Unions and Intersections of Leray Complexes*

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Abstract

A simplicial complex X is *d-Leray* if $\tilde{H}_i(Y;\mathbb{Q}) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. Let $L(X)$ denote the minimal d such that X is d-Leray.

Theorem: Let X, Y be simplicial complexes on the same vertex set. Then

$$L(X \cap Y) \leq L(X) + L(Y) \quad ,$$

$$L(X \cup Y) \leq L(X) + L(Y) + 1 \quad .$$

1 Introduction

Let X be a simplicial complex on the vertex set V. The *induced* subcomplex on a subset of vertices $S \subset V$ is $X[S] = \{\sigma \in X : \sigma \subset S\}$. The *link* of a simplex $\sigma \in X$ is $\mathrm{lk}(X,\sigma) = \{\tau \in X \ : \ \tau \cap \sigma = \emptyset \ , \ \tau \cup \sigma \in X \ \}$. All (co)homology groups considered in this note are with rational coefficients.

A simplicial complex X is *d-Leray* if $\tilde{H}_i(Y) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. Equivalently X is d-Leray if $\tilde{H}_i(\mathrm{lk}(X,\sigma)) = 0$ for all $\sigma \in X$ and $i \geq d$. Let $L(X)$ denote the minimal d such that X is d-Leray.

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Note that $L(X) = 0$ iff X is a simplex. $L(X) \leq 1$ iff X is the clique complex of a chordal graph.

The class $\mathcal{L}^d$ of d-Leray complexes arises naturally in the context of Helly type theorems [3]. The *Helly number* $\mathrm{h}(\mathcal{F})$ of a family of sets $\mathcal{F}$ is the minimal positive integer h such that if $\mathcal{K} \subset \mathcal{F}$ is finite and $\bigcap_{K\in\mathcal{K}'} K \neq \emptyset$ for all $\mathcal{K}' \subset \mathcal{K}$ of cardinality $\leq h$, then $\bigcap_{K\in\mathcal{K}} K \neq \emptyset$. The *nerve* $N(\mathcal{K})$ of a family of sets $\mathcal{K}$, is the simplicial complex whose vertex set is $\mathcal{K}$ and whose simplices are all $\mathcal{K}' \subset \mathcal{K}$ such that $\bigcap_{K\in\mathcal{K}'} K \neq \emptyset$. It is easy to see that

$$\mathrm{h}(\mathcal{F}) \leq 1 + L(N(\mathcal{F})).$$

For example, if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$, then by the Nerve Lemma (see e.g. [2]) $N(\mathcal{F})$ is d-Leray, hence follows Helly's Theorem: $\mathrm{h}(\mathcal{F}) \leq d+1$. This argument actually proves the Topological Helly Theorem: If $\mathcal{F}$ is a finite family of closed sets in $\mathbb{R}^d$ such that the intersection of any subfamily of $\mathcal{F}$ is either empty or contractible, then $\mathrm{h}(\mathcal{F}) \leq d+1$.

Nerves of families of convex sets however satisfy a stronger combinatorial property called *d-collapsibility* [7], that leads to some of the deeper extensions of Helly's Theorem. It is of considerable interest to understand which combinatorial properties of nerves of families of convex sets in $\mathbb{R}^d$ extend to arbitrary d-Leray complexes. For some recent work in this direction see [1, 4].

In this note we are concerned with unions and intersections of Leray complexes.

Theorem 1.1. *Let $X_1, \ldots, X_r$ be simplicial complexes on the same finite vertex set. Then*

$$L\Big(\bigcap_{i=1}^{r} X_i\Big) \leq \sum_{i=1}^{r} L(X_i) \qquad (1)$$

$$L\Big(\bigcup_{i=1}^{r} X_i\Big) \leq \sum_{i=1}^{r} L(X_i) + r - 1 \qquad (2)$$

Example: Let $V_1, \ldots, V_r$ be disjoint sets of cardinalities $|V_i| = a_i$, and let $V = \bigcup_{i=1}^r V_i$. Consider the complexes $X_i = \{\sigma \subset V \ : \ \sigma \not\supset V_i\}$. Then $L(X_i) = a_i - 1$, $L(\cap_{i=1}^r X_i) = \sum_{i=1}^r a_i - r$ and $L(\cup_{i=1}^r X_i) = \sum_{i=1}^r a_i - 1$, so equality is attained in both (1) and (2).

