\alpha \in \Phi$, the set Φ is closed under reflection σ_α through the hyperplane perpendicular to α where

$$\sigma_\alpha(v) := v - 2\frac{(\alpha, v)}{(\alpha,\alpha)}\alpha, v \in E$$

(4) For any roots $\alpha, \beta \in \Phi$ we have $a_{\alpha,\beta} := (\alpha^\vee, \beta) \in \mathbb{Z}$ where $\alpha^\vee := \frac{2}{(\alpha,\alpha)}\alpha \in E$ is the corresponding *coroot*. So $\sigma_\alpha(v) = v - (\alpha^\vee, v)\alpha$.

DEFINITION 0.2. (1) A subset Φ^+ of Φ is *a set of positive roots* if
 (a) for each root $\alpha \in \Phi$ exactly one of the roots -α, α belongs to Φ^+ and
 (b) for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Phi^+$.
(2) The group of isometries of V generated by reflections $\sigma_\alpha, \alpha \in \Phi$ is called the *Weyl group* W_Φ of Φ. As the group W_Φ acts faithfully on the finite set Φ we see that it is finite.
(3) If a subset of positive roots Φ^+ is chosen, elements of $-\Phi^+$ are called negative roots.
(4) A positive root is *simple* if it cannot be written as the sum of two positive roots. We denote by $\Delta \subset \Phi^+$ the set of simple roots.
(5) The *rank* of root system $\Phi \subset E$ is the dimension of E.

Low rank examples.

There is only one root system of rank 1, consisting of two nonzero vectors $\{\alpha, -\alpha\}$ such that $(\alpha, \alpha) = 2$. This root system is called A_1.

In rank 2 there are four possibilities, which are called $A_1 \times A_1, A_2, B_2$ and G_2. The root system $A_1 \times A_1$ is of the form $\{\pm\alpha, \pm\beta\}$ where $\alpha, \beta \in V$ are orthogonal vectors. The set Δ of simple roots for systems A_2, B_2 and G_2 consists of vectors $\alpha, \beta \in V$ such that $a_{\beta,\alpha} = 1, a_{\alpha,\beta} = -1, -2$ or -3 correspondingly, $(\alpha, \alpha) = 1, (\beta, \beta) = -a_{\alpha,\beta}$. Therefore $\sigma_\alpha(\beta) = \beta + n\alpha$, where $n = 1, 2, 3$ and $\sigma_\beta(\alpha) = \beta + \alpha$.

Whenever Φ is a root system in E, and U is a subspace of E spanned by $\Psi = \Phi \cap U$ then Ψ is a root system in U. Thus, the exhaustive list of four root systems of rank 2 shows that any two roots meet at an angle of $0, 30, 45, 60, 90, 120, 135, 150$ or 180 degrees.

CLAIM 0.3. (1) *Given a root system Φ we can always choose a set Φ^+ of positive roots.*
(2) *The Weyl group W_Φ acts simply-transitively on the set $\{\Phi^+\}$ of subsets of positive roots.*

(3) *The set* Δ *of simple roots is a basis of* E *with the property that every vector in* Φ^+ *is a linear combination of elements of* Δ *with integral non-negative coefficients.*
(4) *For any two different simple roots* $\alpha, \beta \in \Phi^+$ *we have* $-(\alpha, \beta) \in \{0, 1, 2, 3\}$.
(5) *The group* W_Φ *is generated by reflections* $s_\alpha, \alpha \in \Delta$ *and relations*

$$s_\alpha^2 = e, \alpha \in \Delta, (s_\alpha s_\beta)^{m_{\alpha,\beta}} = e, \alpha \neq \beta \in \Delta$$

where $m_{\alpha,\beta} = 2, 3, 4$ *or* 6 *in the case when* $-(\alpha, \beta) = 0, 1, 2$ *or* 3.
(6) *For any two different roots* $\alpha, \beta \in \Phi$ *the roots of the form* $\alpha + k\beta, k \in \mathbb{Z}$ *form an unbroken chain* $\{\alpha - r\beta, ..., \alpha, ..., \alpha + s\beta\}$ *where* $r, s \in \mathbb{Z}^+$ *and* $r + s \leq 4$.
(7) *For any* $\beta \in \Phi^+$ *there exists* $\alpha \in \Delta$ *such that* $(\alpha^\vee, \beta) \in \mathbb{Z}^+$
(8) *The set* $\{\alpha^\vee\}, \alpha \in \Phi$ *of coroots forms the dual root system* $\Phi^\vee$ *in* V.
(9) $s_{\alpha^\vee} = s_\alpha$ *for all* $\alpha \in \Phi$. *So both root systems* Φ *and* $\Phi^\vee$ *have the same Weyl group.*
(10) *Any reflection* $s \in W$ *is equal to* s_α *for some* $\alpha \in \Phi$.

Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. The Killing form of $\mathfrak{g}$ induces the Euclidean inner product $(,)$ on the dual space $\mathfrak{h}^\vee$. For any $\alpha \in \mathfrak{h}^\vee$ we define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} | [h, x] = \alpha(h)x, h \in \mathfrak{h}$. Let $\Phi_\mathfrak{g} \subset \mathfrak{h}^\vee$ be the the set of non-zero functionals $\alpha \in \mathfrak{h}^\vee$ such that $\mathfrak{g}_\alpha \neq \{0\}$.

THEOREM 0.4. (1) $dim(\mathfrak{g}_\alpha) = 1$ *for all* $\alpha \in \Phi_\mathfrak{g}$.
(2) $\Phi_\mathfrak{g} \subset \mathfrak{h}^\vee$ *is a root system.*
(3) *Let* $\mathfrak{g}, \mathfrak{g}'$ *be semi-simple Lie algebras* $\mathfrak{g}, \mathfrak{g}'$ *such that the root systems* $\Phi_\mathfrak{g}, \Phi_{\mathfrak{g}'}$ *are isomorphic. Then the Lie algebras* $\mathfrak{g}, \mathfrak{g}'$ *are isomorphic.*
(4) *For any root system* Φ *in a vector space* V *there exists a semi-simple Lie algebra* $\mathfrak{g}$ *such that* $\mathfrak{h} = V$ *and* $\Phi_\mathfrak{g} = \Phi$. *Moreover the Lie algebra* $\mathfrak{g}$ *is generated by elements* $h_i, e_i, f_i, i \in \Delta$ *and the Chevalley-Serre relations:*

1.$[h_i, h_j] = 0$
2.$[e_i, f_i] = h_i$ *and* $[e_i, f_j] = 0$ *if* $i \neq j$
3.$[h_i, e_j] = a_{ij} e_j$
4.$[h_i, f_j] = -a_{ij} f_j$
5.$ad(e_i)^{1 - a_{ij}}(e_j) = 0$
6.$ad(f_i)^{1 - a_{ij}}(f_j) = 0$.

DEFINITION 0.5. Let $\Phi \subset E$ be a root system and Φ^+ be a set of positive roots.
(1) For any positive root $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ we define $ht(\beta) := \sum_{\alpha \in \Delta} c_\alpha$.
(2) The *root lattice* Λ_r is the span in E of the set Φ
(3) The *weight lattice* Λ is defined by
$\Lambda := \{\lambda \in E | (\alpha^\vee, \lambda) \in \mathbb{Z}\}$ for all $\alpha \in \Delta$.

REMARK 0.6. It is clear that $\Lambda_r \subset \Lambda$.

Example 0.7 (The root system A_{n-1}). Let $\mathbb{R}^n$ be the standard n-dimensional Euclidean vector space with the basis $e_i, 1 \le i \le n, E \subset \mathbb{R}^n$ be the subspace of vectors $\sum_i c_i e_i, c_i \in \mathbb{R}$ such that $\sum_i c_i = 0, \Phi \subset E$ be the set of vectors of the form $\alpha_{i,j} := e_i - e_j, 1 \le i \ne j \le n$ and $\Phi^+ \subset \Phi$ be the subset of vectors of the form $\alpha_{i,j}, 1 \le i < j \le n$. Then Φ is a root system in E, $\Phi^+ \subset \Phi$ is a set of positive roots, $\Delta = \{\alpha_i\}, 1 \le i < n$ where $\alpha_i := e_i - e_{i+1}, W_\Phi = S_n, \Lambda_r = \mathbb{Z}^n \cap E$ and Λ is the span of Λ_r and any one of vectors $v_k := e_k - 1/n \sum_{i=1}^n e_i, 1 \le k \le n$. So $\Lambda/\Lambda_r = \mathbb{Z}/n\mathbb{Z}$. We have $m_{\alpha_i,\alpha_j} = 2$ if $|i-j| > 1$ and $m_{\alpha_i,\alpha_j} = 3$ if $|i-j| = 1$. The rank of the root system A_{n-1} is equal to $n-1$ and $ht(\alpha_{i,j}) = j - i$

Problem 0.8. Let S_n the symmetric group on n elements. The Weyl group W of the Lie algebra of $n \times n$-matrices is equal to S_n.

