Social choice theory

Topics in Discrete Mathematics: CS/Math 945

Instructor: Gil Kalai

Problem Set I: Expected Utility Theory

Instruction: you can do the problems in groups of at most three. You can use any material you want but please indicate your source if you used a source.

This problem-set is about individual choices and not about social choice. The major new ingredient is that are considering individual preference and choice under uncertainty: we add is to introduce probabilities.

1) The purpose of this problem set is to introduce you to von Neumann Morgensterm utility theory and some related issues. Problems 2 and 3 will depend on the notations of this problem.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n alternatives. Individuals are offered to choose between **lotteries** on the alternatives. For example, they can choose between

Getting alternative a with probability 0.5 and alternative b with probability 0.5

or

getting alternative c.

Every lottery z on the alternatives can be described by a vector of probabilities $(p_1, p_2, \ldots, p_n)$ where for every i, $0 \le p_i \le 1$ and $p_1 + p_2 + \cdots + p_n = 1$.

The lotterry z consists of getting the prize a_1 with probability p_1, getting the prize a_2 with probability p_2 and so on. We will also use the notation

$$z \; = \; p_1 \circ a_1 + p_2 \circ a_2 + \cdots + p_n \circ a_n,$$

and ommit the terms where $p_i = 0$. (The later notation is convenient if we want to consider lotteries which involve a few, usually two alternatives.)

This later notation is also used to consider "lotteries of lotteries" (or compound lotteries). If $z_1, z_2, \ldots, z_k$ are lotteries and $p_1, p_2, \ldots, p_k$ probabilities that sum up to one, the compound lottery

$$z \; = \; p_1 \circ z_1 + p_2 \circ z_2 + \cdots + p_k \circ z_k,$$

stands for having lotterry z_1 with probability p_1, lottery z_2 with probability p_2 etc, and then after the chosen lottery z have been determined chosing the alternatives according to the probabilities for the lottery z. Every alternative a can be regarded as a lottery $1 \circ a$ and every lottery z can be regarded as a compound lottery $1 \circ z$. Suppose that we have a transitive preference relation on the set of all compound lotteries which satisfies two additional axioms.

The Independence Axiom (I).

For any three loterries p, q and r and every α, $0 < \alpha < 1$

$$p \; \succeq \; q$$

if and only if

$$\alpha \circ p + (1 - \alpha) \circ r \; \succeq \; \alpha \circ p + (1 - \alpha) \circ r.$$

The continuity axiom (or Archimedean axiom) (C).

If

$$p \succ q \succ r$$

then there exists α, $0 < \alpha < 1$ so that

$$q \sim \alpha \circ p + (1 - \alpha) \circ r.$$

Theorem:

There is a unique (up to linear functions) function $u : A \to R$ so that for

$$z_1 = p_1 \circ a_1 + p_2 \circ a_2 + \cdots + p_n \circ a_n,$$

and

$$z_2 = q_1 \circ a_1 + q_2 \circ a_2 + \cdots + q_n \circ a_n,$$

We have

$$z_1 \succ z_2$$

if and only if

$$\sum_{i=1}^{n} p_i u(a_i) > \sum_{i=1}^{n} q_i u(a_i).$$

We refer to $u(a_i)$ as the utility of alternative a_i and to

$$U(z_1) = \sum_{i=1}^{n} p_i u(a_i)$$

as the utility (or expected utility) of the lottery z_1.

2) **Totally non archimedean utilities:**

Lets replace (C) by almost the opposite property.

(NA) For every a, b, c such that $a \succ b \succ c$ either

$$[\textbf{Right}] \qquad b \succ p_1 \circ a + p_2 \circ c,$$

for every p, $0 < p < 1$

or

$$[\textbf{Left}] \qquad b \succ p_1 \circ a + p_2 \circ c.$$

for every p, $0 < p < 1$.

Prove that there is a binary tree T and a map from A to the leaves of this tree so that $a \succ b$ if the leave $U(a)$ is to the left of the leave $U(b)$. And for a, b and c, we have equation [**Right**] if the minimal vertex above b and c is strictly below the minimal vertex above a and c, and we have equation [**Left**] if the minimal vertex above a and b is srivtly below the minimal vertex above a and c.

3) **One-sided left totally non archimedean utilities:**

Let us now replace (C) the even more restricted property (always [**Right**]).

For every a, b, c such that $a \succ b \succ c$

$$b \succ p_1 \circ a + p_2 \circ c,$$

for every p, $0 < p < 1$.

Show that you can associate to every alternative a a vector of the form $u(a) = (1, 1, 1, \ldots, 1, 0, 0, 0, \ldots)$ such that for

$$z_1 = p_1 \circ a_1 + p_2 \circ a_2 + \cdots + p_n \circ a_n,$$

and

$$z_2 = q_1 \circ a_1 + q_2 \circ a_2 + \cdots + q_n \circ a_n,$$

we have

$$z_1 \succ z_2$$

If and only if

$\sum_{i=1}^{n} p_i u(a_i)$ is lexicographically larger than $\sum_{i=1}^{n} q_i u(a_i)$.

Can you give a "story" for a decision maker who has preferences which are described by utility functions described in this problem.

4 a) Search the internet and read about the following topics: "Utility," utilitarianism," "choice," "decision".

b) Find some material regarding controversy and critque concerning the expected utility theory.