Define the *rational homological dimension* of a complex X by

$$\mathrm{hd}(X) = \max\{i \geq 0 : H^i(X) \neq 0\}$$

where by convention $\mathrm{hd}(\emptyset) = -1$. If X is not a simplex, then

$$L(X) = 1 + \max\{\mathrm{hd}(X[S]) : S \subset V\} = 1 + \max\{\mathrm{hd}(\mathrm{lk}(X,\sigma)) : \sigma \in X\}.$$

Theorem 1.1 is a consequence of the following

Theorem 1.2. *Let X, Y be complexes on the same finite vertex set. Then*

$$\mathrm{hd}(X \cap Y) \leq 1 + \max_{\sigma \in Y}\big\{\mathrm{hd}(X[\sigma]) + \mathrm{hd}(\mathrm{lk}(Y,\sigma))\big\} \quad .$$

Our main result (Proposition 2.1) gives a spectral sequence for the cohomology of the intersection of two complexes, which directly implies Theorems 1.2 and 1.1. The derivation of this sequence involves a simple use of the method of simplicial resolutions. For more advanced applications of simplicial resolutions see Vassiliev's papers [6, 5].

2 A Spectral Sequence for $H^*(X \cap Y)$

Let K be a simplicial complex. The subdivision $\mathrm{sd}(K)$ is the order complex of the set of simplices of K ordered by inclusion. For $\sigma \in K$ let $D_K(\sigma)$ denote the order complex of the interval $[\sigma, \cdot] = \{\tau \in K \ : \ \tau \supset \sigma\}$. $D_K(\sigma)$ is called the *dual cell* of σ. Let $\dot{D}_K(\sigma)$ denote the order complex of the interval $(\sigma, \cdot] = \{\tau \in K \ : \ \tau \supsetneqq \sigma\}$. Note that $\dot{D}_K(\sigma)$ is isomorphic to $\mathrm{sd}(\mathrm{lk}(K,\sigma))$ via the simplicial map $\tau \to \tau - \sigma$. Since $D_K(\sigma)$ is contractible, it follows that $H^i(D_K(\sigma), \dot{D}_K(\sigma)) \cong \tilde{H}^{i-1}(\mathrm{lk}(K,\sigma))$ for all $i \geq 0$. Write $K(p)$ for the family of p-dimensional simplices in K.
Our main observation is the following

Proposition 2.1. *For two complexes X and Y on the same finite vertex set, there exists a spectral sequence $\{E_r\}$ converging to $H^*(X \cap Y)$ such that*

$$E_1^{p,q} = \bigoplus_{\sigma \in Y(n-p)} \bigoplus_{\substack{i,j \geq 0 \\ i+j=p+q}} H^i(X[\sigma]) \otimes \tilde{H}^{j-1}(\mathrm{lk}(Y,\sigma)) \quad .$$

Proof: Let

$$K = \bigcup_{\sigma \in Y} X[\sigma] \times D_Y(\sigma) \subset X \times \mathrm{sd}(Y)$$

and let

$$\pi : K \to \bigcup_{\sigma \in Y} X[\sigma] = X \cap Y$$

denote the projection on the first coordinate. For $z \in X \cap Y$, let $\tau = \mathrm{supp}(z)$ denote the minimal simplex in $X \cap Y$ containing z. The fiber $\pi^{-1}(z) = \{z\} \times D_Y(\tau)$ is a cone, hence π is a homotopy equivalence. For $0 \le p \le \dim Y = n$ let

$$F_p = \bigcup_{\substack{\sigma \in Y \\ \dim \sigma \ge n-p}} X[\sigma] \times D_Y(\sigma) \quad .$$

By excision

$$H^*\big(F_p, F_{p-1}) \cong H^*(\bigcup_{\sigma \in Y(n-p)} X[\sigma] \times D_Y(\sigma), \bigcup_{\sigma \in Y(n-p)} X[\sigma] \times \dot{D}_Y(\sigma)\big) \quad . \tag{3}$$

Next note that for distinct $\sigma, \tau \in Y(n-p)$

$$\big(X[\sigma] \times D_Y(\sigma) - X[\sigma] \times \dot{D}_Y(\sigma)\big) \cap X[\tau] \times D_Y(\tau) = \emptyset \quad .$$