(1) For any permutation $w \in S_n$ and a pair $(i,j), 1 \le i,j \le n$ we define $w[i;j]$ as the number of elements $k, 1 \le k \le i$ such that $w(i) \le j$.
   Show that $w' \le w; w', w \in W$ iff $w[i;j] \le w'[i;j]$ for all pairs $(i,j), 1 \le i,j \le n$.
(2) Let $\kappa : W \hookrightarrow SL_n(\mathbb{C})$ be the imbedding of S_n as the subgroup of permutation matrices. For any $w \in W$ we define $\mathcal{O}_w := B\kappa B \subset SL_n(\mathbb{C})$ where $B \subset SL_n(\mathbb{C})$ is the subgroup of upper-triangular matrices. Show that $w' \le w; w', w \in W$ iff $\mathcal{O}_{w'} \subset \bar{\mathcal{O}}_w$ where $\bar{\mathcal{O}}_w$ is the closure of $\mathcal{O}_w$ in $SL_n(\mathbb{C})$.

Definition 0.9. (1) Let $\mathfrak{g}$ be a semi-simple Lie algebra of a connected algebraic $\mathbb{C}$-group G which acts by conjugation on $\mathfrak{g}$. We denote by $\mathcal{B}$ the *flag variety* of Borel subalgebras of $\mathfrak{g}$.
(2) For any $\mathfrak{b} \in \mathcal{B}$ we denote by $\mathfrak{h}_\mathfrak{b}$ the quotient of $\mathfrak{b}$ by the unipotent radical and by $q_\mathfrak{b}$ the projection $q_\mathfrak{b} : \mathfrak{b} \to \mathfrak{h}_\mathfrak{b}$.
(3) For any $\mathfrak{b} \in \mathcal{B}$ we denote by $B_\mathfrak{b} \subset G$ the stabilizer of $\mathfrak{b}$ in G. As well known $B_\mathfrak{b}$ is a Borel subgroup of G.
(4) For any pair $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$ there exist $g \in G$ such that $Ad(g)(\mathfrak{b}') = \mathfrak{b}$ and such element g defines an isomorphism i_g between and $\mathfrak{h}_\mathfrak{b}$ and $\mathfrak{h}'_\mathfrak{b}$. Since g is uniquely defined up to a left multiplication by an elements of the Borel subgroup $B_\mathfrak{b}$ which acts trivially on the quotient $\mathfrak{h}_\mathfrak{b}$ we see that the isomorphism i_g does not depend on a choice of g. So we have a canonical identification of commutative Lie algebras $\mathfrak{h}_\mathfrak{b}, \mathfrak{b} \in \mathcal{B}$. We denote this Lie algebra $\tilde{\mathfrak{h}}$ and call it the *abstract Cartan algebra*.
(5) One can define the Weyl group W as the set of G-orbits on $\mathcal{B} \times \mathcal{B}$. Given a representative $(\mathfrak{b}, \mathfrak{b}')$ of $w \in W$ the restrictions of the projections $q_\mathfrak{b}, q_{\mathfrak{b}'}$ to the intersection $\mathfrak{b} \cap \mathfrak{b}'$ defines isomorphisms $t : (\mathfrak{b} \cap \mathfrak{b}')/\mathfrak{n} \to \mathfrak{h}_\mathfrak{b}$ and $t' : (\mathfrak{b} \cap \mathfrak{b}')/\mathfrak{n} \to \mathfrak{h}_{\mathfrak{b}'}$ where $\mathfrak{n}$ is the nilpotent radical of the Lie algebra $\mathfrak{b} \cap \mathfrak{b}'$ and therefore an isomorphism $h \to h^w := t' \circ t(h), h \in \tilde{\mathfrak{h}}$. There exists unique group structure on W such that the map $w \to \hat{w}$ is an action of the group W on $\tilde{\mathfrak{h}}$.

(6) For any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ a choice of a Borel subalgebra $\mathfrak{b}$ defines an isomorphism $\mathfrak{h} \to \tilde{\mathfrak{h}}$ well defined up to a composition with $w \in W$.
(7) As well known the characteristic polynomial of $ad(x) \in \mathrm{End}(\mathfrak{g}), x \in \mathfrak{g}$ is divisible by $t^r, r := dim(\tilde{\mathfrak{h}})$ and so has a form $t^r D(x,t)$ where $D(x,t)$ is a polynomial in t of degree $dim(\mathfrak{g}) - r$. Moreover [please check] $D(x,t)$ is polynomial function on $\mathfrak{g} \times \mathbb{A}^1$ and we define $D(x) := D(x,0)$.
(8) We define $\mathfrak{g}_{rs} := \{x \in \mathfrak{g} | D(x) \neq 0\}, \mathfrak{h}_{rs} := \mathfrak{g}_{rs} \cap \mathfrak{h}$ and use the identification of $\mathfrak{h}$ with the abstract Cartan algebra $\tilde{\mathfrak{h}}$ to define the open subset $\tilde{\mathfrak{h}}_{rs}$ of $\tilde{\mathfrak{h}}$. Please check that
$$\tilde{\mathfrak{h}}_{rs} = \{h \in \tilde{\mathfrak{h}} | St_W(h) = \{e\}\}$$
(9) We define $\tilde{\mathfrak{g}}$ as the subvariety in $\mathfrak{g} \times \mathcal{B}$ of pairs $(x, \mathfrak{b}) \subset \mathfrak{g} \times \mathcal{B}$ such $x \in \mathfrak{b}$ and denote by $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ and by $\tau : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{h}}$ the natural projections. Since the space $\mathcal{B}$ is compact we see that the morphism π is proper.
(10) We define $\tilde{\mathfrak{g}}_{rs} := \pi^{-1}(\mathfrak{g}_{rs})$ and denote by π_{rs}, τ_{rs} the restrictions of π and τ on $\tilde{\mathfrak{g}}_{rs}$.

Claim 0.10. (1) *The Weyl group W acts freely on $\tilde{\mathfrak{g}}_{rs}$ and π_{rs} is a W-torsor [that it W acts simply-transitively on fibers of π_{rs}].*
(2) *The projection τ_{rs} is W-equivariant.*

Let $\mathfrak{g}$ be a semi-simple Lie algebra of an algebraic $\mathbb{C}$-group $G, \mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, W the Weyl group of $\mathfrak{g}$. Any semi-simple element $x \in \mathfrak{g}$ is conjugate to an element $h_x \in \mathfrak{h}$ under the adjoint action of G on $\mathfrak{g}$ and the W orbit $h_x^W \subset \mathfrak{h}$ of h_x does not depend on a choice of $h_x \in \mathfrak{h}$. In other words we have a well defined map $\tilde{r}_{\mathbb{C}} : \mathfrak{g}(\mathbb{C}) \to \mathfrak{h}/W(\mathbb{C})$ of sets. On the other hand any element $x \in \mathfrak{g}$ can be written uniquely as the sum $x = x_s + x_n$ where x_s, x_n are commuting semi-simple and unipotent elements. So we can define a map $r_{\mathbb{C}} : \mathfrak{g}(\mathbb{C}) \to \mathfrak{h}/W(\mathbb{C})$ of sets where $r_{\mathbb{C}}(x) := \tilde{r}_{\mathbb{C}}(x_s)$. The Chevalley] theorem says that this map is algebraic. To be more precise let $S(\mathfrak{g}^\vee)$ and $S(\mathfrak{h}^\vee)$ be the rings of polynomials functions on $\mathfrak{g}$ and $\mathfrak{h}$. We denote by $A \subset S(\mathfrak{g}^\vee)$ the subring of Ad-invariant polynomials and by $C_{\mathfrak{h}} \subset S(\mathfrak{h}^\vee)$ the subring of W-invariant polynomials. As we know we can identify the ring $C_{\mathfrak{h}}$ with the ring C of polynomial functions on the abstract Cartan algebra $\tilde{\mathfrak{h}}$. It is clear [please check] that

(1) the restriction map defines a ring homomorphism $r_{\mathfrak{h}} : A \to C_{\mathfrak{h}}$.
(2) The induced ring homomorphism $r : A \to C$ does not depend on a choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Theorem 0.11 (Chevalley). *The ring homomorphism $r : A \to C$ is an isomorphism.*

Proof. Since the set of semi simple elements is dense in $\mathfrak{g}$ and any semi simple element is conjugate to one in $\mathfrak{h}$ we see that r is injective. To prove the surjectivity consider the ring $\hat{A}$ of Ad-invariant regular on $\mathfrak{g}_{rs}$ and the ring $\hat{C}$ of W-invariant regular functions on $\tilde{\mathfrak{g}}_{rs}$. As before we have a ring homomorphism $\hat{r}:\hat{A}\to\hat{C}$.

LEMMA 0.12. *The ring homomorphism $\hat{r}$ is an isomorphism.*

PROOF. We want to show that any $f\in\hat{C}$ is of the form $\hat{r}(F)$ for $F\in\hat{A}$. Let $F':=\tau^{\star}_{rs}(f)$. As follows from Claim 2 the function F' is W-invariant and therefore has a form $F'=\pi^{\star}_{rs}(F_f),F_f\in\hat{A}$. But this implies that $f=\hat{r}(F_f)$. □

To finish the proof of the theorem it is sufficient to show that the function F_f on $\mathfrak{g}_{rs}$ extends to a regular function on $\mathfrak{g}$ in the case when f extends to a regular function on $\tilde{\mathfrak{h}}$. But this follows immediately from the properness of π and the following well known result.

CLAIM 0.13. *Let X be a smooth algebraic variety, $Y\subset X$ a proper closed subvariety F a regular function on $X-Y$ which is bounded [as an analytic function] near any point $y\in Y$. Then F extends to a regular function on X.*

[Please check that you know a proof of this Claim.] □