Together with (3) this implies a direct sum decomposition

$$H^*(F_p, F_{p-1}) \cong \bigoplus_{\sigma \in Y(n-p)} H^*(X[\sigma] \times D_Y(\sigma), X[\sigma] \times \dot{D}_Y(\sigma)) \quad . \tag{4}$$

Let $\{E_r\}$ be the cohomology spectral sequence associated with the filtration $\emptyset \subset F_0 \subset \cdots \subset F_n = K$. Then by (4) and the Künneth formula

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}) \cong$$

$$\bigoplus_{\sigma \in Y(n-p)} H^{p+q}(X[\sigma] \times D_Y(\sigma), X[\sigma] \times \dot{D}_Y(\sigma)) \cong$$

$$\bigoplus_{\sigma \in Y(n-p)} \bigoplus_{\substack{i,j \ge 0 \\ i+j=p+q}} H^i(X[\sigma]) \otimes H^j(D_Y(\sigma), \dot{D}_Y(\sigma)) \cong$$

$$\bigoplus_{\sigma \in Y(n-p)} \bigoplus_{\substack{i,j \ge 0 \\ i+j=p+q}} H^i(X[\sigma]) \otimes \tilde{H}^{j-1}(\dot{D}_Y(\sigma)) \cong$$

$$\bigoplus_{\sigma \in Y(n-p)} \bigoplus_{\substack{i,j \ge 0 \\ i+j=p+q}} H^i(X[\sigma]) \otimes \tilde{H}^{j-1}(\mathrm{lk}(Y, \sigma)) \quad .$$

□

Proof of Theorem 1.2: We have to show that $H^k(X \cap Y) = 0$ for $k \geq 2 + \max\big\{\mathrm{hd}(X[\sigma]) + \mathrm{hd}(\mathrm{lk}(Y,\sigma)) : \sigma \in Y\big\}$. By Proposition 2.1 it suffices to check that

$$H^i(X[\sigma]) \otimes \tilde{H}^{j-1}(\mathrm{lk}(Y,\sigma)) = 0$$

for all $\sigma \in Y$ and $i, j \geq 0$ such that $i + j = k$. Indeed, if $i \geq 1 + \mathrm{hd}(X[\sigma])$ then $H^i(X[\sigma]) = 0$. Otherwise $j \geq 2 + \mathrm{hd}(\mathrm{lk}(Y,\sigma))$ and $\tilde{H}^{j-1}(\mathrm{lk}(Y,\sigma)) = 0$.

□

Remark: For an r-tuple $(X_1, \ldots, X_r)$ of simplicial complexes on the same finite vertex set V, let $\mathcal{F}$ denote the set of all chains

$$\emptyset = \sigma_1 \prec \sigma_2 \prec \ldots \prec \sigma_r \prec \sigma_{r+1} = V$$

such that $\sigma_i \in X_i$ for all $1 \leq i \leq r$. Iterating Theorem 1.2 we obtain the following

Theorem 2.2.

$$\mathrm{hd}\big(\bigcap_{i=1}^{r} X_i\big) \leq \max_{(\sigma_1,\ldots,\sigma_{r+1}) \in \mathcal{F}} \sum_{i=1}^{r} \mathrm{hd}\big(\mathrm{lk}(X_i[\sigma_{i+1}], \sigma_i)\big) + r - 1 \quad .$$

□

Proof of Theorem 1.1: By induction it suffices to consider the $r = 2$ case. If X or Y are simplices, then $L(X \cap Y) \leq \max\{L(X), L(Y)\}$. We may thus assume that neither X nor Y are simplices. By Theorem 1.2

$$\mathrm{hd}(X \cap Y) \leq 1 + \max_{\sigma \in Y}\big(\mathrm{hd}(X[\sigma]) + \mathrm{hd}(\mathrm{lk}(Y,\sigma))\big) \leq L(X) + L(Y) - 1.$$

Hence, by the monotonicity of L

$$L(X \cap Y) \leq L(X) + L(Y) \quad . \tag{5}$$

Next, let $k \geq L(X) + L(Y) + 1$. Then by (5) and the Mayer-Vietoris sequence

$$\to \tilde{H}^{k-1}(X \cap Y) \to H^k(X \cup Y) \to H^k(X) \oplus H^k(Y) \to$$

it follows that $H^k(X \cup Y) = 0$. Again by monotonicity we obtain

$$L(X \cup Y) \leq L(X) + L(Y) + 1.$$

□

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